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Modular Forms ①

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Books

Diamond, Shurman, "A first course in modular forms", GTM

Chapters 1 - 5 contain most course material.

Zagier's Chapter of "The 1-2-3 of modular forms"

Concise introduction and many examples/applications.

Milne Lecture Notes.

1. Introduction

Some notation:

$\Gamma(1)$ $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc = 1 \right\}$

"modular group"

$\mathbb{H} = \{ \tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0 \}$

$\Gamma(1)$ acts on $\mathbb{H}$ by:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \dfrac{a\tau + b}{c\tau + d}$ (exercise: this is a group action)

Definition

Let $k \in \mathbb{Z}$, $\Gamma \subset \Gamma(1)$ a subgroup of finite index

For now, a weakly modular function (weight k, level Γ) is a modular form if it is holomorphic on $\mathbb{H}$ together with some other condition (*).

If $\Gamma = \Gamma(1)$ "level 1", then (*) is equivalent to:

$\exists$ constants $y, c \in \mathbb{R}_{>0}$ such that $|f(\tau)| \leq c$ for $\mathrm{Im}(\tau) >$

Motivating Examples

A) Representation numbers for quadratic forms

If $k \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 0}$

Definition

$r_k(n) :=$ # distinct ways to write n as a sum of k squares

We count different orderings/signs as distinct.

e.g. $r_2(1) = 4$: $1 = 1^2 + 0^2 = 0^2 + 1^2 = (-1)^2 + 0^2 = 0^2 + (-1$

$r_2(n)$, $x^2 + y^2$, $r_k(n)$, $x_1^2 + \ldots + x_k^2$

Studied by Gauss, Jacobi.

Jacobi Θ-function is defined on $\mathbb{H}$ by $\Theta(\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 \tau}$

Formal manipulation: if $k \geq 1$ then

$$\Theta(\tau)^k = \sum_{n \geq 0} r_k(n) e^{2\pi i n \tau}$$

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B) Complex Uniformisation of Elliptic Curves

If we have a lattice Λ in $\mathbb{C}$ (discrete subgroup $\cong \mathbb{Z}^2$), then the Weierstrass $\wp$-function is

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

$$\wp(z;\Lambda) : \mathbb{C}/\Lambda \to \mathbb{C}$$

$$\wp(z+\omega;\Lambda) = \wp(z;\Lambda), \quad \omega\in\Lambda$$

If we define an elliptic curve

$$E_\Lambda : y^2 = 4x^3 - g_4(\Lambda)x - g_6(\Lambda)$$

$$g_4(\Lambda) = 60\sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^4}, \quad g_6(\Lambda) = 140\sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^6}$$

Then the map $z \mapsto (\wp(z;\Lambda), \wp'(z;\Lambda))$

gives an isomorphism $\mathbb{C}/\Lambda \xrightarrow{\sim} E_\Lambda(\mathbb{C})$

If $\tau\in\mathbb{H}$, define $\Lambda_\tau = \mathbb{Z}\oplus\tau\mathbb{Z} \subset \mathbb{C}$

$\tau \mapsto \begin{matrix} g_4(\Lambda_\tau) \\ g_6(\Lambda_\tau)\end{matrix}$ give modular forms of weight $\begin{matrix}4\\6\end{matrix}$

Homogeneous polynomials in these are also modular forms, e.g.

$g_4(\Lambda_\tau)^3 - 27(g_6(\Lambda_\tau))^2$ is a modular form of weight 12

$= \mathrm{disc}(E_{\Lambda_\tau}) \propto \Delta(\tau)$, Ramanujan Δ-function.

functional equation relating $\zeta(s)$ and $\zeta(1+s)$

Euler Product $\zeta(s) = \prod_p (1-p^{-s})^{-1}$

We will study how to associate an L-function $L(f,s)$ to a modular form f, with similar properties to $\zeta(s)$.

Hecke's Converse Theorem:

Suppose we have a function (Dirichlet series)

$Z(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $a_n \in \mathbb{C}$, converging absolutely for $Re(s) \gg 0$

If $Z(s)$ has meromorphic continuation to $\mathbb{C}$ and the right functional equation, then

$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau}$ is a modular form.

D) Hasse-Weil L-functions

For $E/\mathbb{Q}$ an elliptic curve, one can define a function

$$L(E,s) = \prod_{p \text{ prime}} L_p(E,s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

For p a prime of good reduction with $\tilde{E}_p/\mathbb{F}_p$ the reduction,

$$L_p(E,s) = (1 - a_p p^{-s} + p^{1-2s})^{-1}$$

where $a_p = p+1 - \#\tilde{E}_p(\mathbb{F}_p)$ = trace of Frobenius

Absolutely convergent for $Re(s) > 2$.

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Modular forms of level 1, $\Gamma(1) = SL_2(\mathbb{Z})$

Eisenstein series.

Modular forms of weight k, level 1 $\Rightarrow$ finite dimensional vector space

$\Rightarrow$ Weight k, level Γ : finite dimensionality.

Recall : weakly modular function of weight k (level 1)

meromorphic $f : \mathbb{H} \to \mathbb{C}$, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

for $z \in \mathbb{H}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$f : \mathbb{H} \to \mathbb{C}$ weakly modular, weight 0

$\Updownarrow$

holomorphic functions $\mathbb{C} \to \mathbb{C}$

With f as above, then $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$

so $f(\tau+1) = f(\tau)$

Suppose moreover that f is holomorphic on the region

$\{Im(\tau) > Y\} \in \mathbb{H}$, some $Y \in \mathbb{R}_{>0}$.

Denote by D^* the punctured unit disc, $\{0 < |q| < 1, q \in \mathbb{C}$

The map $\tau \mapsto e^{2\pi i \tau}$ gives a holomorphic surjection

$\mathbb{H} \to D^*$. We may define a function

$F : D^* \to \mathbb{C}$ $\quad F(e^{2\pi i \tau}) = f(\tau)$

Then we have $f(\tau) = F(e^{2\pi i \tau}) = \sum_{n \in \mathbb{Z}} a_n(f) q^n$
where $q = e^{2\pi i \tau}$ (valid where f is holomorphic).
This series is the <u>Fourier expansion</u> or <u>q-expansion</u> of f.

<u>Definition</u>

Suppose that f is a weakly modular function of weight k.

1. f is <u>meromorphic at ∞</u> if f is holomorphic for $\mathrm{Im}(\tau) \gg 0$, and in the q-expansion $f(\tau) = \sum_{n \in \mathbb{Z}} a_n(f) q^n$, there exists $N \in \mathbb{Z}$ such that $a_n(f) = 0 \; \forall n < N$.
   (i.e. F extends to a meromorphic function at $0 \in \Delta$)
   $\hookrightarrow$ F has at worst a pole at 0
2. Similarly, f is <u>holomorphic at ∞</u> if $a_n(f) = 0 \; \forall n < 0$
   $\hookrightarrow$ F has no singularities at 0
3. If f is meromorphic at ∞, we say that f is a <u>meromorphic form of weight k</u>.
4. If f is holomorphic on $\mathbb{H}$ and at ∞, then f is a <u>modular form</u> (of weight k, level 1).
5. If f is a modular form, and $a_0(f) = 0$, then we say that f is a <u>cusp form</u> (or <u>cuspidal</u> modular form).

<u>Lemma</u>

Weakly modular function of weight k (level 1)

$f : \mathbb{H} \to \mathbb{C}$, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

f meromorphic.

Want : For f weakly modular

f holomorphic at ∞ $\Leftrightarrow$ $\exists\, C, Y \in \mathbb{R}_{>0}$ such that

$|f(\tau)| \leq C$ for $\mathrm{Im}(\tau) > Y$.

($\Rightarrow$) Suppose f holomorphic at ∞.

$$f(\tau) = F(q) = \sum_{n \geq 0} a_n q^n$$

i.e. F holomorphic at $q = 0$

$\therefore$ $\exists\, C, r \in \mathbb{R}_{>0}$ such that

$|F(q)| \leq C$ for $|q| \leq r$

$$|e^{2\pi i \tau}| = e^{-2\pi t} \leq r$$

$$\tau = \sigma + it$$

$$-2\pi t \leq \log r$$

$$\mathrm{Im}(\tau) = t \geq \frac{\log r}{2\pi}$$

($\Leftarrow$) $\exists\, C, Y \in \mathbb{R}_{>0}$ such that $|f(\tau)| \leq C$ $\forall\, \mathrm{Im}(\tau) > Y$.

Set of modular forms, level 1, weight k is M_k
" " cusp " " " " S_k

Want to show that $f, g \in M_l \Rightarrow f \cdot g \in M_{k+l}$ (with $f \in M_k$)

$f \in M_k,\ g \in S_l \Rightarrow f \cdot g \in S_{k+l}$

If f, g are modular forms of weight k then they are holomorphic at ∞ and on $\mathbb{H}$.

$\Rightarrow f \cdot g$ holomorphic on $\mathbb{H}$.

By combing constants from f, g, $\exists\ c, Y \in \mathbb{R}_{>0}$ such that $|f(\tau)g(\tau)| \leq c$ for $\mathrm{Im}(\tau) > Y$.

Hence fg is holomorphic at ∞ (since clearly $f \cdot g$ weakly modular)

By uniqueness of Laurent series, fg has q-expansion $\left(\sum_{n\in\mathbb{Z}} a_n(f) q^n\right)\left(\sum_{m\in\mathbb{Z}} b_n(g) q^n\right)$

$\therefore$ If f is a cusp form, then so is $f \cdot g$.

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Notation

i) Set of modular forms of weight k, level 1 is denoted by $M_k(\Gamma(1))$, or M_k for short.

ii) Set of cusp forms of weight k, level 1, is denoted $S_k(\Gamma(1))$ or S_k.

M_k and S_k are vector spaces under the obvious scalar multiplication and addition.

Exercise: $f \in M_k$, $g \in M_l$, then $f \cdot g \in M_{k+l}$

$f \in M_k$, $g \in S_l$, $f \cdot g \in S_{k+l}$

i.e. $\bigoplus_{k \in \mathbb{Z}} M_k$ is a graded $\mathbb{C}$-algebra. $= \mathbb{C}[G_4, G_6]$

$\bigoplus_{k \in \mathbb{Z}} S_k$ is a graded ideal.

Remark

Consider the action of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(1)$ on $\mathbb{H}$, $\tau \mapsto \frac{-\tau}{-1} = \tau$.

So a weakly modular function of weight k satisfies

$f(\tau) = (-1)^k f(\tau)$

Therefore, if k is odd, $f \equiv 0$.

(If $\Gamma \subset \Gamma(1)$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma$, we could have odd weight)

Lemma

This series for $G_K(\tau)$ is absolutely convergent for $\tau \in \mathbb{H}$, and it converges uniformly for τ in a compact subset of $\mathbb{H}$.
($\Rightarrow G_K(\tau)$ is holomorphic)

Proof

$(c,d) \in \mathbb{Z}^2$.

Do the sum in layers, around each parallelogram.

At the n^{th} parallelogram, we get $(2m+1)^2 - (2m-1)^2 = 8m$ lattice points. If $\tau \in$ (some compact subset of $\mathbb{H}$), we have some r independent of τ such that $|z| \geq r$ as z ranges over the boundary of the first parallelogram. (z ranges over ▱ not jus

If $c\tau + d$ is in the m^{th} layer, $|c\tau + d| \geq mr$.

$$\sum_{(c,d)\in\mathbb{Z}^2\setminus \underline{0}} \frac{1}{|c\tau+d|^K} \leq \sum_{m=1}^{\infty} \frac{8m}{(mr)^K} = \frac{8}{r^K} \sum_{m=1}^{\infty} \frac{1}{m^{K-1}}$$

This converges for $K > 2$. $\square$

Remark

Summing in this way, the sum diverges for $K = 2$.

We will show next time that G_K is weakly modular of weight K.

Clearly $G_K(\tau) = G_K(\tau + 1)$

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Then if $\hat{h}(t) = \int_{-\infty}^{\infty} h(x) e^{-2\pi i x t} dx$

we have $S(x) = \sum_{d \in \mathbb{Z}} h(x+d) = \sum_{m \in \mathbb{Z}} \hat{h}(m) e^{2\pi i m x}$

Sketch: $S(x) = S(1+x)$

<u>Ideas for Poisson Summation</u> (assume necessary convergence)

$f : (-\infty, \infty)$ continuous

$g(x) : \sum_{n \in \mathbb{Z}} f(x+n)$

Clearly $g(x) = g(x+1)$, so g is periodic.

Fourier expansion of $g := \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}$

$$a_k = \int_0^1 g(x) e^{-2\pi i k x} dx$$

$$= \int_0^1 \left[\sum_{n \in \mathbb{Z}} f(x+n) \right] e^{-2\pi i k x} dx$$

$$= \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i k x} dx = \sum_{n \in \mathbb{Z}} \int_n^{n+1} f(x) e^{-2\pi i k x} dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x} dx$$

$$\therefore \sum_{n \in \mathbb{Z}} f(x+n) = \sum_{k \in \mathbb{Z}} e^{2\pi i k x} \int_{-\infty}^{\infty} f(t) e^{-2\pi i k t} dt$$

$$x = 0 \Rightarrow \sum_{n \in \mathbb{Z}} f(n) = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} f(t) e^{-2\pi i k t} dt$$

<u>Assumptions</u> (sufficient)

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<u>Proposition</u>

G_k is weakly modular of weight k.

<u>Proof</u>

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\underline{0}} \frac{(c\tau+d)^k}{(m(a\tau+b)+n(c\tau+d))^k}$$

$$= (c\tau+d)^k \sum_{(m,n)\in\mathbb{Z}^2\setminus\underline{0}} \frac{1}{((ma+nc)\tau+(mb+nd))^k} \qquad (*)$$

Right multiplication by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives a bijection

$\mathbb{Z}^2\setminus\underline{0} \to \mathbb{Z}^2\setminus\underline{0}, \quad (m,n)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (am+cn, bm+dn)$

$$\therefore (*) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\underline{0}} \frac{1}{(m\tau+n)^k} = G_k(\tau) \qquad \square$$

<u>Proposition</u>

$$G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$

<u>Proof</u>

Claim: Fix $c \geqslant 1$, $\tau \in \mathbb{H}$.

Then $$\sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^k} = \sum_{m>0} \frac{(-2\pi i)^k m^{k-1}}{(k-1)!} q^{cm}$$

Assume this claim.

by definition of Fourier Transform

$$\hat{h}_c(m) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i m x}}{(c\tau + x)^k}\, dx \overset{x = cu}{=} \frac{1}{c^{k-1}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i c m u}}{(\tau + u)^k}\, du$$

For $n \in \mathbb{Z}$, set $f_n(z) = \frac{1}{z^k} e^{-2\pi i n z}$

$$\hat{h}_c(m) = \frac{1}{c^{k-1}} e^{2\pi i c m \tau} \int_{-\infty+\tau}^{\infty+\tau} f_{cm}(z)\, dz$$

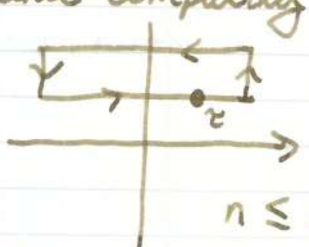
Write $I_n = \int_{-\infty+\tau}^{\infty+\tau} f_n(z)\, dz$

To prove the claim, we must show that $I_n = 0 \quad \forall n \leq 0$

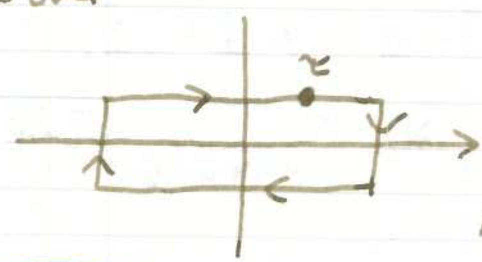
and $I_n = \frac{(-2\pi i)^k n^{k-1}}{(k-1)!} \qquad \forall n > 0$

We do this using $\mathrm{Res}_{z=0} f_n(z) = \frac{1}{(k-1)!}(2\pi i n)^{k-1}$

and computing contour integrals over



$n \leq 0$



$n > 0$

$\square$

Remark

For k a positive even integer, $\zeta(k) = -\frac{(2\pi i)^k B_k}{2(k!)}$

The B_k are Bernoulli numbers : $\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$

$B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$

Definition

$E_k(\tau) := \frac{G_k(\tau)}{2\zeta(k)}$

Consequence : the q-expansion of $E_k(\tau)$ has rational

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Goal: Show that $M_k(\Gamma(1))$ are finite dimensional and compute their dimensions.

We study the group action of $\Gamma(1)$ on $\mathbb{H}$.

Fundamental Domains

Definition

Suppose that a group G acts continuously on a topological space X. Then a fundamental domain for this action is an open subset $F \subset X$ such that:

i) No two distinct points of F are in the same $(G\cdot)$ orbit.

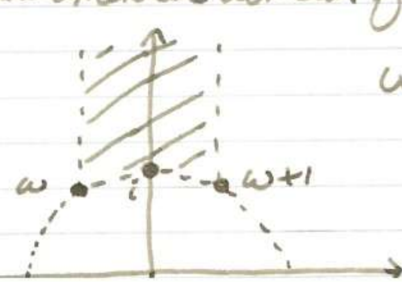
ii) Every point $x \in X$ lies in the orbit of (is G-equivalent to) a point in $\overline{F} \subset X$.

closure. important or this wouldn't work.

Proposition

The set $\{\tau \in \mathbb{H} : |\tau| > 1, |\mathrm{Re}(\tau)| < \frac{1}{2}\} \subset \mathbb{H}$ is a fundamental domain for the action of $\Gamma(1)$ on $\mathbb{H}$.

$\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$



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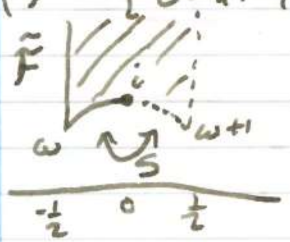
Proposition

The set $\widetilde{\mathcal{F}} \subset \mathbb{H}$ contains a unique element of every $\Gamma(1)$ orbit.

$(\widetilde{\mathcal{F}} = \{\tau \in \mathbb{H} : |\tau| > 1, |\mathrm{Re}(\tau)| < \frac{1}{2}\} \cup \{\mathrm{Re}(\tau) = -\frac{1}{2}, |\tau| \geq 1\} \cup \{|\tau| = 1, \mathrm{Re}(\tau) \in (-\frac{1}{2},$



$$\Gamma(1) = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$$

$\widetilde{\mathcal{F}}$ is a strict fundamental domain for the $\Gamma(1)$ action

$\Rightarrow \mathcal{F}$ is a fundamental domain.

Proof

Recall that $S = \underset{\tau \mapsto -\frac{1}{\tau}}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$, $T = \underset{\tau \mapsto \tau + 1}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$

Let $G = \langle S, T \rangle$. Fix $\tau \in \mathbb{H}$. We will show that $\widetilde{\mathcal{F}}$ contains an element of $G\tau$: for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$,

$$\mathrm{Im}(g\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2} \quad (*)$$ check explicitly, use $ad - bc = 1$

As $(c,d) \in \mathbb{Z}^2$ vary over the possible bottom rows for $g \in G$, $|c\tau + d|$ will attain a minimum, so $(*)$ attains a maximum.

So fix $g_0 \in G$ such that $\mathrm{Im}(g_0\tau)$ is maximal. In particular, $\mathrm{Im}(Sg_0\tau) = \mathrm{Im}\left(-\frac{1}{g_0\tau}\right) = \frac{\mathrm{Im}(g_0\tau)}{|g_0\tau|^2}$

$\nearrow |T^n g_0 \tau| \geq 1 \quad \forall n \in \mathbb{Z}$
see previous page

First we find n such that ~~Re~~ $Re(T^n g_0 \tau) \in [-\frac{1}{2}, \frac{1}{2})$

Then if $|T^n g_0 \tau| = 1$, $Re(T^n g_0 \tau) > 0$, take $ST^n g_0 \tau \in \tilde{F}$,

otherwise $T^n g_0 \tau \in \tilde{F}$.

This shows that $\tilde{F}$ contains a point of every G-orbit (hence every $\Gamma(1)$ orbit).

For uniqueness, suppose we have $\tau_1, \tau_2 \in \tilde{F}$, $\tau_1 \neq \tau_2$, with

$\tau_2 = \gamma \tau_1$ for some $\gamma \in \Gamma(1)$.

$-\frac{1}{2} \leq Re(\tau_i) < \frac{1}{2}$, so $\gamma \neq \pm T^n$ for any $n \in \mathbb{Z}$.

or $\gamma = \pm T^n$ by considering that $\det \gamma = 1$

So If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $c \neq 0$. Also, $Im(\tau_i) \geq \frac{\sqrt{3}}{2}$ trivially since $\tau_i \in \tilde{F}$

$$\frac{\sqrt{3}}{2} \leq Im(\tau_2) = \frac{Im(\tau_1)}{|c\tau_1 + d|^2} \leq \frac{Im(\tau_1)}{c^2 \, Im(\tau_1)^2} \leq \frac{2}{c^2\sqrt{3}}$$

see geometrically that $|c\tau_1 + d|^2 \geq c^2 Im(\tau_1)^2$ $\forall d$ or explicitly

$$\Rightarrow c^2 \leq 4/3 \Rightarrow c = \pm 1 \quad \begin{pmatrix} c \in \mathbb{Z} \\ c \neq 0 \end{pmatrix}$$

$$\text{So } Im(\tau_2) = \frac{Im(\tau_1)}{|\pm\tau_1 + d|^2} \leq \frac{Im(\tau_1)}{|\tau_1|^2} \leq Im(\tau_1)$$

Similarly, $Im(\tau_1) \leq Im(\tau_2)$, so equality above forces

and also $Im(\tau_1) = Im(\tau_2)$

$|\tau_1| = |\tau_2| = 1$. Hence $\tau_1 = \tau_2$ ※

and $d = 0$ forced

$\hookrightarrow Im(\tau_1) = Im(\tau_2)$, $|\tau_1| = |\tau_2|$, $\tau_i \in \tilde{F}$ implies this

So $\tilde{F}$ contains a unique element of every ~~$\Gamma(1)$~~ G, or $\Gamma(1)$ THINK orbit.

Lastly, we show that $G = \Gamma(1)$.

Take $\gamma \in \Gamma(1)$. For any $\tau \in \mathbb{H}$, there exists $g \in G$ such that

$g\gamma\tau \in \tilde{F}$

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Remark

This proposition is an example of "reduction theory". It is a re-interpretation of Gauss' theory of reduced forms for binary quadratic forms. (See Sheet 1)

Exercise

Follow reasoning of previous page.

$-\frac{1}{2} \leq Re(\tau) < \frac{1}{2}$

$\tau = \gamma\tau \Rightarrow \gamma \neq \pm T^n$

$\Rightarrow \gamma = \pm I$ or $c \neq 0$

$\Rightarrow c = \pm 1 \Rightarrow |\tau| = 1$

i) Suppose $\tau \in \tilde{\mathcal{F}}$. Then

$$\mathrm{Stab}_{\Gamma(1)}(\tau) = \begin{cases} \{\pm I\} & \tau \neq \omega, i \\ \{\langle S \rangle\} & \tau = i \\ \{\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \rangle\} & \tau = \omega \end{cases}$$

$\tau = \frac{a-d \pm \sqrt{(d-a)^2+4bc}}{2c}$

$\tau \in \mathbb{H}$, $\cancel{Im(\tau) \geq \frac{\sqrt{3}}{2}}$

$\cancel{c = \pm 1, ad-bc = 1}$

$\cancel{\Rightarrow (a+d)^2 \leq 1}$

τ an integer in an im. quad. field

$|\tau| = 1 \Rightarrow \tau$ a unit $\Rightarrow \tau = i, \omega$

Notation

We call $\tau \in \mathbb{H}$ *elliptic* if it is in the $\Gamma(1)$ orbit of i or ω.
$\Leftrightarrow$ Stab has order 4 or 6.

ii) Compute $\mathrm{Stab}_{\Gamma(1)}(\tau)$ for general τ.

If τ is an elliptic point, its order is defined to be

$\frac{1}{2}|\mathrm{Stab}_{\Gamma(1)}(\tau)|. = n_\tau$

Note that $\Gamma(1) \circlearrowright \mathbb{H}$ factors through $\Gamma(1)/\{\pm I\}$

so order $(\tau) = \#\mathrm{Stab}_{\Gamma(1)/\{\pm I\}}(\tau)$.

Remark

Zeroes and Poles of Modular Forms

Definition

Let f be a meromorphic form of weight k, $\tau \in \mathbb{H}$, with order of vanishing $\mathrm{ord}_\tau(f)$ at τ.

$$f(\tau) = \sum_n a_n(f) q^n \qquad (a_n(f) = 0 \text{ for } n < N, \text{ some } N \in \mathbb{Z}$$

$\mathrm{ord}_\infty(f) = N$, smallest n with $a_n(f) \neq 0$.

$\mathrm{ord}_\tau(f)$ depends only on $\Gamma(1) \cdot \tau$.

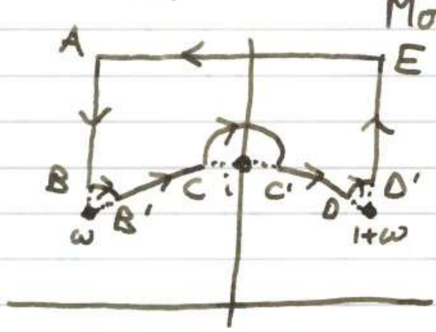
Proposition

Let f be a non-zero meromorphic form of weight k. Then

$$\mathrm{ord}_\infty(f) + \sum_{\substack{\tau\Gamma(1) \\ \text{cosets of} \\ \mathbb{H}/\Gamma(1)}} \frac{\mathrm{ord}_\tau(f)}{\underbrace{\text{order } \tau}_{= n_\tau}} = k/12$$

7/01/14

Modular Forms ⑤



We will prove that for a meromorphic form of weight k,

$$\mathrm{ord}_\infty(f) + \sum_{\Gamma(1)\tau \in \mathbb{H}/\Gamma(1)} \frac{\mathrm{ord}_\tau(f)}{n_\tau} = \frac{k}{12} \quad (*)$$

n_τ = order of τ $= \frac{1}{2}|\mathrm{Stab}_{\Gamma(1)}(\tau)|$

$f(\tau) = F(q) = \sum_{n\in\mathbb{Z}} a_n q^n$, $\quad a_n = 0$ for $n \ll 0$

$\mathrm{ord}_\infty(f) = \min\{n : a_n \neq 0\}$

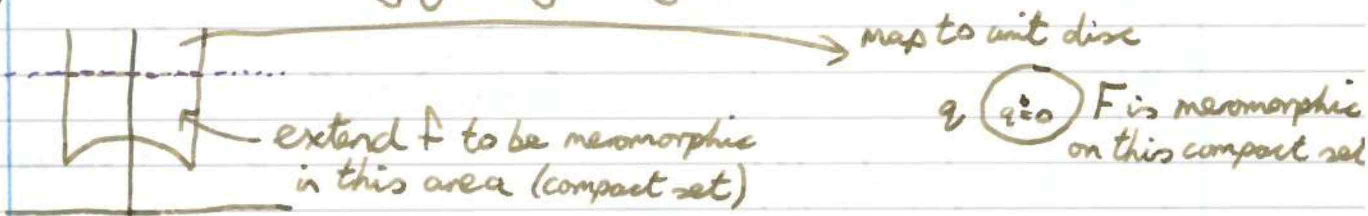
For $\tau \in \mathbb{H}$, $\mathrm{ord}_\tau(f)$ = degree of f at τ

$$n_\tau = \begin{cases} 1 & \tau \notin \Gamma(1)i,\ \Gamma(1)\omega \\ 2 & \tau \in \Gamma(1)i \\ 3 & \tau \in \Gamma(1)\omega \end{cases}$$

<u>Remark</u>

i) Since $f(\gamma\tau) = (c\tau+d)^k f(\tau)$, and $(c\tau+d)^k$ is non-vanishing and holomorphic on $\mathbb{H}$, $\mathrm{ord}_\tau(f)$ only depends on the orbit $\Gamma(1)\tau$.

ii) The sum $(*)$ has only finitely many non-zero terms.



Meromorphic functions on compact sets

$\Rightarrow$ finitely many zeroes and poles $\Rightarrow$ $(*)$ is a finite sum.

Proof

Assume for simplicity that f has no zeroes/poles on $\partial\tilde{F}$ apart from possibly at $i, \omega, 1+\omega$.

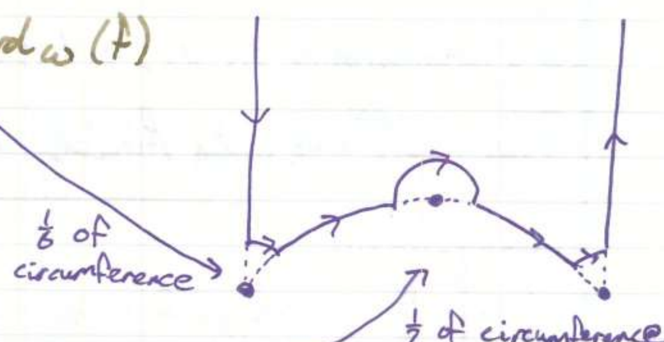
Choose points $A = -\frac{1}{2} + iY$, $E = \frac{1}{2} + iY$ with Y large enough such that the interior of our contour C contains all zeroes/poles in the interior of $\tilde{F}$. BB', CC', DD' are arcs of circles with small radius r, centred on the elliptic points.

$$\frac{1}{2\pi i}\int_C \frac{f'(\tau)}{f(\tau)}\,d\tau = \sum_{\Gamma(1)\tau \in \mathbb{H}/\Gamma(1),\, \Lambda_\tau = 1} \mathrm{ord}_\tau(f)$$

First $\int_B^{B'}$: As $r \to 0$, $f(\tau) \sim a(z-\omega)^{\mathrm{ord}_\omega(f)}$ and

$$\frac{1}{2\pi i}\int_B^{B'} \frac{f'(\tau)}{f(\tau)}\,d\tau \to \frac{1}{6}\left(\frac{1}{2\pi i}\int_{\circlearrowleft \omega}\right) = -\frac{1}{6}\mathrm{ord}_\omega(f)$$

Similarly, $\int_D^{D'} \to -\frac{1}{6}\mathrm{ord}_\omega(f)$



$f(\tau) = f(1+\tau)$

$$\int_{D'}^{E} = \int_B^A = -\int_A^B$$

$$\int_C^{C'} \to -\frac{1}{2}\mathrm{ord}_i(f)$$

$f(-\frac{1}{\tau}) = \tau^k f(\tau)$, $\tau \mapsto -\frac{1}{\tau}$ maps $B'C$ to DC'.

$$\int_{B'}^{C} \frac{f'(\tau)}{f(\tau)}\,d\tau \overset{u=-\frac{1}{\tau}}{=} \int_D^{C'} \frac{f'(u)}{f(u)} + \frac{k}{u}\,du$$

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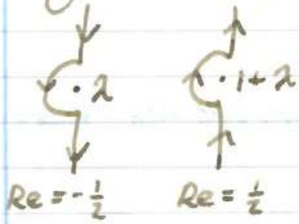
Modular Forms (5)

If f has zeroes/poles on $\partial\hat{F} \setminus \{i, \omega, 1+\omega\}$, modify C:



Contributions from circle arcs cancel out.

Some Consequences

Recall $\Delta = \dfrac{E_4^{\,3} - E_6^{\,2}}{1728}$, $\Delta \in S_{12}(\Gamma(1))$

$\text{ord}_\infty(\Delta) = 1$ because Δ has no poles because $k = 12$

$1 = \frac{k}{12} - (\text{terms} \geq 0)$. So proposition $\Rightarrow$ Δ has no zeroes on $\mathbb{H}$.

(Recall: in the introduction, we saw $\Delta(\tau) = \text{constant} \times \text{disc}\left(\mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}\right.$

Definition

$j(\tau) := \dfrac{E_4(\tau)^3}{\Delta(\tau)}$, weakly modular of weight 0, and holomorphic on $\mathbb{H}$, with $\text{ord}_\infty(j) = -1$. i.e. constant

because $\text{ord}_\infty(\Delta) = 1$
$\text{ord}_\infty(E_4) = 0$

Corollary

$j(\tau)$ gives a bijection $\mathbb{H}/\Gamma(1) \to \mathbb{C}$ (j weakly modular of weight $0 \Rightarrow j(\gamma\tau) = j(\tau)\ \forall \gamma \in \Gamma(1)$).

Proof

For $z \in \mathbb{C}$, $f_z : \mathbb{H} \to \mathbb{C}$, $f_z(\tau) = E_4(\tau)^3 - z\Delta(\tau) \in M_{12}(\Gamma$

$f_{\mathbb{Z}}$ has $\begin{cases} \text{simple zero at } \Gamma(1)\tau, \ \tau \text{ non-elliptic} \\ \text{double " } \quad \Gamma(1)i \\ \text{triple " } \quad \Gamma(1)\omega \end{cases}$

<u>Remark</u> by deciding which terms in the sum are non-zero
simply knowing $\mathrm{ord}_\tau(f) \in \mathbb{Z}_{\geq 0}$

This implies that every elliptic curve over $\mathbb{C}$ is isomorphic to $\mathbb{C}/\Lambda$ for a lattice Λ.

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ONLINE NOTES

Dimension Formula

Recall that for f a non-zero meromorphic form of weight k:

$$\mathrm{ord}_\infty(f) + \sum_{\Gamma(1)\tau \in \mathbb{H}/\Gamma(1)} \frac{\mathrm{ord}_\tau(f)}{n_\tau} = \frac{k}{12} \quad (*)$$

Lemma

i) If $k<0$, then $M_k(\Gamma(1)) = 0$

ii) $M_0(\Gamma(1)) = \{\text{constant functions on } \mathbb{H}\} \cong \mathbb{C}$

Proof

i) Immediate from the formula above. for holomorphic f, ord ≥ 0 including at ∞

ii) If $f \in M_0(\Gamma(1))$, $f(\tau) = a_0(f) + \sum_{n\geq 1} a_n(f) q^n$

$\Rightarrow f - a_0(f) \in S_0(\Gamma(1))$ and so the proposition $\Rightarrow f - a_0(f) \equiv 0$

i.e. f is constant. because if $f-a_0(f)$ is non-constant then $\mathrm{ord}_\infty(f) > 0$ ✗

Lemma

For even $k \geq 0$, $\dim_{\mathbb{C}} M_k(\Gamma(1)) \leq \lfloor k/12 \rfloor + 1$

Moreover, if $k \equiv 2 \ (12)$, then $\dim_{\mathbb{C}} M_k(\Gamma(1)) \leq \lfloor k/12 \rfloor$

Proof

Set $m = \lfloor k/12 \rfloor + 1 > k/12$. Fix $P_1, \dots, P_m \in \mathbb{H}/\Gamma(1)$, linearly independent distinct, non-elliptic orbits. Suppose that $f_1, \dots, f_{m+1} \in M_k(\Gamma(1))$.

to have equality in (*). Hence $\sum_{i=1}^{l} \lambda_i f_i = 0$. $\square$

$M = \bigoplus_{k \geq 0} M_k(\Gamma(1))$ (graded ring)

$\cap$

holomorphic functions on $\mathbb{H}$.

<u>Theorem</u>

$R : \mathbb{C}[X, Y] \to M$, $\begin{array}{l} X \longmapsto E_4 \\ Y \longmapsto E_6 \end{array}$ is an isomorphism.

<u>Corollary</u>

a) $\left\{ E_4^{\,a} E_6^{\,b} : \begin{array}{c} a, b \in \mathbb{Z}_{\geq 0} \\ 4a + 6b = k \end{array} \right\}$ is a basis for $M_k(\Gamma(1))$

b) $\dim M_k(\Gamma(1)) = \begin{cases} \lfloor k/12 \rfloor + 1 & k \not\equiv 2 \ (12) \\ \lfloor k/12 \rfloor & k \equiv 2 \ (12) \end{cases}$

<u>Proof</u> (Theorem $\Rightarrow$ Corollary)

a) Immediate from the Theorem.

b) Exercise. $\square$

<u>Proof</u> (of Theorem)

give X degree 4, Y degree 6

exercise $\Rightarrow \dim_{\mathbb{C}} M_k(\Gamma(1)) \leq \dim_{\mathbb{C}} \mathbb{C}[X, Y]_{\deg = k}$

So it is sufficient to prove that R is injective i.e. E_4 and E_6 are algebraically independent.

(think of $E_4, E_6 \subset \mathrm{Frac}(M) \subset$ {meromorphic functions on $\mathbb{H}$})

$[E_4, E_6]$

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Meromorphic function satisfies a polynomial $\Rightarrow$ constant. c.f. complex analysis

$\therefore \; E_4^3 / E_6^2 = \lambda \in \mathbb{C}$, constant

$\Rightarrow \left(E_6 / E_4 \right)^2 = \frac{1}{\lambda} E_4$, holomorphic. $\dim_{\mathbb{C}} M_2(\Gamma(1)) \leq 0$ by Lemma

$\Rightarrow E_6 / E_4 \in M_2(\Gamma(1)) = 0$ # Or see Serre's Course in Arithmetic Ch 7

Hence E_4, E_6 are algebraically independent. $\square$

Congruence Subgroups

Definition

$N \in \mathbb{Z}_{\geq 1}$. The principal congruence subgroup $\Gamma(N) \subset SL_2(\mathbb{Z})$ is $\Gamma(N) = \{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \}$

Lemma

$\Gamma(N)$ is a finite index normal subgroup of $SL_2(\mathbb{Z})$.

Proof

$\Gamma(N) = \ker \left(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z}) \right)$ $\square$

Remark

$SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ is surjective. see Sheet 1 So $[SL_2(\mathbb{Z}) : \Gamma(N)] = \# SL_2(\mathbb{Z}/N\mathbb{Z})$

Definition

A subgroup $\Gamma \subset SL_2(\mathbb{Z})$ is a congruence subgroup if subgroup is a congruence subgroup "congruence subgroup property".

<u>Definition</u>

$N \in \mathbb{Z}_{\geq 1}$, $\Gamma_0(N) = \{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \ (N) \} \supset \Gamma($

$\Gamma_1(N) = \{ \gamma \in SL_2(\mathbb{Z}) : \begin{matrix} a \equiv d \equiv 1 \ (N) \\ c \equiv 0 \ (N) \end{matrix} \} \supset \Gamma(N)$

$SL_2(\mathbb{Z}) = \Gamma(1) \cup \Gamma_0(N) \cup \Gamma_1(N) \cup \Gamma(N)$

(exercise : compute the index in $SL_2(\mathbb{Z})$ of $\Gamma_0(N)$, $\Gamma_1(N)$)

<u>Modular Forms with Level a Congruence Subgroup</u>

Recall : $f : \mathbb{H} \to \mathbb{C}$ is weakly modular of weight k, level Γ (meromorphic)

if $f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$\mathcal{F}$  Fundamental Domain for $\Gamma(1)$

For $\Gamma \subset \Gamma(1)$, if $\alpha_1, \dots, \alpha_d$ are coset representatives for $SL_2(\mathbb{Z})/\Gamma$, then $\bigcup_i \alpha_i \mathcal{F}$ is a fundamental domain for Γ.

If $\Gamma = \Gamma_0(p)$, p prime :

Coset reps $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & \\ & \end{pmatrix}$ $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, ..., $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$S \qquad ST^k, \ k = 1, \dots, p-1$

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Modular Forms ⑦

$\Gamma \supset \Gamma(N) \ni \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$, $f(\tau + N) = f(\tau)$

Let Γ be a congruence subgroup. $\exists$ a minimal $h \in \mathbb{Z}_{\geq 1}$ such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$. h = "period of the cusp ∞".

$f(\tau + h) = f(\tau)$ for any weakly modular function of level Γ.

<u>Definition</u>

$f : \mathbb{H} \to \mathbb{C}$ weakly modular, weight k, level Γ.

f holomorphic for $\mathrm{Im}(\tau) \gg 0$. Let $q_h = e^{2\pi i \tau / h}$.

Define $F_{(h)}(q_h) = f(\tau)$. F is a meromorphic function on a punctured disc with centre 0. $\longleftrightarrow \tau = i\infty$

So $F(q_h) = \sum_{n \in \mathbb{Z}} a_n(f) q_h^n$. We say that f is meromorphic (resp. holomorphic) at ∞ if F extends to a meromorphic (resp. holomorphic) function at 0.

<u>Notation</u>

"slash operator"

$\gamma \in SL_2(\mathbb{Z})$, $k \in \mathbb{Z}$, $f : \mathbb{H} \to \mathbb{C}$.

Define $f|_{\gamma,k} : \mathbb{H} \to \mathbb{C}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by

$f|_{\gamma,k}(\tau) = (c\tau + d)^{-k} f(\gamma\tau)$

f is weakly modular of weight k, level Γ

$\Leftrightarrow$ f is meromorphic and $f|_{\gamma,k} = f$ for all $\gamma \in \Gamma$.

Suppose that f is weakly modular of weight k, level Γ, and $\alpha \in SL_2(\mathbb{Z})$.

Then $f|_{\alpha,k}$ is weakly modular of weight k, level $\alpha^{-1}\Gamma\alpha$.

Definition

$f: \mathbb{H} \to \mathbb{C}$ weakly modular, weight k, level Γ.

i) We say that f is a meromorphic form (of weight k, level Γ) if $f|_{\alpha,k}$ is meromorphic at ∞ for all $\alpha \in SL_2(\mathbb{Z})$.

ii) We say that f is a modular form (of weight k, level Γ) if f is holomorphic on $\mathbb{H}$ and $f|_{\alpha,k}$ is holomorphic at ∞.

Moreover, f is a cusp form if $a_0(f|_{\alpha,k})$ is 0 $\forall \alpha \in SL_2(\mathbb{Z})$.

Notation

The $\mathbb{C}$-vector space of modular forms is $M_k(\Gamma)$.

The $\mathbb{C}$-vector space of cusp forms is $S_k(\Gamma)$.

Lemma

$f: \mathbb{H} \to \mathbb{C}$, $\alpha, \beta \in SL_2(\mathbb{Z})$, $k \in \mathbb{Z}$

$$(f|_{\alpha,k})|_{\beta,k} = f|_{\alpha\beta,k}$$

Proof

Exercise. $\square$

This is the same as saying that $|_{\cdot,k}$ defines a right-action of $SL_2(\mathbb{Z})$ on the space of functions from $\mathbb{H}$ to $\mathbb{C}$.

Consequences

1. To show that f is weakly modular of weight k, level Γ, it suffices to show that $f|_{r_i,k} = f$ for $r_1, \dots, r_n$ generators of Γ (finite index subgroups of finitely generated groups are finitely generated).

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2. Let f be as above, (i.e. weakly modular of level Γ, $\gamma \in \Gamma$) $\alpha \in SL_2(\mathbb{Z})$. $f|_{\alpha,k}$ only depends on $\Gamma\alpha$ $\quad \left(f|_{\gamma\alpha,k} = (f|_{\gamma,k})|_{\alpha,k} = f|_{\alpha,k} \right)$

Proposition 1

Let $\Gamma(N) \subset \Gamma$. Suppose that we have $f: \mathbb{H} \to \mathbb{C}$ holomorphic, and weakly modular of weight k, level Γ.

Write $f(\tau) = \sum_{n \geq 0} a_n(f) q_N^n$

Suppose that $\exists\, c \in \mathbb{R}_{\geq 0}$, such that $|a_n(f)| \leq cn^r$, some $r \in \mathbb{R}$ (with c, r independent of n). Then f is a modular form of weight k and level Γ.

Proof

Exercise in 1.2.6, Diamond + Shurman. $\square$

Theta Functions

Jacobi Θ-function : $\Theta(\tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 \tau} = 1 + 2\sum_{n \geq 1} q^{n^2}$

It is easy to check that the series is absolutely + uniformly convergent on compact subsets of $\mathbb{H}$.

$\therefore$ $\Theta(\tau)$ is holomorphic and $\Theta(\tau) = \Theta(\tau + 1)$.

Definition

$\Theta(\tau, k) = \Theta(\tau)^k = \sum_{n=0}^{\infty} r(n,k) q^n \qquad (\text{for } k \in \mathbb{Z}_{\geq 1})$

where $r(n,k) = \#$ representations of n as a sum of k squares of integers.

Proposition 2

$\Theta\left(-\frac{1}{4\tau}\right) = \underbrace{\sqrt{\frac{2\tau}{i}}}_{} \Theta(\tau)$

$\mathrm{Re} > 0$, use usual branch of $\sqrt{\ }$ on $\mathbb{R}_{>0}$

Corollary

$$\Theta\left(\frac{\tau}{4\tau+1}\right)^2 = (4\tau+1)\,\Theta(\tau)^2$$

Proof

$$\Theta\left(\frac{\tau}{4\tau+1}\right)^2 = \frac{2(-1-\frac{1}{4\tau})}{i}\,\Theta\left(-\frac{1}{4\tau}-1\right)^2 \qquad \Theta(\tau+1)=\Theta(\tau)$$

$$= \frac{2}{i}\left(-1-\frac{1}{4\tau}\right)\Theta\left(-\frac{1}{4\tau}\right)^2$$

$\frac{\tau}{4\tau+1} = -\frac{1}{4(-\frac{1}{4\tau}-1)}$, then use the Proposition. $= (4\tau+1)\,\Theta(\tau)^2$ $\square$

Suppose that $k \geq 2$, even. Then $\Theta(\tau,k)\big|_{\gamma,k/2} = \Theta(\tau,k)$

for $\gamma \in \left\langle \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\4&1\end{pmatrix} \right\rangle$.

If further $k \equiv 0\ (4)$, this holds for $\gamma \in \left\langle \pm\begin{pmatrix}1&1\\0&1\end{pmatrix}, \pm\begin{pmatrix}1&0\\4&1\end{pmatrix} \right\rangle$

On the example sheets we prove that the 2nd group is $\Gamma_0(4)$.

So $k \equiv 0\ (4) \Rightarrow \Theta(\tau,k)$ is weakly modular, weight $\frac{k}{2}$, level $\Gamma_0(4)$

If $k \equiv 2\ (4)$, $\Theta(\tau,k)$ is weakly modular, weight $k/2$, level $\Gamma_1(4)$.

Proposition 1 $\Rightarrow$ $\Theta(\tau,4k) \in M_{2k}(\Gamma_0(4))$. because coefficients growth is bounded

(enough to show that $\Theta(\tau,4) \in M_2(\Gamma_0(4))$)

Recall that $r(n,4) = 8\sum_{\substack{0<d\mid n\\ 4\nmid d}} d$

We will deduce this next time from $\dim_{\mathbb{C}} M_2(\Gamma_0(4)) = 2$

+ weight two Eisenstein series.

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Modular Forms ⑧

Proposition

$\Theta(-\frac{1}{4\tau}) = \sqrt{\frac{2\tau}{i}}\,\Theta(\tau)$

Proof

$h(x) = e^{-\pi t x^2}$, $t \in \mathbb{R}_{>0}$, $x \in \mathbb{R}$

$\hat{h}(y) = \frac{1}{\sqrt{t}} e^{-\frac{\pi y^2}{t}}$

$s(x) = \sum_{d\in\mathbb{Z}} h(x+d)$ $\quad \sum \hat{h}(m)$

Main Idea

Poisson summation $\Rightarrow \underbrace{\sum_{d\in\mathbb{Z}} e^{-\pi t d^2}}_{=s(0)} = \frac{1}{\sqrt{t}} \sum_{m\in\mathbb{Z}} e^{-\pi \frac{m^2}{t}}$ (*)

$\to$ evaluate Poisson Formula at $x=0$

$\Theta(\tau) = \sum_{d\in\mathbb{Z}} e^{2\pi i d^2 \tau}$

So for $t \in \mathbb{R}_{>0}$, (*) gives $\Theta(\frac{it}{2}) = \frac{1}{\sqrt{t}}\Theta(\frac{i}{2t})$ this is exactly (*)

Then for $\tau = \frac{it}{2}$, $t \in \mathbb{R}_{>0}$, (so that $\tau = si$, $s \in \mathbb{R}_{>0}$) the formula holds

Uniqueness of analytic continuation

$\Rightarrow$ equality holds for $\tau \in \mathbb{H}$. $\square$

From proposition 1 last time:

f holomorphic at ∞, q-expansion coefficients $a_n(f) \leq cn^r$

$\Rightarrow f|_{\alpha,k}$ holomorphic at ∞ for $\alpha \in SL_2(\mathbb{Z})$.

e.g. $\Theta(\tau,k) = \sum_{n\geq 0} r(n,k) q^n$

$r(n,k) \leq 2^k n^k \quad \Rightarrow \Theta(\tau,4k) \in M_{2k}(\Gamma_0(4))$ by Prop. 1.

Old Forms

Lemma

$f \in M_k(\Gamma_0(N))$ (e.g. $N=1$). Let $M \geq 1$. Then

$f_M : \mathbb{H} \to \mathbb{C}$, $\tau \mapsto f(M\tau)$ is in $M_k(\Gamma_0(MN))$.

Proof

$f_M(\tau) = f(\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}\tau)$

Suppose that $\gamma \in \Gamma_0(MN)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

1

$$f_M|^{(\tau)}_{\gamma,k} = (c\tau+d)^{-k} f\left(\begin{pmatrix} M & 0 \\ 0 & 1\end{pmatrix}\gamma\tau\right)$$

$$= (c\tau+d)^{-k} f\left(\begin{pmatrix} M & 0 \\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix} M & 0 \\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix} M & 0 \\ 0 & 1\end{pmatrix}\tau\right)$$

$$= (c\tau+d)^{-k} f\Bigg(\underbrace{\begin{pmatrix} a & bM \\ c/M & d\end{pmatrix}}_{\in\Gamma_0(N)}\begin{pmatrix} M & 0 \\ 0 & 1\end{pmatrix}\tau\Bigg)$$

since f has level $\Gamma_0(N)$

$c \equiv 0\ (MN)$

$$= f(M\tau) = f_M(\tau)$$

We check that it is holomorphic at ∞:

$f(\tau) = \sum a_n q^n$, $\quad f_M(\tau) = \sum a_n q^{nM}$

$f_M|_{\alpha,k}(\tau) = \left(\frac{cM\tau+d}{c\tau+d}\right)^k f|_{\alpha,k}(M\tau)$ for $\alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$

So $f|_{\alpha,k}(\tau)$ is bounded for $\mathrm{Im}(\tau) \gg 0$ (by holomorphy at ∞ of $f|_{\alpha,k}(M\tau)$)

$\Rightarrow$ the same holds for $f_M|_{\alpha,k}(\tau)$

So $f_M|_{\alpha,k}$ is holomorphic at ∞. $\square$

Now $r(n,4) = 8 \sum_{\substack{d|n,\, d>0 \\ 4\nmid d}} d$ $\quad\leftarrow$ from weight 2 Eisenstein series, next page

$\Theta(\tau,4) = \sum r(n,4)\, q^n$

<u>Weight 2 Eisenstein Series</u>

For $k>2$, even, $\quad G_k(\tau) = \sum_{(c,d)\in\mathbb{Z}^2\setminus 0} \frac{1}{(c\tau+d)^k}$

But this sum does not converge absolutely for $k=2$.

But we can define

$$G_2(\tau) = 2\zeta(2) + 2(2\pi i)^2 \sum_{n\geq 1} \sigma_1(n)\, q^n$$

This converges absolutely

$$G_2(\tau) = \sum_{d\neq 0} \frac{1}{d^2} + \sum_{c\neq 0}\left(\sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^2}\right)$$

<u>Note</u>

Caution! <u>$G_2(\tau)$ is <u>not</u> weakly modular</u>!

(We showed that $G_k(\tau)$, $k>2$ was weakly modular, by considering for $SL_2(\mathbb{Z})$, $\gamma: \mathbb{Z}^2\setminus 0 \xrightarrow{\sim} \mathbb{Z}^2\setminus 0$ and reordering $\sum_{\mathbb{Z}^2\setminus 0}$)

but here, the $\sum$ does not converge absolutely

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Modular Forms ⑧

<u>Fact</u>

$G_2^*(\tau) = G_2(\tau) - \frac{\pi}{\mathrm{Im}(\tau)}$ satisfies

$G_2^*\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2^*(\tau)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

<u>Exercise</u>

$// \; G_2^*(\tau) - 2G_2^*(2\tau)$

$G_2(\tau) - 2G_2(2\tau) \in M_2(\Gamma_0(2))$

$G_2^*(\tau) - 4G_2^*(4\tau) = G_2(\tau) - 4G_2(4\tau) \in M_2(\Gamma_0(4))$.

<u>Fact</u>

$\Gamma_0(4) \subset \Gamma_0(2)$

$\dim M_2(\Gamma_0(4)) = 2$

$M_2(\Gamma_0(2)) \hookrightarrow M_2(\Gamma_0(4))$

$f \longmapsto f$

$G_2 - 2G_2(2\tau)$, $G_2 - 4G_2(4\tau)$ are a basis.

True in general $\Gamma \subset \Gamma'$

$M_k(\Gamma') \hookrightarrow M_k(\Gamma)$

<u>Corollary</u>

$\Theta(\tau, 4) = -\frac{1}{\pi^2}\left(G_2(\tau) - 4G_2(4\tau)\right)$

$\Rightarrow r(n,4) = 8 \sum_{d \mid n,\, 4 \nmid d} d$ ← this is what we get from coefficients of G_2

<u>Theorem</u>

Let Γ be a congruence subgroup. $M_k(\Gamma) = \{0\}$, $k < 0$.

$M_0(\Gamma) = \mathbb{C}$ (constants)

$M_k(\Gamma)$ for $k > 0$ is a finite dimensional vector space.

<u>Moreover, $\bigoplus_{k \geq 0} M_k(\Gamma)$ is a finitely generated $\mathbb{C}$-algebra.</u>

Suppose that $\Gamma' \triangleleft \Gamma$, congruence subgroups.

~~G~~ $G = \Gamma' \backslash \Gamma$ (elements $\Gamma'\gamma$)

G acts on $M_k(\Gamma')$:

$f \in M_k(\Gamma')$, $g \in G$, $g = \Gamma'\gamma$

Define $f^g = f|_{\gamma,k}$

<u>Check</u> $M_k(\Gamma')^G = M_k(\Gamma)$

$f \in M_k(\Gamma')$

Consider $\prod_{g \in G} (f - f^g) = 0$ (in $\bigoplus_k M_k(\Gamma)$)

$\parallel$

$f^n + h_1 f^{n-1} + \ldots + h_{n-1} f + h_n$

h_i are symmetric

$\Rightarrow h_i \in M_\bullet(\Gamma')^G = M_\bullet(\Gamma)$

<u>Lemma</u>

$f \in M_k(\Gamma') \Rightarrow f$ satisfies a polynomial with coefficients in $M_\bullet(\Gamma)$.

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Lemma

$\Gamma' \trianglelefteq \Gamma$ congruence subgroups. $f \in M_k(\Gamma')$

$\exists h_i \in M_{ik}(\Gamma)$, $i = 1, \dots, n = [\Gamma : \Gamma']$

such that $f^n + h_1 f^{n-1} + \dots + h_{n-1} f + h_n = 0$

(equality in $M_{nk}(\Gamma')$)

Proof

$G = \Gamma' \backslash \Gamma$, $M_k(\Gamma') \circlearrowleft G$, $M_k(\Gamma')^G = M_k(\Gamma)$

Consider $\prod_{g \in G} (f - f^g) = 0$ $\square$

$M(\Gamma) = \bigoplus_k M_k(\Gamma) \subset \bigoplus_k M_k(\Gamma') = M(\Gamma')$

$M(\Gamma) \subset M(\Gamma')$ is integral.

Corollary

Γ a congruence subgroup, $k < 0$. Then $M_k(\Gamma) = \{0\}$

$M_0(\Gamma)$ = constant functions

Proof

Suppose that $\Gamma \trianglelefteq SL_2(\mathbb{Z})$, $k < 0$.

according to Theorem last lecture

Apply the lemma to $f \in M_k(\Gamma)$: $h_i \in M_{ik}(\Gamma(1)) = \{0\}$

$\Rightarrow f^n = 0$

according to Theorem last lecture

For $k = 0$, apply the lemma, $h_i \in M_0(\Gamma(1)) =$ constants.

$\Rightarrow f$ is a root of a polynomial in $\mathbb{C}[X]$

$\Rightarrow f$ is a constant function. c.f. Complex Analysis

For general Γ, $\Gamma(N) \subset \Gamma \subset SL_2(\mathbb{Z})$, $\Gamma(N) \trianglelefteq SL_2(\mathbb{Z})$

~~$M_k(\Gamma(N))$~~ $M_k(\Gamma) \subset M_k(\Gamma(N))$, so Γ case

follows from $\Gamma(N)$ case. $\square$

Theorem

Γ a congruence subgroup. $M(\Gamma) = \bigoplus_{k \geq 0} M_k(\Gamma)$ is a finitely generated $\mathbb{C}$-algebra.

We will make use of:

Lemma

F a field. $A \subset B$ integral domains and F-algebras.
Assume that B is integral over A i.e. for all $b \in B$, $\exists a_i \in A$ such that $b^n + a_1 b^{n-1} + \dots + a_n = 0$.
Assume $\mathrm{Frac}(B) / \mathrm{Frac}(A)$ is a field extension of finite degree.
Then A is a finitely generated F-algebra
$\iff$ B is a finitely generated F-algebra.

Proof (of Lemma)

($\Rightarrow$) Let A be a finitely generated F-algebra. Let $\tilde{A}$ be the integral closure of A in $\mathrm{Frac}(B)$.
(i.e. $\tilde{A}$ = elements of $\mathrm{Frac}(B)$ satisfying monic polys. with coefficients in A).

as B is integral over A

$A \subset B \subset \tilde{A}$. Noether: $\tilde{A}$ is an f.g. A-module.
A Noetherian $\Rightarrow$ B an f.g. A-module.
by assumption
$\Rightarrow B$ an f.g. F-algebra

($\Leftarrow$) Suppose that $b_1, \dots, b_n$ generate B as an F-algebra.
Let $C \subset A \subset B$ be the F-algebra generated by the coefficients of the polynomials satisfied by the b_i.
B is a f.g. and integral C-algebra

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$\Rightarrow B$ is a f.g. C-module

b_i generate B as a C-algebra but they satisfy polynomials in C so we need only take $b_i^{k_i}$ up to a certain k_i to generate B as a C-module

$\Rightarrow A$ is a f.g. C-module

$\Rightarrow A$ is a f.g. F-algebra. as C is an f.g. F-algebra □

Proof (of Theorem) Γ a congruence subgroup. Want to show $M(\Gamma) = \bigoplus_{k\geq 0} M_k(\Gamma)$ is an f.g $\mathbb{C}$-algebra

$\Gamma(N) \subset \Gamma$, $\Gamma(N) \trianglelefteq SL_2(\mathbb{Z})$

$A = M(\Gamma(1))$ (generated by E_4, E_6 over $\mathbb{C}$)

$B = M(\Gamma(N))$ ~~$B \subseteq G \subseteq A$~~ $\boxed{A \subset G \subset B}$

$G = M(\Gamma)$

Claim: $Frac(B)/Frac(A)$ is finite ($\Rightarrow Frac(B)/Frac(C)$ is finite)

Then A is a f.g. $\mathbb{C}$ algebra.

Lemma $\Rightarrow B$ is a f.g. $\mathbb{C}$-algebra $\Rightarrow G$ is a f.g. $\mathbb{C}$-algebra

---

Proof of Claim: We show $Frac\,B/Frac\,A$ finite, so it is important that Γ is a congruence subgroup

$G = \Gamma(N) \backslash \Gamma(1)$ $(\cong SL_2(\mathbb{Z}/N\mathbb{Z}))$

G acts on $M_k(\Gamma(N))$ for each k. ($f \mapsto f^g = f|_{\gamma,k}$ where $\Gamma(N)\gamma = g$

$\Rightarrow G$ acts on $M(\Gamma(N)) = B$. $B^G = A$.

$\Rightarrow G$ acts on $Frac(B)$

Let $x = p/q \in Frac(B)$. Suppose $x^g = x \;\forall g \in G$.

$x \prod_{g\in G} q^g \in B^G = A$

$\prod_{g\in G} q^g \in B^g = A \Rightarrow x \in Frac(A)$.

So $Frac(B)^G = Frac(A)$

Artin's Lemma $\Rightarrow Frac(B)/Frac(A)$ is Galois with Galois group G

(in particular, finite). □

Some versions: $[Frac(B) : Frac(A)] \leq |G|$

?

Corollary

$M_k(\Gamma)$ is finite dimensional over $\mathbb{C}$.

Proof   Idea: $M(\Gamma)$ f.g., restrict to correct weight

$M(\Gamma)$ is finitely generated over $\mathbb{C}$ and has generating set $f_1, \dots, f_n$, $f_i \in M_{k_i}(\Gamma(N))$. Weight k products of the f_i give a finite set of generators spanning $M_k(\Gamma)$. □

To show that $\Theta(\tau, 4) =$ sum of weight two Eisenstein series we needed that $\dim_{\mathbb{C}} M_2(\Gamma_0(4)) = 2$.

Aim: Prove a general formula for $\dim M_k(\Gamma)$.

We do this by studying $\Gamma \backslash \mathbb{H}$ as a Riemann surface, compactify, then associate divisors on this compact Riemann surface to modular forms. Riemann-Roch will then allow us to compute dimensions.

References: Farkas, Kra "Riemann Surfaces"

Miranda "Algebraic Curves and Riemann Surfaces"

The compact Riemann surfaces $\Gamma \backslash \mathbb{H}$ are the $\mathbb{C}$-points of an algebraic curve $C/\mathbb{C}$. $C = \mathrm{Proj}(M(\Gamma))$ $M(\Gamma)$ normal, graded f.g. $\mathbb{C}$-algebra

We will work complex analytically to get information about C.

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Definition

Let X be a topological space.

A presheaf (of Abelian groups) on X is a pair $(\mathcal{F}, \rho)$ comprising:

Think of $\mathcal{F}$ as symmetries of $\mathcal{U}$ or an action on $\mathcal{U}$

- For $\mathcal{U} \subset X$ open, an abelian group $\mathcal{F}(\mathcal{U})$

Think of ρ as a transition map

- If $V \subset \mathcal{U} \subset X$ open sets, $\rho^{\mathcal{U}}_{V} : \mathcal{F}(\mathcal{U}) \to \mathcal{F}(V)$ a group hom.

with $\rho^{\mathcal{U}}_{\mathcal{U}} = id$, and for $W \subset V \subset \mathcal{U}$, $\rho^{V}_{W} \circ \rho^{\mathcal{U}}_{V} = \rho^{\mathcal{U}}_{W} : \mathcal{F}(\mathcal{U}) \to \mathcal{F}(W)$

Notation

If $f \in \mathcal{F}(\mathcal{U})$, $V \subset \mathcal{U}$, write $f|_V$ for $\rho^{\mathcal{U}}_{V}(f)$

Example

1. Let X be any topological space.

$(\mathcal{O}^{cts}_X, \rho)$ is a pre-sheaf, where

$\mathcal{O}^{cts}_X(\mathcal{U}) = \{\text{continuous functions } \mathcal{U} \to \mathbb{C}\}$ ← group under addition of functions

For $V \subset \mathcal{U}$, $\rho^{\mathcal{U}}_{V} : \mathcal{O}^{cts}_X(\mathcal{U}) \to \mathcal{O}^{cts}_X(V)$, $f \mapsto f|_V$

2. $X \subset \mathbb{C}$ open.

$(\mathcal{O}_X, \rho) : \mathcal{O}_X(\mathcal{U}) = \{\text{holomorphic functions on } X\}$

$\rho^{\mathcal{U}}_{V}$ = restriction. $\mathcal{O}_X \subset \mathcal{O}^{cts}_X$

Definition

A presheaf $\mathcal{F}$ on a topological space X is a sheaf if for any $\mathcal{U} \subset X$ open and $\{\mathcal{U}_i\}_{i \in I}$ an open cover of $\mathcal{U}$,

think of $\mathcal{U}_i$ as an atlas

i) then if $f, g \in \mathcal{F}(\mathcal{U})$, $f|_{\mathcal{U}_i} = g|_{\mathcal{U}_i}$ $\forall i$, then $f = g$

so functions agreeing on the whole atlas are equal

ii) if $f_i \in \mathcal{F}(\mathcal{U}_i)$, $i \in I$, with $f_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = f_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$ for all $j \in I$, then $\exists!\ f \in \mathcal{F}(\mathcal{U})$ with $f|_{\mathcal{U}_i} = f_i$ for each $i \in I$.

compatible charts can be patched together to form a function on the whole atlas.

Examples 1 and 2 are both sheaves.

<u>Definition</u>

(for us)

A Riemann surface is a <u>connected</u>, Hausdorff topological space X equipped with a sheaf $\mathcal{O}_X$ with $\mathcal{O}_X \subset \mathcal{O}_X^{cts}$ (i.e. $\mathcal{O}_X(U) \subset \mathcal{O}_X^{cts}(U)$ for each U), such that there exists an open cover $\bigcup_{i \in I} U_i = X$ and homeomorphisms $\phi_i : U_i \xrightarrow{\sim} V_i$ open in $\mathbb{C}$. (charts ϕ_i and an atlas U_i), with, for $U \subset U_i$ open,

functions $V_i \to \mathbb{C}$, $V_i \subset \mathbb{C}$ open

$$\left\{ \begin{matrix} \text{holomorphic functions} \\ \underset{V_i}{\phi_i(U)} \to \mathbb{C} \end{matrix} \right\} \xrightarrow[\sim]{f \mapsto f \circ \phi_i} \mathcal{O}_X(U) \text{ an isomorphism.}$$

functions $U \to \mathbb{C}$ for $U \subset X$ open

A holomorphic map between Riemann surfaces is a continuous map $f : X \to Y$ such that for $U \subset Y$ open, precomposing with f gives a homomorphism

$$\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$$
$$\varphi \mapsto \varphi \circ f$$

~~This is the map~~

i.e. every "good" map in $\mathcal{O}_Y(U)$ (holomorphic $U \to \mathbb{C}$) pulls back nicely to a "good" map of $\mathcal{O}_X$

This is the map $f^\# : \mathcal{O}_Y \to \underbrace{f_* \mathcal{O}_X}_{U \mapsto \mathcal{O}_X(f^{-1}(U))}$ (sheaf on Y)

Suppose that X is a Riemann surface with a nice action of a group G (e.g. a congruence subgroup $\Gamma \curvearrowright \mathbb{H}$).

Then the topological space $G\backslash X$, $\mathcal{O}_{G\backslash X}(U) = \mathcal{O}_X(\pi^{-1}(U))^G$

$$X \xrightarrow{\pi} G\backslash X$$

will be a Riemann Surface, $(G\backslash X, \mathcal{O}_{G\backslash X})$.

$\pi : X \twoheadrightarrow G\backslash X$ will be holomorphic, and in local coordinates, at a point $x \in X$, π looks like $z \mapsto z^{n_x}$.

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Modular Forms (10)

Groups acting on Riemann Surfaces

Today, we just use the fact that a Riemann surface is Hausdorff and locally compact, i.e. $\forall x \in X$, $\exists\, U_x \subset X$ open, and compact $K \subset X$ with $x \in U \subset K$ (a compact neighbourhood).

Every point has an open neighbourhood homeomorphic to an open disc in $\mathbb{C}$.

Let G be a group, X a Riemann surface.

A holomorphic group action of G on X is a group action

$$g_*: X \to X \quad \text{with all maps } g_* \text{ holomorphic.}$$

$$\left(e = id,\ e_*: X \xrightarrow{id} X,\ (gh)_* = g_* \circ h_*\right)$$

i.e. a homomorphism $r: G \to \mathrm{Aut}^{hol}(X)$, group of holomorphic bijections $X \to X$ with holomorphic inverse. biholomorphic maps

Definition

$G \circlearrowright X$ a holomorphic group action. We say that G acts properly if for any pair $A, B \subset X$ of compact subsets, the set $S_{A,B} = \{g \in G : g_*(A) \cap B \neq \emptyset\}$ is finite.
i.e. almost all elements of G separate A and B

Exercise

Give G the discrete topology. Consider $f: G \times X \to X \times X$
$(g, x) \mapsto (x, g_* x)$.

G acts properly $\Leftrightarrow$ f is proper i.e. $f^{-1}(\text{compact}) = \text{compact}$

(One way, consider $f^{-1}(A \times B)$).

Remark

If $G \circlearrowright X$ is a proper action then for $x \in X$, the stabiliser G_x is finite (apply definition to the compact sets $\{x\}, \{x\}$).

Lemma

an open set that can be properly shifted providing x is not fixed

$G \circlearrowright X$ properly. Then for $x \in X$, there is an open neighbourhood $x \in U_x$ (connected with compact closure) satisfying

$$g_*(U_x) \cap U_x \neq \emptyset \Leftrightarrow g_* x = x \text{ (i.e. } g \in G_x).$$

Proof

First, we find $U \ni x$ an open neighbourhood such that $g_*(U) \cap U \neq \emptyset$ for only finitely many $g \in G$.

(Apply the definition of proper action to a compact neighbourhood K of x, and take $U = \text{interior}(K)$). non-trivial since $x \in$ open $\subset K$

see end of page

We have $g_1, \dots, g_n$ with $g_{i*} U \cap U \neq \emptyset$.

If $g_{i*} x \neq x$, take V_i, V_i^* disjoint open neighbourhoods of x, $g_{i*} x$ in U. because X is Hausdorff

Let W_i be an open neighbourhood of x in X with $g_{i*}(W_i) \subset V_i^*$.

possible as G has a holomorphic group action

Set $U_i = V_i \cap W_i$. Check that $U_i \cap g_i U_i = \emptyset$.

Let U_x = connected component of $\bigcap_i U_i$ containing x $\square$

Has compact closure since $\underbrace{\text{closed subspaces}}_{\overline{U_x}}$ of $\underbrace{\text{compact spaces}}_{K}$ are compact

$$\text{Interior}(A) = \{x \in A \mid x \in \underset{\text{open}}{U} \subset A, \text{ some } U\}$$

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<u>Lemma</u> (Refined Version)

$G \circlearrowright X$ as above. Then $\exists\, U_x$ as before with $gU_x = U_x$ for $g \in G_x$.

This Lemma $\Rightarrow$ the G-orbit of U_x in X $= \bigcup_{g \in G} gU_x$ is $\coprod_{gG_x \in G/G_x} gU_x$ (here, gU_x depends only on gG_x).

<u>Proof</u>

Let $U \ni x$ be as in the previous lemma. Set $U_x = \underbrace{\bigcap_{g \in G_x} gU}$ connected component of $\bigcap_{g\in G_x} gU$ containing x $\square$

<u>Quotient Topology</u>

$G\backslash X$ = set of G-orbits

$\pi : X \to G\backslash X$, $x \mapsto Gx$

Topology: A set $V \subset G\backslash X$ is open $\Leftrightarrow \pi^{-1}(V)$ is open in X.

Exercise: This is the coarsest topology such that every continuous map $f : X \to Y$ ($\in$ topological space) with $f(gx) = f(x)$ $\forall g \in G, x \in X$ factors through a continuous map.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ G\backslash X \arrow[r, "f'"'] \arrow[ur, phantom, "\circlearrowleft"] & Y \end{tikzcd}$$

From now on, $G\backslash X$ is a topological space (with quotient topology).

<u>Lemma</u>

$\pi : X \to G\backslash X$ is continuous and open, and $G\backslash X$ is Hausdorff.

<u>Proof</u>

π is continuous by definition of quotient topology. so $\pi^{-1}(\text{open}) = \text{open}$

We must show that ($U \subset X$ open $\Rightarrow \pi^{-1}(\pi U)$ is open.)

But this is just $\bigcup_g gU$. as $\pi : X \to G\backslash X$, $x \mapsto Gx$

It remains to show that $G \backslash X$ is Hausdorff. Suppose that $Gx \neq Gy$ are distinct G-orbits. so that $gx \neq y \ \forall g \in G$

Take $K_x \supset U_x$, $K_y \supset U_y$, distinct compact neighbourhoods of $x, y \in X$.

because this is only true for the finite set $g_1, \dots, g_n$

Properness $\Rightarrow (gK_x \cap K_y \neq \emptyset) \Leftrightarrow g = g_1, \dots, g_n$.

We can shrink K_x, K_y to ensure that $y \notin g_i K_x$ for $i = 1, \dots, n$

Now set $V_y = U_y \cap \left(X \setminus \bigcup_{i=1}^{n} g_i K_x \right)$ compact in Hausdorff is closed

Now, $gU_x \cap V_y = \emptyset \ \ \forall g \in G$.

So $\pi(U_x)$ and $\pi(V_y)$ are disjoint open neighbourhoods of Gx and Gy. $\square$

<u>Corollary</u>

Let $x \in X$, $U_x \ni x$ a G_x-stable open neighbourhood as in Lemma 2. Then we have a diagram:

by construction of U_x the union is disjoint

$$\begin{tikzcd}
\bigcup_{g \in G} gU_x = \pi^{-1}(\pi U_x) \arrow[d, "\pi"'] \arrow[r, equal] \arrow[dr, phantom, "\circlearrowleft"] & \coprod_{g \in G / G_x} gU_x \arrow[d] \\
\pi U_x \arrow[r, dashed] & G_x \backslash U_x
\end{tikzcd}
\qquad
\begin{tikzcd}
gy \arrow[d, mapsto] \\
G_x y
\end{tikzcd}$$

check well defined and continuous

quotient topology

with the bottom map a homeomorphism.

In fact, this holds for all $g \in G$ as $\pi : X \to G \backslash X$ but we have restricted to U_x and $g_x \in G_x$ are the only $g \in G$ that fix U_x

<u>Proof</u>

$U_x \xrightarrow{\pi} \pi U_x$ is a continuous map, and $\pi(g_x y) = \pi(y)$ for all $y \in U_x$, so it factors through $U_x \xrightarrow{\pi} \pi U_x$, $U_x \to G_x \backslash U_x \dashrightarrow \pi U_x$

Likewise, the right hand map is <u>constant on G-orbits</u>, hence factors through πU_x.

since $y \in U_x$

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i.e. the quotient of $\bigcup_{g \in G} g\mathcal{U}_x = \coprod_{g \in G/G_x} g\mathcal{U}_x$ by G is given by $\coprod_{g \in G/G_x} gG_xg^{-1} \backslash g\mathcal{U}_x$ and $G_x \circlearrowright \mathcal{U}_x \simeq \underset{g \mapsto gg}{gG_xg^{-1} \circlearrowright g\mathcal{U}_x}$ $\square$

<u>Definition</u>

X is a Riemann surface, $G \circlearrowright X$ a proper group action.

$(X, \mathcal{O}_X)$, $\mathcal{O}_X(\mathcal{U}) = \{\text{holomorphic functions } \mathcal{U} \to \mathbb{C}\}$

Let $V \subset G\backslash X$ be open. $\pi^{-1}(V)$ has an action of G.

So $\mathcal{O}_X(\pi^{-1}(V))$ has a (right) action of G:

for $f : \pi^{-1}(V) \to \mathbb{C}$, define f^g by $f^g(x) := f(gx)$.

Denote the G-invariants in $\mathcal{O}_X(\pi^{-1}(V))$ by $\mathcal{O}_X(\pi^{-1}(V))^G$.

<u>Definition</u>

Denote by $\mathcal{O}_{G\backslash X}$ the presheaf $V \mapsto \mathcal{O}_X(\pi^{-1}(V))^G$.

If $W \subset V$, $\mathcal{O}_X(\pi^{-1}(V))^G \xrightarrow{\text{restriction}} \mathcal{O}_X(\pi^{-1}(W))^G$.

It is easy to check that $\mathcal{O}_{G\backslash X}$ is a sheaf.

Also, if $V \subset G\backslash X$ open,

$\mathcal{O}_{G\backslash X}(V) \subset \mathcal{O}_X^{cts}(\pi^{-1}(V))^G = \mathcal{O}_{G\backslash X}^{cts}(V)$.

<u>Theorem</u>

$(G\backslash X, \mathcal{O}_{G\backslash X})$ is a Riemann Surface. i.e.
- $\mathcal{O}_{G\backslash X}$ a sheaf
- Open cover of $G\backslash X$, $\mathcal{U}_i$
- Charts $\phi_i : \mathcal{U}_i \xrightarrow{\sim} V_i \subset \mathbb{C}$

<u>Remark</u>

X connected $\Rightarrow G\backslash X$ connected.

<u>Proof</u> (of Theorem)

We need to find charts on open neighbourhoods of every point in $G\backslash X$

By the corollary? check online, we can reduce to $0 \in X \subset \mathbb{C}$ (connected, open), $G \to \mathrm{Aut}^{hol}(X)$.

with G a finite group stabilising 0 and for $x \neq 0$, $G_x = \{id\}$)

- Claim : $\exists$ an open neighbourhood U of 0 with $gU = U$ $\forall g \in G$, and a biholomorphism $U \cong D$, open unit disc such that
$$\begin{array}{ccc} U & \xrightarrow{g} & U \\ \wr\vert & & \wr\vert \\ D & \xrightarrow{\times \zeta(g)} & D \end{array}$$
$\forall g \in G$ for a root of unity $\zeta(g)$.

- Claim $\Rightarrow$ Theorem : Now, $G \cong \mathbb{Z}/n\mathbb{Z}$, generator acting by $\times \zeta$ on D, ζ a primitive n^{th} root of unity.

$$G \backslash U \simeq \begin{array}{c} (\mathbb{Z}/n\mathbb{Z}) \backslash D \\ \wr \big\downarrow \\ D \end{array} \qquad \begin{array}{c} z \\ \big\downarrow \\ z^n \end{array}$$

(a holomorphic function on D is invariant under multiplication by ζ $\Leftrightarrow$ it is a holomorphic function in z^n).

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Theorem

$(G\backslash X, \mathcal{O}_{G\backslash X})$ is a Riemann Surface. Moreover, $\pi : X \to G\backslash X$ is holomorphic, π has local form $z \mapsto z^{n_x}$, around x. Here, $n_x = \#Stab(x)$, $Stab(x)$ cyclic, order n_x.

Proof

- Idea: reduce to the case of a finite group acting on the unit disc, fixing 0.
- Schwarz's Lemma: If $f : D \to D$ is a biholomorphism which fixes 0 then $f(z) = \zeta z$, some $\zeta \in \mathbb{C}$, $|\zeta| = 1$.

So if $G \subset Aut^{hol}(D)$, $g(0) = 0 \ \forall g \in G$, with G finite, then $G \cong \mathbb{Z}/n\mathbb{Z}$, and a generator acts by $z \mapsto \zeta z$, ζ a primitive n^{th} root of unity.

because for each $G \ni g : z \mapsto \zeta_g z$, g finite order n $\Rightarrow \zeta_g^n = 1$, ζ_g a primitive n^{th} root. This generates all n^{th} roots. Then all elements of G generate some cyclic group since $C_a \times C_b \cong C_{ab}$ for $(a,b) = 1$ (and Schwarz $\Rightarrow G$ abelian)

Now consider the map $\alpha : G\backslash D \to D$, $Gz \mapsto z^n$.

- Claim: for $U \subset G\backslash D$ open, this identifies $\mathcal{O}_{G\backslash D}(U)$ with $\mathcal{O}_D(\alpha(U))$, i.e. $(G\backslash D, \mathcal{O}_{G\backslash D})$ is a Riemann surface which is in fact isomorphic to the unit disc.

- Proof of Claim:

in terms of $\mathcal{O}_D(U)$ — extra condition since we are in the quotient

$\mathcal{O}_{G\backslash D}(U) = \{ f : \pi^{-1}(U) \to \mathbb{C} \text{ holomorphic} \mid f(\zeta z) = f(z), z \in \pi^{-1}(U)\}$

$\Rightarrow f \in \mathcal{O}_{G\backslash D}(U)$ is a holomorphic function in z^n, with $f(z) = g(z^n)$, some g which is holomorphic

$\Rightarrow g \in \mathcal{O}_D(\alpha(U))$

i.e. any $f \in \mathcal{O}_{G\backslash D}(U)$ factors through $\mathcal{O}_D(\alpha(U))$ to give G

- Proof of Theorem:

We may assume that $G \subset Aut(X)$. We saw that for $x \in X$, we had a G_x-invariant open neighbourhood $U_x \ni x$ with $G_x\backslash U_x \cong$ a neighbourhood of $\pi(x) \in G\backslash X$

otherwise, trying to form $G\backslash X$ from X is already silly

by last lecture's Corollary

because we reduce $\begin{matrix} X \\ G \end{matrix}$ to $\begin{matrix} U_x \\ G_x \end{matrix}$

U_x has compact closure, and G acts properly

This reduces the Theorem to the case where G is <u>finite</u> and fixes a point $x \in X$.
Use convexity arguments and the open mapping theorem $\Rightarrow$ we can find a neighbourhood $x \in U_x$ which is G-invariant, and $U_x \cong D$, which reduces to the case studied previously. $\square$

<u>Applications to Modular Forms</u>

<u>Proposition</u>

$\Gamma(1) \circlearrowright \mathbb{H}$ is a proper group action (so that every congruence subgroup acts properly).

<u>Proof</u>

We want to show that if $A, B \in \mathbb{H}$ are compact, then $\#\{\gamma \in \Gamma(1) : \gamma A \cap B \neq \emptyset\}$ is finite.

actual proof

- $\boxed{\text{Idea:}}$ $A \subset \bigcup_{i=1}^{n} \gamma_i \bar{F}$, $B \subset \bigcup_{j=1}^{m} \delta_j \bar{F}$
so $\gamma A \cap B \neq \emptyset \Rightarrow \gamma \gamma_i = \pm \delta_j$, some j
$\Rightarrow \gamma$ is a finite set.

for each i, some j depending on i

can take finitely many since A, B are compact

- 'Easier' proof: We may assume that A, B are rectangles. $\mathrm{Im}(A) \subset [a_1, a_2]$, similarly for $\mathrm{Re}(A)$
$\mathrm{Im}(B) \subset [b_1, b_2]$. $\mathrm{Re}(B)$

A, B are compact so closed and bounded and therefore contained inside distinct rectangles

Suppose $\tau \in A$, $\gamma\tau \in B$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$b_1 \leq \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2} \leq b_2$
$\mathrm{Re}(\tau)$ bounded $\Rightarrow$ only finitely many possibilities for c, d.
$\Rightarrow \gamma = \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \times \{$finitely many elements of $\Gamma(1)\}$

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$\mathrm{Re}(r\tau)$ bounded $\Rightarrow$ finitely many possibilities for n. $\square$

Definition

Γ a congruence subgroup. $Y(\Gamma) := \Gamma \backslash \mathbb{H}$ (Riemann Surface)
"open modular curve of level Γ"

Corollary

$j : \mathbb{H} \to \mathbb{C}$ a meromorphic form of weight 0, induces a biholomorphism $j : Y(\Gamma(1)) \to \mathbb{C}$.

Proof

We saw that j is a bijective map $\Gamma(1) \backslash \mathbb{H} \to \mathbb{C}$.
$j : \mathbb{H} \to \mathbb{C}$ is in $\mathcal{O}_{\mathbb{H}}(\mathbb{H})^{\Gamma(1)} = \mathcal{O}_{Y(\Gamma(1))}(Y(\Gamma(1)))$
Therefore j induces a holomorphic bijection $Y(\Gamma(1)) \to \mathbb{C}$, hence a biholomorphism. $\square$

$\mathbb{H} \xrightarrow{\pi} Y(\Gamma(1)) \simeq \mathbb{C}$. π has local form:

$z \mapsto z$, $\tau \in \mathbb{H}$ non-elliptic
$z \mapsto z^2$, $\tau \in \Gamma(1) i$
$z \mapsto z^3$, $\tau \in \Gamma(1) \omega$

Weakly modular functions of weight $2k$, level Γ
$\updownarrow$
meromorphic (k-fold) differentials on $Y(\Gamma)$.

Cusps and Compact Modular Curves

Weight 0 weakly modular functions of level $\Gamma(1)$
$\parallel$
meromorphic functions on $Y(\Gamma(1))$

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- Idea: a weight 0 modular form

$\qquad\qquad\qquad\qquad\qquad\qquad ||$

a holomorphic function on $X(\Gamma(1)) \supset Y(\Gamma(1))$

$\nearrow$ compact

$\mathbb{C} \cup \{\infty\}$ $\qquad \mathbb{C}$

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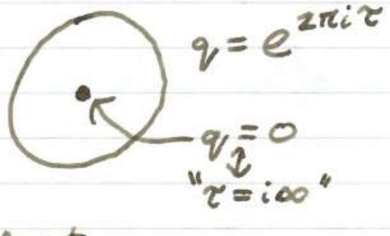
Cusps and Compactifying Modular Forms

Today: $Y(\Gamma) \cup \{\text{cusps}\} = X(\Gamma)$

We need to control the behaviour of $f(\tau)$ for

$\tau \in \{\mathrm{Im}(\tau) > A\}$ and for $\tau \in \alpha(\{\mathrm{Im}(\tau) > A\})$, $\alpha \in SL_2(\mathbb{Z})$



$\alpha(\{\mathrm{Im}(\tau) > A\})$ meets the real axis at a/c for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Definition

Let Γ be a congruence subgroup.

(The cusps of level Γ) $= C_\Gamma := {}_\Gamma \backslash \mathbb{P}^1(\mathbb{Q})$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$ $\qquad$ $[\,\cdot\,]$ projective coordinates

The cusp $\infty \Leftrightarrow \Gamma\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Note that the action of $SL_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ is transitive.

Definition

Suppose that $S = \Gamma x \in C_\Gamma$.

The width h_S of S is defined to be the index of

$\{\pm I\}\mathrm{Stab}_\Gamma(x)$ in $\mathrm{Stab}_{SL_2(\mathbb{Z})}(x)$.

(or $\mathrm{Stab}_{\bar\Gamma}(x)$ in $\mathrm{Stab}_{PSL_2(\mathbb{Z})}(x)$)

$\bar\Gamma$ = Image of Γ in $PSL_2(\mathbb{Z})$, $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$

Exercise

i) h_S is independent of the choice of $x \in S$

ii) If $\gamma \in SL_2(\mathbb{Z})$, $\gamma x = \infty$, then h_S is the width of the cusp ∞ at level $\gamma\Gamma\gamma^{-1}$ ($\Leftrightarrow \pm\begin{pmatrix} 1 & h_S \\ 0 & 1 \end{pmatrix}$ generates $\mathrm{Stab}_{\gamma\Gamma\gamma^{-1}}(\infty)$)

because conjugation by γ $\Leftarrow$

Examples

i) $\Gamma = \Gamma(1)$, $C_\Gamma = \{ \Gamma[\begin{smallmatrix}1\\0\end{smallmatrix}] \} = \{\infty\}$, $h_\infty = 1$

ii) p prime , $\Gamma = \Gamma_0(p)$

Exercise : $C_\Gamma = \{ \Gamma[\begin{smallmatrix}1\\0\end{smallmatrix}] , \Gamma[\begin{smallmatrix}0\\1\end{smallmatrix}] \}$

width 1 as $(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}) \in \Gamma_0(p)$ — width p

We would like to construct compact Riemann Surfaces with underlying set $Y(\Gamma) \cup C_\Gamma = X(\Gamma)$.

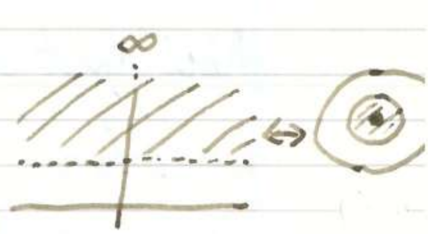
<u>$X(\Gamma)$ as a topological space</u>

Definition

$\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, the <u>extended upper half plane</u>

$SL_2(\mathbb{Z}) \circlearrowright \mathbb{H}^*$

We define a topology on $\mathbb{H}^*$ by specifying some open sets and taking the topology generated by them.

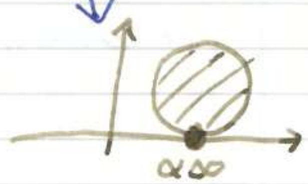


We have the following open sets :

- $U \subset \mathbb{H}$ open
- $\{\infty\} \cup U_A$, $U_A = \{ \tau \mid Im(\tau) > A \}$, $\forall A \in \mathbb{R}_{>0}$.
- $\alpha(\{\infty\} \cup U_A)$ $\forall \alpha \in SL_2(\mathbb{Z})$, $\forall A \in \mathbb{R}_{>0}$

open neighbourhood of the point $\alpha\infty \in \mathbb{P}^1(\mathbb{Q})$

Exercise : $\alpha(\{\infty\} \cup U_A)$ looks like



→ Just think about how Möbius maps transform $\mathbb{C}$

$\mathbb{H}^*$ is connected and Hausdorff.

Definition

As a topological space, $X(\Gamma)$ is defined to be $\Gamma \backslash \mathbb{H}^*$ with quotient topology.

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Warning

$\Gamma \circlearrowright \mathbb{H}^*$ is not proper. $\mathrm{Stab}_\Gamma(\infty)$ is infinite.

e.g. $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, any n with n | width of ∞

Lemma

If $x \in \mathbb{P}^1(\mathbb{Q})$, then $\exists$ an open neighbourhood $\mathcal{U}$ of $x \in \mathbb{H}^*$, such that $(\gamma \in \Gamma,\ \gamma\mathcal{U} \cap \mathcal{U} \neq \emptyset) \Leftrightarrow \gamma \in \mathrm{Stab}_\Gamma(x) = \Gamma_x$ (*)

Proof

$\alpha x = \infty$ for some $\alpha \in SL_2(\mathbb{Z})$. Translating by α, it is enough to prove the lemma for $x = \infty$. conjugate all of (*) by α

Exercise:

$\mathrm{Im}(\gamma\tau) = \dfrac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$

If $c \neq 0$ then $|\tau| \gg 0 \Rightarrow \mathrm{Im}(\gamma\tau)$ small

$\exists A$ large enough such that $(\gamma \mathcal{U}_A \cap \mathcal{U}_A \neq \emptyset) \Rightarrow \gamma \in \Gamma_\infty$ (upper triangular)

so $\mathcal{U}_A$ is what we need $\square$

Now, if $x \in \mathbb{P}^1(\mathbb{Q})$, then there is an open neighbourhood $\mathcal{U} \ni x$ in $\mathbb{H}^*$ such that for $\pi : \mathbb{H}^* \to X(\Gamma)$,

$\mathcal{U}_A$?

$Y(\Gamma) \cap \pi(\mathcal{U})$ is biholomorphic to $\left\langle \begin{pmatrix} 1 & h_S \\ 0 & 1 \end{pmatrix} \right\rangle \backslash \mathcal{U}_A \xrightarrow{\sim}$ (punctured open disc)

$\tau \mapsto \exp(2\pi i \tau/h_S) = q_{h_S}$

Here h_S = width of cusp at x, $S = \Gamma_x$

by corollary with commuting diagram

"around" ∞

Proposition

$X(\Gamma)$ is connected, Hausdorff and compact.

Proof

$\mathbb{H}^*$ connected $\Rightarrow X(\Gamma)$ connected.

by contradiction. pull back disconnection of $X(\Gamma)$ to $\mathbb{H}^*$

We have a continuous map $X(\Gamma) \to X(\Gamma(1)) \cong \mathbb{P}^1_{\mathbb{C}}$

To show that $X(\Gamma)$ is Hausdorff:

$x \neq y$

$x, y \in Y(\Gamma) \Rightarrow$ we can separate x, y. by normal topology on $\mathbb{C}$

$S, S' \in C_\Gamma \Rightarrow$ we can separate S, S' by the previous lemma. applied in $\Gamma(1)$!

Take $\alpha(S) = S'$ ~~because with~~ the statement of the lemma in $\mathbb{H}^*$ merely says that $\mathcal{U}$ and $\gamma\mathcal{U}$ are disjoint in $X(\Gamma)$.

$s \in C_\Gamma$, $x \in Y(\Gamma)$ $\Rightarrow$ We can separate the images of s and x in $X(\Gamma(1))$. Take inverse images to separate s, x in $X(\Gamma)$.

$\mathcal{F} \subset \mathbb{H}$, $\mathcal{F}^* = \mathcal{F} \cup \{\infty\} \subset \mathbb{H}^*$.

$\mathcal{F}^*$ is compact. $X(\Gamma) = $ image of $\bigcup_{\Gamma\alpha \in \Gamma \backslash \Gamma(1)} \alpha \mathcal{F}^*$ under $(\mathbb{H}^* \to X(\Gamma))$

Hence $X(\Gamma)$ is compact. $\square$

<u>Definition</u>

$U \subset X(\Gamma)$ open.

$$\mathcal{O}_{X(\Gamma)}(U) := \{ f \in \mathcal{O}^{cts}_{X(\Gamma)}(U) \mid f|_{U \cap Y(\Gamma)} \in \mathcal{O}_{Y(\Gamma)}(U \cap Y(\Gamma)) \}$$

<u>Proposition</u>

$(X(\Gamma), \mathcal{O}_{X(\Gamma)})$ is a Riemann Surface.

<u>Proof</u> previous proposition shows that $X(\Gamma)$ is connected, Hausdorff

$\mathcal{O}_{X(\Gamma)}$ is a sheaf. Every point of $Y(\Gamma)$ has an open neighbourhood U with a chart because $\mathcal{O}_{Y(\Gamma)}$ is a sheaf

For the cusp ∞, there is a homeomorphism from $\pi(U_A \cup \{\infty\})$ to an open disc, restricting to a biholomorphism from $\underset{\subset Y(\Gamma)}{\pi(U_A)}$ to a punctured unit disc

$\Rightarrow$ we get a chart on $\pi(U_A \cup \{\infty\})$, since a continuous function on any open disc, holomorphic on the punctured disc, is actually holomorphic on the whole open disc. $\square$

<u>Remark</u>

$X(\Gamma) \to X(\Gamma(1))$. In local coordinates at a cusp $s \in C_\Gamma$, this has the form $z \mapsto z^{h_s}$ (local coordinates $= q_{h_s}$, $q = (q_{h_s})^{h_s}$

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Differentials and Divisors

We will relate modular forms to meromorphic differentials on $X(\Gamma)$, and use the Riemann-Roch formula to compute the dimensions of $M_{2k}(\Gamma)$ for any congruence subgroup Γ.

$f : \mathbb{H} \to \mathbb{C}$, weakly modular, weight 2, level Γ.

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\dfrac{d(\gamma\tau)}{d\tau} = \dfrac{1}{(c\tau+d)^2}$

So $f(\tau)\,d\tau$ satisfies $f(\gamma\tau)\,d(\gamma\tau) = f(\tau)\,d\tau$

Similarly, if f has weight $2k$, $f(\gamma\tau)(d(\gamma\tau))^k = f(\tau)\cdot(d\tau)^k$

Meromorphic Differentials on Riemann Surfaces

Definition

1. $U \subset \mathbb{C}$ open, $n \in \mathbb{Z}_{>0}$. Define the $\mathbb{C}$-vector space of meromorphic differentials of degree n on U to be

$\Omega^{\otimes n}(U) = \{ f(z)(dz)^n : f \text{ meromorphic on } U \}$

If $\phi : U_1 \to U_2$ is holomorphic ($U_1, U_2 \subset \mathbb{C}$ open), define

$\phi^* : \Omega^{\otimes n}(U_2) \to \Omega^{\otimes n}(U_1)$ by

$$f(z_2)(dz_2)^n \longmapsto f(\phi(z_1))(\phi'(z_1))^n (dz_1)^n$$

2. X a Riemann Surface. If U_1, U_2 are two open subsets of X with charts $\phi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{C}$,

$$\tau_{ij} := \phi_j \circ \phi_i^{-1} : \underbrace{\phi_i(U_i \cap U_j)}_{\subset V_i} \xrightarrow{\sim} \underbrace{\phi_j(U_i \cap U_j)}_{\subset V_j}$$

transition map? ϕ_i^{0} is a homeomorphism

Then, a meromorphic differential of degree n on X is a rule sending charts $\phi : \underset{\cap\, X}{U} \xrightarrow{\sim} \underset{\cap\, \mathbb{C}}{V}$ to meromorphic differentials $\omega(\phi)$ on V

such that for two charts ϕ_1, ϕ_2 — i.e. the differentials are compatible; they match where charts overlap

$$\tau_{ij}^*\left(\omega(\phi_j)\big|_{\phi_j(U_i\cap U_j)}\right) = \omega(\phi_i)\big|_{\phi_i(U_i\cap U_j)}$$

Check that a meromorphic differential on X is uniquely determined by $\omega(\phi_i)$, with $\phi_i : U_i \xrightarrow{\sim} V_i$, and $X = \bigcup_i U_i$, and vice-versa.

Consequence: $U \overset{\subset X}{\longmapsto}$ {meromorphic differentials of degree n on U}

where ρ_V^U are given by pull-back as follows:

$U \supset V$
ρ_V^U maps differentials on U to differentials on V

If $f : X \to Y$ is holomorphic, we can define

f^* : {meromorphic differentials on Y} $\to$ {meromorphic differentials on X}

using
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \cup & & \cup \\ U_x & \longrightarrow & U_y \\ \wr\wr & & \wr\wr \\ \mathbb{C} \supset V_x & \longrightarrow & V_y \subset \mathbb{C} \end{array} \qquad \omega(\phi_y) \mapsto f^*\omega(\phi_y)$$
differential on V_x.

Notation

$\Omega^{\otimes n}(X)$ = {meromorphic differentials of degree n on X}

For $U \subset X$, $\Omega^{\otimes n}(U)$ = meromorphic differentials on U.

This gives a sheaf $\Omega_X^{\otimes n}$ on X.
1. $\Omega^{\otimes n}(U)$ is a group under addition
2. $\Omega^{\otimes n}(U)$ has sheaf properties by definition of differentials on X

Definition $\quad \mathbb{H} \xrightarrow{\pi} Y(\Gamma) \hookrightarrow X(\Gamma)$

Suppose that $\omega \in \Omega^{\otimes n}(X(\Gamma))$. We get $\pi^*\omega \in \Omega^{\otimes n}(\mathbb{H})$ $\quad \left(= f_\omega(\tau)(d\tau)^n\right)$

Theorem

Let $\omega \in \Omega^{\otimes k}(X(\Gamma))$. $\pi^*\omega = f_\omega(\tau)(d\tau)^k$

Then f_ω is a meromorphic form of weight $2k$ and level Γ.

Moreover, $\omega \mapsto f_\omega$ gives a $\mathbb{C}$-vector space isomorphism

$$\Omega^{\otimes k}(X(\Gamma)) \xrightarrow{\sim} \mathrm{Mero}_{2k}(\Gamma)$$
meromorphic forms, weight $2k$, level Γ.

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Proof

i) f_ω is weakly modular:

If $\gamma \in \Gamma$,
$$\begin{array}{ccc} & \tau \to \gamma\tau & \\ \gamma : \mathbb{H} & \xrightarrow{\sim} & \mathbb{H} \\ \downarrow \pi & & \downarrow \pi \\ Y(\Gamma) & \xrightarrow{id} & Y(\Gamma) \end{array}$$
diagram commutes

$$(\pi \circ \gamma)^* \omega = \gamma^*(\pi^* \omega)$$
$$\overset{\shortparallel}{\pi^*\omega} \quad (\text{since } \pi \circ \gamma = \pi)$$
remember $\pi^*\omega = f_\omega(\tau)(d\tau)^k$

$$\text{So } \underbrace{\gamma^*(f_\omega(\tau)(d\tau)^k)}_{\gamma^*(\pi^*\omega)} = \frac{1}{(c\tau+d)^{2k}} f_\omega(\gamma\tau)(d\tau)^k$$
$$= f_\omega(\tau)(d\tau)^k \longleftarrow \pi^*\omega$$
$$\Rightarrow f_\omega(\gamma\tau) = (c\tau+d)^{2k} f_\omega(\tau)$$

---

ii) We want to show that for any $\alpha \in SL_2(\mathbb{Z})$, $f_\omega|_{\alpha,2k}$ is meromorphic at ∞. If $\alpha : X(\alpha^{-1}\Gamma\alpha) \simeq X(\Gamma)$ then $f_{\alpha^*\omega} = f_\omega|_{\alpha,2k}$. So it suffices to show that f_ω is meromorphic at ∞. In a neighbourhood of $\infty \in X(\Gamma)$,

so that f is a meromorphic form

we have $\Gamma_\infty \backslash U_A \xrightarrow{\sim}$ punctured open disc
$$\tau \longmapsto \exp\left(\frac{2\pi i \tau}{h_\infty}\right) = q_h$$
stabiliser of ∞

This is a chart in a neighborhood of ∞.

ω gives a meromorphic differential $g(q_h)(dq_h)^k$ on the open disc, pulls back to $f_\omega(\tau)(d\tau)^k$ on U_A, via the map $\tau \mapsto \exp\left(\frac{2\pi i\tau}{h}\right)$

$$\frac{dq_h}{d\tau} = \frac{2\pi i}{h} q_h$$

We can see that $f_\omega(\tau) = \left(\frac{2\pi i q_h}{h}\right)^k g(q_h)$, hence f_ω is meromorphic at ∞ since g is meromorphic on the disc

iii) We write down an inverse to the map $\omega \mapsto f_\omega$ to show that this is an isomorphism.

$f \in \mathrm{Mero}_{2k}(\Gamma)$. We will construct $\omega(f)$.

Suppose that $x \overset{= \Gamma\tau}{\in} Y(\Gamma)$. We have charts around τ, x such that π has the form $\overset{\text{in local coordinates}}{z \mapsto z^{n_x}}$, and Γ_τ acts on z by multiplication by a n_x^{th} root of unity, ζ

$f = g(z)$ in these coordinates. — local coordinates

f a meromorphic form $\Rightarrow f(\tau)(d\tau)^k$ is Γ-invariant

$\Rightarrow g(z)(dz)^k$ is Γ_τ-invariant

$$\Rightarrow \underset{\overset{\shortparallel}{g(z)(dz)^k}}{g(\zeta z)(d\zeta z)^k} = g(\zeta z)\zeta^k (dz)^k$$

i.e. $g(\zeta z) = \zeta^{-k} g(z)$, so $z^k g(z)$ is Γ_τ invariant.

so $z^k g(z) = h(z^{n_x})$, h meromorphic.

Now define $\omega(f)$ on this chart, by

$\omega(f) = (n_x z)^{-k} h(z)(dz)^k$ → fits in with local form of π

Exercise : This pulls back to $g(z)(dz)^k$ under the map $z \mapsto z^{n_x}$.

i.e. $\pi^*(\omega(f)) = g(z)(dz)^k$

$$(n_x z)^{-k} h(z)(dz)^k \mapsto (n_x z^{n_x})^{-k} h(z^{n_x})(n_x z^{n_x-1})^k (dz)^k$$

$$= z^{-k} h(z^{n_x})(dz)^k$$

$$= g(z)(dz)^k$$

i) Shows f weakly modular
ii) Shows f meromorphic at ∞

iii) Produces an inverse for $\Omega^{\otimes k}(X(\Gamma)) \to M_{2k}(\Gamma)$: $\omega(f) \leftarrow f$
constructs charts for $\omega(f)$ for $x \in X(\Gamma)$ not a cusp

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Proof (continued)

To show that $\omega \mapsto f_\omega$ is an isomorphism, we will write down an inverse $f \mapsto \omega(f)$. To write down $\omega(f)$ we need to write down compatible differentials $\omega(f)(\phi)$ on V, where $\phi: \overset{\subset \times}{U} \to \overset{\in \mathbb{C}}{V}$ for charts ϕ in an atlas.

Last time we wrote down $\omega(f)(\phi)$ for $\phi: U_x \xrightarrow{\sim} V$ a chart on some open neighbourhood of ~~x~~ x for every $x \in Y(\Gamma)$.

In a neighbourhood of ∞, we have a chart ϕ_∞ to an open disc, induced by $\tau \mapsto e^{\frac{2\pi i \tau}{h}} = q_h$ as in part (ii), h = width of ∞.

$f(\tau) = F(q_h)$, F a meromorphic function on the disc.

Now set $\omega(f)(\phi_\infty) = \left(\frac{2\pi i q_h}{h}\right)^{-k} F(q_h)(dq_h)^k$

This differential pulls back to $f(\tau)(d\tau)^k$ under the map $\tau \mapsto e^{2\pi i \tau / h}$.

again see part (ii)

We must check that the $\omega(f)(\phi)$ are compatible.

By construction, everything pulls back to $f(\tau)(d\tau)^k$ on $\pi^{-1}(U) \subset \mathbb{H}$ (for $\phi: U \to V$). Compatibility follows from our next lemma. $\square$

$U_1, U_2 \to$ two differentials on $U_1 \cap U_2$ with pullback to $\pi^{-1}(U_1 \cap U_2)$ the same, so Lemma $\Rightarrow$ Equality

Lemma

If $X \xrightarrow{\pi} Y$ is a non-constant, holomorphic map of Riemann Surfaces, with Y connected, then

$\pi^*: \Omega^{\otimes n}(Y) \longrightarrow \Omega^{\otimes n}(X)$ is injective.

$f(z)(dz)^n \longmapsto f(\pi_i(z))(\pi_i'(z))^n (dz)^n$ π_i charts for π

Proof

Y connected $\Rightarrow \pi_i'(z)$ non-zero everywhere

Exercise. $\Rightarrow \ker \pi^* = \{ f(z)(dz)^n \in \Omega^{\otimes n}(Y) \mid f(\pi_i(z)) = 0 \ \forall i, \forall z \}$ $\square$

$\Rightarrow f \equiv 0$, $\ker \pi^* = \{0 \text{ map}\}$, π injective

$$\begin{array}{ccc}
\mathrm{Mero}_{2k}(\Gamma) & \xrightarrow{\sim} & \Omega^{\otimes k}(X(\Gamma)) \\
\cup\!\!\mid & & \cup\!\!\mid \\
M_{2k}(\Gamma) & \xrightarrow{\sim} & ? \\
\cup\!\!\mid & & \cup\!\!\mid \\
S_{2k}(\Gamma) & \xrightarrow{\sim} & ?
\end{array}$$

Recall that for $f \in \mathrm{Mero}_{2k}(\Gamma)$, we defined $\mathrm{ord}_x(f)$ for $x \in Y(\Gamma)$. $x = \Gamma\tau$, $\tau \in \mathbb{H}$.

$$\mathrm{Ord}_x(f) = \begin{cases} \text{order of zero of } f \text{ at } \tau & f \text{ holomorphic at } \tau \\ (\text{order of pole of } f \text{ at } \tau) \times (-1) & f \text{ not holomorphic at} \end{cases}$$

For $s \in C(\Gamma)$, $s \in \Gamma x$, $x \in \mathbb{P}^1(\mathbb{Q})$, then if $\alpha\infty = x$ for some $\alpha \in SL_2(\mathbb{Z})$,

$$\mathrm{ord}_s(f) = \mathrm{ord}_\infty(f|_{\alpha,2k}) = \begin{cases} \text{least } n \text{ such that coefficient} \\ \text{of } q_h^n \text{ is non-zero in } q \text{ expansion} \\ \text{of } f|_{\alpha,2k} \end{cases}$$

h = width of ∞ for $\alpha^{-1}\Gamma\alpha$.

$\Rightarrow \pm\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \alpha^{-1}\Gamma\alpha$

$2k$ even $\Rightarrow f|_{\alpha,2k}(\tau+h) = f|_{\alpha,2k}(\tau)$

So $f|_{\alpha,2k}(\tau) = \sum_{n\in\mathbb{Z}} a_n q_h^n$, $q_h = e^{2\pi i \frac{\tau}{h}}$

$f \in M_{2k}(\Gamma) \Leftrightarrow \mathrm{ord}_x(f) \geq 0 \quad \forall x \in X(\Gamma)$ i.e. holomorphic everywhere

$f \in S_{2k}(\Gamma) \Leftrightarrow \begin{cases} \mathrm{ord}_x(f) \geq 0 & \forall x \in Y(\Gamma) \\ \mathrm{ord}_x(f) \geq 1 & \forall s \in C(\Gamma) \end{cases}$ i.e. holomorphic everywhere and a "zero" at each cusp

<u>Orders of Vanishing for Meromorphic Differentials</u>

X a Riemann Surface, $\omega \in \Omega^{\otimes n}(X)$

<u>Definition</u>

For $x \in X$, $V_x(\omega)$ = order of vanishing of $f(z)$ ω_f at z_0 where $\omega(\phi) = f(z)(dz)^n$ for a chart $\phi: \underset{x}{\overset{\cup\!\!\mid}{U}} \to \underset{z_0}{\overset{\cup\!\!\mid}{V}}$, $x \mapsto z_0$

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Exercise

For $f \in \mathrm{Mero}_{2k}(\Gamma)$, $\omega(f) \in \Omega^{\otimes k}(X(\Gamma))$

$\omega(f) = (n_x z)^{-k} h(z)(dz)^k$ where $f = g(z)$ locally, $z^k g(z) = h(z^{n_x})$

- $x \in Y(\Gamma)$, $V_x(\omega(f)) = \dfrac{\mathrm{ord}_x(f) - k(n_x - 1)}{n_x}$

$x = \Gamma\tau$, $n_x = \#\{\text{Stabiliser of } \tau \text{ in } \Gamma/\{\pm I\}\}$

($\pi : \mathbb{H} \to X(\Gamma)$ has local form $z \mapsto z^{n_x}$ from a neighbourhood of τ to a neighbourhood of x).

- $s \in C(\Gamma)$, $V_s(\omega(f)) = \mathrm{ord}_s(f) - k$ $\left(\Leftrightarrow \left(\frac{2\pi i q_h}{h}\right)^{-k} F(q_h)(dq_h)^k\right)$

Corollary

i) $\omega \in \Omega^{\otimes k}(X(\Gamma))$

Just look at previous discussion in terms of $V_x(\omega)$ instead of $\mathrm{ord}_x(f)$

$$f_\omega \in M_{2k}(\Gamma) \Leftrightarrow \begin{array}{ll} V_x(\omega) \geq \frac{-k(n_x-1)}{n_x} & \forall x \in Y(\Gamma) \\ V_s(\omega) \geq -k & \forall s \in C(\Gamma) \end{array}$$

ii) $$f_\omega \in S_{2k}(\Gamma) \Leftrightarrow \begin{array}{ll} V_x(\omega) \geq \frac{-k(n_x-1)}{n_x} & \forall x \in Y(\Gamma) \\ V_s(\omega) \geq 1-k & \forall s \in C(\Gamma) \end{array}$$

Now we can use the Riemann-Roch formula to compute the dimensions of those vector spaces.

Divisors

Definition

X a Riemann Surface

$\mathrm{Div}(X) :=$ Free abelian group generated by points of X

$\ni D = \sum_{x \in X} a_x [x]$ ($a_x \in \mathbb{Z}$, 0 for all but finitely many x)

$\deg(D) = \sum_{x \in X} a_x$.

D is effective, "$D \geq 0$" if $a_x \geq 0$ $\forall x \in X$.

For f a meromorphic function on X, a compact Riemann Surface

non zero

$div(f) = \sum_{x \in X} v_x(f)[x]$ order of vanishing of f at x.

$\omega \in \Omega^{\otimes n}(X)$, $div(\omega) = \sum_{x \in X} v_x(\omega)[x]$

Recall that for f a meromorphic function on a compact Riemann Surface, $\deg(div(f)) = 0$.

Definition

D a divisor on X (compact).

$L(D) := \{f \text{ non-zero meromorphic functions} : div(f) + D \geq 0\} \cup \{0\}$

Remark

a sheaf

$L(D) = \mathcal{O}_X(D)(X)$ where if $U \subset X$ is open

$\mathcal{O}_X(D)(U) = \{f \text{ non-zero, meromorphic on } U : div(f) + D|_U \geq 0\} \cup \{0\}$

$\sum_{x \in U} a_x[x]$

Example

Ω_X^{hol} = sheaf of holomorphic differentials on X.

ω_0 a non-zero meromorphic differential on X. (degree 1)

$\Omega_X^{hol} \simeq \mathcal{O}_X(div(\omega_0))$

$\omega \mapsto \omega/\omega_0$

$f \in \mathrm{Mero}_{2k}(\Gamma)$. $f \in M_{2k}(\Gamma) \iff V_x(\underset{\uparrow}{\omega(f)}) \geq \frac{-k(n_x - 1)}{n_x}$ $\forall x \in Y(\Gamma)$

$\in \Omega^{\otimes k}(X(\Gamma))$

$V_x(\omega(f)) \geq -k \quad \forall x \in C_\Gamma$

Definition

ω_0 a non-zero meromorphic differential of degree 1 on $X(\Gamma)$.

$K = \mathrm{div}(\omega_0) = \sum_{x \in X(\Gamma)} V_x(\omega_0)[x]$

$k \in \mathbb{Z}_{\geq 0}$: $D(k) := k\,\mathrm{div}(\omega_0) + k\sum_{s \in C_\Gamma}[s] + \sum_{x \in Y(\Gamma)} \left\lfloor \frac{k(n_x - 1)}{n_x} \right\rfloor [x]$

$D_c(k) := D(k) - \sum_{s \in C_\Gamma}[s]$

Theorem

The map $f \mapsto \frac{\omega(f)}{\omega_0^k}$ induces isomorphisms: $(\omega_0 = g(z)\,dz)$

$M_{2k}(\Gamma) \xrightarrow{\sim} L(D(k)) = \{0\} \cup \{\text{meromorphic functions } g \text{ on } X(\Gamma) : D(k) + \mathrm{div}(g) \geq 0\}$

$\cup$

$S_{2k}(\Gamma) \xrightarrow{\sim} L(D_c(k))$

orders of vanishing no worse than specified

Proof

Immediate by considering the definition of $D(k)$, $D_c(k)$ and the orders of vanishing of $\omega(f)$.

The inverse map is $g \mapsto f_{\underbrace{g\omega_0^k}_{\in \Omega^{\otimes k}(X(\Gamma))}} \in \mathrm{Mero}_{2k}(\Gamma)$ □

So to compute $\dim M_{2k}$, $\dim S_{2k}$, we just need to compute $\dim L(D(k))$, $\dim L(D_c(k))$

Remark

There does exist a non-zero differential ω_0. This follows from a general result on compact Riemann Surfaces. In this case we have (more simply) that $\underset{\text{non-constant}}{\pi} : X(\Gamma) \to X(\Gamma(1)) \cong \mathbb{P}^1_{\mathbb{C}}$

So we can just pull back a non-zero differential from $\mathbb{P}^1$.

Riemann-Roch

X a (connected) compact Riemann Surface.

ω_0 a non-zero meromorphic differential on X. (degree 1)

$K = \mathrm{div}(\omega_0)$, $D \in \mathrm{div}(X)$

Theorem

$$\dim L(D) - \dim L(K-D) = \deg D + 1 - \underbrace{g(X)}_{\text{genus}}$$

Euler Characteristic Formula for $\mathcal{O}_X(D)$:

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim \underset{H^0(X,\, \mathcal{O}_X(K-D))^*}{\underset{\wr\wr}{H^1(X, \mathcal{O}_X(D))}} = \deg(D) + 1 - g(X)$$

Exercise: $\deg K = 2(g-1)$, $\dim L(K) = g$

Genus of Modular Curves

Theorem (Riemann-Hurwitz)

$\pi : X \to Y$ a map between compact, connected Riemann Surfaces. Then

$$2(1-g(X)) = (\deg \pi)\cdot 2(1-g(Y)) - \sum_{x\in X} (e_x - 1)$$

π has local form $z \mapsto z^{e_x}$ in a neighbourhood of x

Apply this to $\pi : X(\Gamma) \to X(\Gamma(1)) \simeq \mathbb{P}^1$, genus 0.

Definition

$d = \deg(\pi) = [PSL_2(\mathbb{Z}) : \bar{\Gamma}]$

($\bar{\Gamma}$ = image of Γ in $PSL_2(\mathbb{Z})$)

$r_2 = \#\{x \in Y(\Gamma) : n_x = 2\}$ $\quad (n_x = 2 \Rightarrow \pi(x) = \Gamma(1)i)$

$r_3 = \#\{x \in Y(\Gamma) : n_x = 3\}$ $\quad (n_x = 3 \Rightarrow \pi(x) = \Gamma(1)\omega)$

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$r_\infty = \# C_\Gamma = \# \pi^{-1}(\infty)$

<u>Theorem</u>

$$g(X(\Gamma)) = 1 + \frac{d}{12} - \frac{r_2}{4} - \frac{r_3}{3} - \frac{r_\infty}{2}$$

<u>Remark</u>

$$2(g(x)-1) + \frac{r_2}{2} + \frac{2r_3}{3} + r_\infty = d/6 > 0 \quad (*)$$

<u>Proof</u>

$\deg(\pi) = d$. For $x \in Y(\Gamma)$, $e_x = \frac{n_{\pi(x)}}{n_x} = \begin{cases} 1 & \text{if } \pi(x) \neq \Gamma(1)i, \Gamma(1)\omega \text{ or if } x \text{ contributes to } r_2 \text{ or } r_3 \\ n_{\pi(x)} & \text{otherwise} \end{cases}$

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{z \mapsto z^{n_x}} & Y(\Gamma) \\ & \searrow^{z \mapsto z^{n_{\pi(x)}}} & \downarrow \\ & & Y(\Gamma(1)) \end{array} \qquad \begin{array}{c} z \\ \downarrow \\ z^{e_x} \end{array}$$

(For $s \in C_\Gamma$, $e_s = h_s$, the width of the cusp)

$g = g(X(\Gamma))$

$$2 - 2g = 2d - \sum_{\pi(x) = \Gamma(1)i} (e_x - 1) - \sum_{\pi(x) = \Gamma(1)\omega} (e_x - 1) - \sum_{s \in C_\Gamma} (e_s - 1)$$

Suppose that $y = \Gamma(1)i$ or $y = \Gamma(1)\omega$

$$\sum_{\pi(x) = y} (e_x - 1) = (n_y - 1)(\# \pi^{-1}(y) - r_{n_y})$$

$$d = \sum_{\pi(x) = y} e_x = n_y (\# \pi^{-1}(y) - r_{n_y}) + r_{n_y}$$

$$\Rightarrow \sum_{\pi(x) = y} (e_x - 1) = \frac{n_y - 1}{n_y} (d - r_{n_y})$$

Also $\sum_{s \in C_\Gamma} (e_s - 1) = d - r_\infty$ Substitute in □

<u>Corollary</u>

g = genus of $X(\Gamma)$

i) $k \geq 1$, $\dim M_{2k}(\Gamma) = k r_\infty + r_2 \lfloor \frac{k}{2} \rfloor + r_3 \lfloor \frac{2k}{3} \rfloor + (2k-1)(g-1)$

ii) $k \geq 2$, $\dim S_{2k}(\Gamma) = \dim M_{2k}(\Gamma) - r_\infty$

3, iii) $\dim S_2(\Gamma) = g$

Proof

i) $\dim M_{2k}(\Gamma) = \dim \mathcal{L}(D(k))$

$$\deg(D(k)) = 2k(g-1) + kr_\infty + r_2 \lfloor \tfrac{k}{2} \rfloor + r_3 \lfloor \tfrac{2k}{3} \rfloor \quad (*)$$

$$\geq 2k(g-1) + kr_\infty + \tfrac{k-1}{2} r_2 + \tfrac{2k-2}{3} r_3$$

$$(*) \quad = (k-1)(d/6) + 2g - 2 + r_\infty$$

$$(\text{for } k \geq 1) \quad > 2g - 2 = \deg(K)$$

$$\Rightarrow \deg(K - D(k)) < 0 \Rightarrow \dim \mathcal{L}(K - D(k)) = 0$$

Riemann-Roch: $\underset{\dim M_{2k}(\Gamma)}{\underset{\parallel}{\dim \mathcal{L}(D(k))}} = \deg(D(k)) + 1 - g$ □

For ii), the same with $\mathcal{L}(D_c(k))$

~~For (iii), substitute k=1 into ii)~~

Corollary

$$\dim M_{2k}(\Gamma_0(4)) = k+1$$

Proof

$[\Gamma(1) : \Gamma_0(4)] = 6 = d$. We constructed 2 linearly independent elements of $M_2(\Gamma_0(4))$; $E_{2,2}, E_{2,4}$

$$\Rightarrow \underbrace{\dim M_2(\Gamma) - \dim S_2(\Gamma)}_{r_\infty - 1} \geq 2$$

$k=1 \Rightarrow M_2(\Gamma) = r_\infty + g - 1$

So $\underset{= \# \pi^{-1}(\infty)}{r_\infty} \geq 3$. $g = \frac{3}{2} - \frac{r_2}{4} - \frac{r_3}{3} - \frac{r_\infty}{2}$

$\Rightarrow g = 0,\ r_\infty = 3,\ r_2 = r_3 = 0$ since $g, r_i \geq 0$

$\Rightarrow \dim M_{2k}(\Gamma) = k+1$ by ~~induction~~ Riemann-Roch □

We can also get formulae for all $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$.

$$\dim M_{2k}(\Gamma) = \dim \mathcal{L}(D(k))$$

$$= \deg(D(k)) + 1 - g - \dim \mathcal{L}(K - D(k))$$

0 as above

0 by previous proof

$$= k+1$$

(substitute $k, r_\infty = 3, r_2 = r_3 = 0$ into (*))

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Hecke Operators

The idea is to construct a collection of commuting endomorphisms of $M_k(\Gamma_1(N))$. The simultaneous eigenvectors for these operators will give a nice basis for $M_k(\Gamma_1(N))$.

Later, for $f \in S_k(\Gamma_1(N))$, $f = \sum_{n \geq 1} a_n q^n$, we will define $L(f,s) = \sum_{n \geq 1} \frac{a_n}{n^s}$ (convergent for $s \in \mathbb{C}$ with $Re(s) \gg 0$)

$L(f,s)$ has a functional equation and analytic continuation to the whole complex plane.

f is a Hecke eigenvector $\Leftrightarrow L(f,s) = \prod_{p \text{ prime}} \left(\text{local } L\text{-factor at } p\right)^{-1}$

(Hecke eigenvectors $\rightarrow$ representations of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$)

Weight 2 Hecke operators with $a_n \in \mathbb{Q}$ $\rightarrow$ $E/\mathbb{Q}$, an elliptic curve.
$\leftarrow$ "modularity theorem"

Automorphic representations are certain representations of $GL_2(\mathbb{A}_\mathbb{Q}) \circlearrowright$ infinite dimensional $\mathbb{C}$-vector space.

Hecke Eigenvector $\longrightarrow$ Automorphic Representation

Modular Forms and Functions on Lattices

Definition

1. A *lattice* in $\mathbb{C}$ is a $\mathbb{Z}$-module $L \subset \mathbb{C}$, free of rank 2, with generators $w_1, w_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$. (i.e. $L \subset \mathbb{C}$ is a discrete subgroup with $L \otimes_\mathbb{Z} \mathbb{R} = \mathbb{C}$)
2. For $N \in \mathbb{Z}_{\geq 1}$, a *$\Gamma_1(N)$-level structure* on a lattice L is a point $t \in \mathbb{C}/L$ of *exact order N*. We write:
$\mathcal{L}_N = \{ (L,t) \mid L \text{ a lattice}, t \text{ a } \Gamma_1(N) \text{ level structure} \}$

collection of $\Gamma_1(N)$-level structures

3. For $k \in \mathbb{Z}$, $F: \mathcal{L}_N \to \mathbb{C}$ a function, then F has weight k if $F(\lambda L, \lambda t) = \lambda^{-k} F(L, t) \quad \forall \lambda \in \mathbb{C}^*$

For example, $G_k(L) = \sum_{0 \neq l \in L} l^{-k}$ for $k > 2$ even, is a weight k function on $\mathcal{L}_1$.

Observe that if for $\tau \in \mathbb{H}$ we write $L_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$, then $G_k(L_\tau) = G_k(\tau)$.

4. $\mathcal{M} := \{ (\omega_1, \omega_2) \mid \omega_1, \omega_2 \in \mathbb{C}^*, \ \omega_1/\omega_2 \in \mathbb{H} \}$

Observe that there is a surjective map $\mathcal{M} \to \mathcal{L}_N$, $(\omega_1, \omega_2) \mapsto (\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \omega_2/N)$

$\mathcal{M}$ has actions of $GL_2^+(\mathbb{R})$, $\mathbb{C}^*$:
+ve determinant

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$, $\gamma(\omega_1, \omega_2) = (a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)$

$\lambda \in \mathbb{C}^*$, $\lambda(\omega_1, \omega_2) = (\lambda\omega_1, \lambda\omega_2)$

<u>Lemma</u>

i) The map $\mathcal{M} \twoheadrightarrow \mathcal{L}_N$ identifies $\mathcal{L}_N$ with the quotient $\Gamma_1(N) \backslash \mathcal{M}$

ii) The map $\mathcal{M} \twoheadrightarrow \mathbb{H}$, $(\omega_1, \omega_2) \mapsto \omega_1/\omega_2$ identifies $\mathbb{H}$ with the quotient $\mathbb{C}^* \backslash \mathcal{M}$

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "\Gamma_1(N)"'] \arrow[dr, "\mathbb{C}^*"] & \\ \mathcal{L}_N \arrow[dr, "\mathbb{C}^*"'] & & \mathbb{H} \arrow[dl, "\Gamma_1(N)"] \\ & Y(\Gamma_1(N)) & \end{tikzcd}$$

(as sets)

<u>Proof</u>

Exercise. ii) is obvious $\square$

<u>Remark</u>

$\mathcal{M} = \{ E/\mathbb{C}$ elliptic curves with an oriented basis $H_1(E, \mathbb{Z})$ and an invariant differential ω on $E \}$ up to isomorphism

$\mathbb{C}/L \simeq E$

$dz \leftrightarrow \omega$

$\underset{(\omega_1, \omega_2)}{L} \leftrightarrow H_1(E, \mathbb{Z})$

$\omega_1/\omega_2 \in \mathbb{H}$.

$\mathbb{H} = \{$elliptic curves $E/\mathbb{C}$ with an oriented basis for $H_1(E, \mathbb{Z})\}$

$\mathcal{L}_N = \{E/\mathbb{C}$ with a point of exact order N and an invariant differential ω on $E\}$

$Y(\Gamma_1(N)) = \{E/\mathbb{C}$ with a point of exact order $N\}$

Definition

F a function $L_N \to \mathbb{C}$.

Define $\tilde{F} : \mathcal{M} \to \mathbb{C}$ by $\tilde{F}(\omega_1, \omega_2) = F(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2, \omega_2/N)$

Define $f : \mathbb{H} \to \mathbb{C}$ by $f(\tau) = \tilde{F}(\tau, 1)$

Lemma

The above definition identifies:

i) Functions $F : L_N \to \mathbb{C}$ of weight k.

ii) $\Gamma_1(N)$-invariant functions $\tilde{F} : \mathcal{M} \to \mathbb{C}$ of weight k

(i.e. $\tilde{F}(\lambda\omega_1, \lambda\omega_2) = \lambda^{-k} F(\omega_1, \omega_2)$;

for $\gamma \in \Gamma_1(N)$, $\tilde{F}(\gamma(\omega_1, \omega_2)) = \tilde{F}(\omega_1, \omega_2)$)

iii) Functions $f : \mathbb{H} \to \mathbb{C}$ (just set-theoretic functions!) invariant under $|_{\gamma,k}$ for $\gamma \in \Gamma_1(N)$.

i.e. not necessarily holomorphic

$f|_{\gamma,k}(\tau) = (c\tau+d)^{-k} f(\gamma\tau)$

Proof

$(i) \leftrightarrow (ii)$ by the previous lemma

~~Exercise~~ $(ii) \leftrightarrow (iii)$ ii) to iii) is clear.
iii) to (ii) follows since $f(\tau)$ specifies $\tilde{F}$ uniquely □

Now we say that a weight k function on L_N is weakly modular / a meromorphic form / a modular form / a cusp form if the associated $f : \mathbb{H} \to \mathbb{C}$ is.

Remark

We can interpret holomorphic, weakly modular functions of weight k and level $\Gamma_1(N)$ as holomorphic sections of a line bundle on $Y(\Gamma_1(N))$ (so long as there are no elliptic points)
ok for $N \geq 4$

Operators

Definition

1. F a function $\mathcal{L}_N \to \mathbb{C}$, $n \in \mathbb{Z}_{\geq 1}$. Define $T_n F$, a function $\mathcal{L}_N \to \mathbb{C}$ by $T_n F = \frac{1}{n} \sum_{L'} F(L', t \bmod L')$
where the sum is over lattices L' with $L' \supset L$, and $\# L'/L = n$, with $t \bmod L'$ having exact order N in $\mathbb{C}/L'$.
($L' \supset L$, $\mathbb{C}/L \overset{\pi}{\twoheadrightarrow} \mathbb{C}/L'$ an isogeny of degree n such that $\pi(t)$ has exact order N, i.e. $\ker(\pi) \cap \langle t \rangle = \{0\}$)
2. For n, N coprime, define
$T_{n,n} F := \frac{1}{n^2} F(\frac{1}{n}L, t \bmod \frac{1}{n}L)$
n coprime to $N \Rightarrow t \bmod \frac{1}{n}L$ has exact order N in $\frac{1}{n}L$

Proposition

We have identities (as endomorphisms of {functions $\mathcal{L}_N \to \mathbb{C}$})

i) m, n coprime, $T_m \circ T_n = T_{mn}$

ii) $p \mid N$, p prime, $r \geq 1$, $T_{p^r} = \underbrace{T_p \circ \dots \circ T_p}_{r \text{ times}} = (T_p)^r$

iii) $p \nmid N$, p prime, $r \geq 1$, $T_{p^r} \circ T_p = T_{p^{r+1}} + p\, T_{p^{r-1}} \circ T_{p,p}$

iv) $T_n \circ T_{m,m} = T_{m,m} \circ T_n$

v) $T_{n,N} \circ T_{m,M} = T_{mn,MN}$

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Proof (from last time)

v) is immediate

iv) works because T_n, $T_{n,N}$ are linear

iv), v) are easy to check.

i) $$(T_m \circ T_n) F(L,t) = \frac{1}{mn} \sum_{\substack{L' \supset L \\ \text{index } m \\ t \bmod L' \text{ has} \\ \text{exact order } N}} \sum_{\substack{L'' \supset L' \\ \text{index } n \\ t \bmod L'' \\ \text{has exact order } N}} F(L'', t \bmod L'')$$

$$T_{mn} F(L,t) = \frac{1}{mn} \sum_{\substack{L'' \supset L \\ \text{index } mn \\ t \bmod L'' \text{ order } N}}$$

So we just need to show that if $L'' \supset L$ with index mn, there is a unique lattice L' with $L'' \supset L' \supset L$ with $[L' : L] = m$,

$[L'':L] = nm$, m, n coprime $\Rightarrow$ L''/L has a unique subgroup of order $[\cancel{L'':L'}] = n$

$L' \subset L''$ is the preimage of this subgroup.

ii) It is enough to show that $T_{p^{r-1}} \circ T_p = T_{p^r}$ for $r \geq 2$.

T_{p^r} is given by a sum over $L'' \supset L$ with index p^r, such that

$(t \bmod L'' \text{ has order } N) \Leftrightarrow (t' = \frac{N}{p} t,\ L''/L \text{ does not contain } t')$

Claim:

$t' \notin L''/L \Rightarrow L''/L$ cyclic (order p^r)

Proof of Claim:

If L''/L is not cyclic, it must contain $\frac{1}{p}L/L \ni t'$.

L''/L cyclic $\Rightarrow \exists! L'$ with $L \subset L' \subset L''$ with

$[L' : L] = p^{r-1}$, $[L'' : L'] = p$.

iii) $T_{p^r} \circ T_p$, $T_{p^{r+1}}$, $T_{p^{r-1}} \circ T_{p,p}$ are all given by

$\frac{1}{p^{r+1}} \times$ sum over $L'' \supset L$ with $[L'' : L] = p^{r+1}$

$$(\ \cdot\ ) F(L,t) = \frac{1}{p^{r+1}} \sum_{L''} a_{L''} F(L'', t)$$

Denote each $a_{L''}$ for the three operators by a, b, c respectively. We want to show that $a = b + pc$.

It is clear that $b = 1$.

For $T_{p^{r-1}} \circ T_{p,p}$, we get two cases:

$$\cancel{a)\ L'' \not\supset \tfrac{1}{p}L}\quad T_{p^{r-1}} \circ T_{p,p} F(L,t) = T_{p^{r-1}}\left((L,t) \mapsto \tfrac{1}{p^2} F(\tfrac{1}{p}L, t)\right)$$

$$= \frac{1}{p^{r+1}} \sum_{\substack{L'' \supset \frac{1}{p}L \\ \text{index } p^{r-1}}} F(L'', t)$$

$$\text{So } c = \begin{cases} 0 & L'' \not\supset \frac{1}{p}L \\ 1 & L'' \supset \frac{1}{p}L \end{cases}$$

a) $c = 0$, $L'' \not\supset \frac{1}{p}L$. Now if $L' \subset L''$ with $[L' : L] = p$
$\Rightarrow L' = L'' \cap \frac{1}{p}L \subsetneqq \frac{1}{p}L$. So $a = 1$, $a = b + pc$

b) $c = 1$, $L'' \supset \frac{1}{p}L$. Now $L' \subset L''$ with $[L' : L] = p$ correspond to lines in the two dimensional $\mathbb{F}_p$-vector space $\frac{1}{p}L / L$, so there are $p+1$ possibilities for L'.

$\Rightarrow a = p+1$, and $a = b + pc$ $\square$

Consequence:

T_p, $T_{p,p}$, $(p \nmid N)$ generate a commutative sub-$\mathbb{C}$-algebra of $\mathrm{End}_{\mathbb{C}}(\text{Functions } \mathcal{L}_N \to \mathbb{C})$, which contain all of the T_n and $T_{n,n}$.

<u>Remark</u>

$$\sum_{n=0}^{\infty} T_{p^n} X = \begin{cases} \dfrac{1}{1 - T_p X} & p \mid N \\[2ex] \dfrac{1}{1 - T_p X + p T_{p,p} X^2} & p \nmid N \end{cases}$$

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Definition

$d \in \mathbb{Z}$, $(d, N) = 1$. We have the "diamond operator":

$\langle d \rangle F(L, t) = F(L, dt)$

This only depends on $\bar{d} \in (\mathbb{Z}/N\mathbb{Z})^*$

Lemma

If F has weight k (i.e. $F(\lambda L, \lambda t) = \lambda^{-k} F(L, t)$ for $\lambda \in \mathbb{C}^*$)

then $T_n F$, $T_{n,N} F$, $\langle d \rangle F$ have weight k.

Proof

Exercise Easy to see as T_n, $T_{n,n}$, $\langle d \rangle$ are linear □

Exercise $t' = t \bmod \frac{1}{n}L$, $nt' \equiv nt \bmod L$ $\frac{1}{n^2} F(\frac{1}{n}L, t') = n^{k-2} F(L, nt') = n^{k-2} F(L, nt) = n^{k-2} \langle n \rangle F$

F has weight $k \Rightarrow T_{n,N} F = n^{k-2} \langle n \rangle F$ $(*)$

Weight k functions on $\mathcal{L}_N \longrightarrow$ functions on $\mathbb{H}$, invariant under $|_{\gamma,k}$ for $\gamma \in \Gamma_1(N)$.

We need to explicate how T_n etc. act on

{functions on $\mathbb{H}$, invariant under $|_{\gamma,k}$ } in order to show that they

stabilise the subspace $M_k(\Gamma_1(N))$.

Matrix Version of Hecke Operators

Definition

$n \in \mathbb{Z}_{\geq 1}$, $S_n^N = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; \begin{matrix} ad = n & (a, N) = 1 \\ 0 \leq b < d & \end{matrix} \right\}$

Lemma

$(L, t) \in \mathcal{L}_N$. Write $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\omega_1/\omega_2 \in \mathbb{H}$ and

$t = \omega_2/N$. For $\sigma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in S_n^N$ define

$L_\sigma :=$ lattice spanned by $\frac{a}{n}\omega_1 + \frac{b}{n}\omega_2$, $\frac{d}{n}\omega_2$

so that $L_\sigma = \frac{1}{n}\sigma L$.

Then $\sigma \mapsto L_\sigma$ is a bijection.

$S_n^N \xrightarrow{\sim} \{\text{lattices } L' \supset L \text{ with index } n \mid t \bmod L' \text{ has order } N\}$

Proof

Exercise. - See Sheet 3 $\square$

Proposition

T_n, $T_{n,N}$, $\langle d \rangle$ preserve $M_k(\Gamma_1(N))$, $S_k(\Gamma_1(N))$

(i.e. they ~~act on~~ preserve holomorphic and holomorphic-at-the-cusps functions $\mathbb{H} \to \mathbb{C}$).

Proof

because $T_{n,N}F = n^{k-2}\langle n \rangle F$

(*): We just need to check ~~that~~ T_n, $\langle n \rangle$.

Suppose $(n,N)=1$. Let $\sigma_n \in SL_2(\mathbb{Z})$ lift $\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix} \in SL_2(\mathbb{Z}/N\mathbb{Z})$ See sheet 1

Suppose $f \in M_k(\Gamma_1(N)) \leftrightarrow F : \mathcal{L}_N \to \mathbb{C}$.

$$f|_{\sigma_n,k}(\tau) = (c\tau+d)^{-k} F(\mathbb{Z}(\sigma_n\tau) + \mathbb{Z}, \tfrac{1}{N})$$

$\sigma_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= F(\mathbb{Z}(a\tau+b) + \mathbb{Z}(c\tau+d), \tfrac{n}{N})$$

$$= F(\mathbb{Z}\tau + \mathbb{Z}, \tfrac{n}{N})$$

$\sigma_n \in SL_2(\mathbb{Z})$

$$= (\langle n \rangle F)(\mathbb{Z}\tau + \mathbb{Z}, \tfrac{1}{N})$$

$$= (\langle n \rangle f)(\tau).$$

$F : \mathcal{L}_N \to \mathbb{C}$

$f(\tau) = F(\mathbb{Z}\tau \oplus \mathbb{Z}, \frac{1}{N})$

should be $\frac{c\tau+d}{N}$

but $\overline{\sigma_n} = \begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix} \in SL_2(\mathbb{Z}/N\mathbb{Z})$

so $c \equiv 0\ (N)$, $d \equiv n\ (N)$

$\frac{c\tau+d}{N} = \frac{n}{N} \bmod (\mathbb{Z}\tau \oplus \mathbb{Z})$

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Proof (continued)

$\tau \in \mathbb{H}$, $L_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$

$f \in M_k(\Gamma_1(N))$, $f \longleftrightarrow F : \mathcal{L}_N \to \mathbb{C}$

$f(\tau) = F(L_\tau, \frac{1}{N})$

$L_{\tau,\sigma} = \frac{1}{n}\sigma L_\tau$, $\sigma \in S_n^N$

By definition, $T_n f : \mathbb{H} \to \mathbb{C}$ given by

$$T_n(f)(\tau) = \frac{1}{n}\sum_{\substack{L' \supset L_\tau \\ \text{index } n}} F(L', \tfrac{1}{N}) = \frac{1}{n}\sum_{\sigma \in S_n^N} F(L_{\tau,\sigma}, \tfrac{1}{N})$$

$$S_n^N = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| (a,N) = 1, ad = 1, 0 \le b < d \right\}$$

$$\sigma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad L_{\tau,\sigma} = \mathbb{Z}\frac{a\tau+b}{n} \oplus \mathbb{Z}\frac{d}{n}$$

$$(L_{\tau,\sigma}, \tfrac{1}{N}) = \frac{1}{a}\left(\mathbb{Z}\frac{a\tau+b}{d} \oplus \mathbb{Z}, \frac{a}{N}\right)$$

so $\frac{1}{n}F(L_{\tau,\sigma}, \frac{1}{N}) = \frac{1}{n}a^k F\left(L_{\frac{a\tau+b}{d}}, \frac{a}{N}\right)$ because F has weight k

$$= \frac{1}{n}a^k (\langle a\rangle f)\left(\frac{a\tau+b}{d}\right)$$

$$= \frac{1}{n}a^k d^k (\langle a\rangle f)\big|_{\sigma,k} = n^{k-1}(\langle a\rangle f)\big|_{\sigma,k}$$

$\left(g : \mathbb{H} \to \mathbb{C},\ g|_{\sigma,k} = d^{-k} g\left(\frac{a\tau+b}{d}\right)\right)$

So $T_n f = \sum_{\sigma \in S_n^N} n^{k-1}(\langle a\rangle f)\big|_{\sigma,k}$

From here, it is easy to check that $f \in M_k(\Gamma_1(N))$ or $S_k(\Gamma_1(N))$ $\Rightarrow T_n f \in M_k(\Gamma_1(N))$ or $S_k(\Gamma_1(N))$. $\square$

Remark

1. For $d \in (\mathbb{Z}/N\mathbb{Z})^*$ we have $\langle d\rangle$ acting on $M_k(\Gamma_1(N))$ giving a representation of $(\mathbb{Z}/N\mathbb{Z})^*$.

$$\Rightarrow M_k(\Gamma_1(N)) = \bigoplus_{\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*} M_k(\Gamma_1(N), \chi)$$

where $M_k(\Gamma_1(N), \chi) = \{ f \in M_k(\Gamma_1(N)) \mid \langle d\rangle f = \chi(d) f \text{ for } d \in (\mathbb{Z}/N\mathbb{Z})^* \}$

f has "character χ".

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2. $\Gamma_1(N)\backslash\Gamma_0(N) \simeq (\mathbb{Z}/N\mathbb{Z})^*$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d$

$\Gamma_0(N)$ acts on $M_k(\Gamma_1(N))$ by $f \mapsto f|_{\gamma,k}$

This action is given by $\langle d \rangle$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

In particular, $M_k(\Gamma_1(N), \text{trivial}) = M_k(\Gamma_0(N))$

<u>Notation</u>

$M_k(N, \chi) = M_k(\Gamma_1(N), \chi)$

How does the q-expansion of $T_n f$ relate to that of f?

<u>Proposition</u>

$f \in M_k(N, \chi)$, $f(\tau) = \sum_{n \geq 0} a_n q^n$

p prime, $T_p f = \sum_{n \geq 0} b_n q^n$.

Then $b_n = a_{np} + \chi(p) p^{k-1} a_{n/p}$

<u>Aside</u>

$\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ extends to $\chi : \mathbb{Z} \to \mathbb{C}$ by setting

$$\chi(m) = \begin{cases} 0 & (m, N) \neq 1 \\ \chi(m \bmod N) & \text{otherwise} \end{cases}$$

$$a_{n/p} = \begin{cases} a_{n/p} & p \mid n \\ 0 & \text{otherwise} \end{cases}$$

<u>Proof</u>

What is S_p^N? If $p \mid N$ then $S_p^N = \left\{ \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \;\middle|\; 0 \leq b \leq p-1 \right\}$

If $p \nmid N$ then $S_p^N = \left\{ \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \;\middle|\; 0 \leq b \leq p-1 \right\} \cup \left\{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\}$

If $p \mid N$, then

$T_p f(\tau) = \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{\tau + b}{p}\right)$

So $b_n = \frac{1}{p} \sum_{b=0}^{p-1} a_{np} e^{2\pi i b n} = a_{np}$

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If $p \nmid N$, we get $T_p f(\tau) = \left(\sum a_{np} q^n\right) + \chi(p) p^{k-1} \underbrace{f(p\tau)}_{= \sum_{n \geq 0} a_{n/p} q^n}$ □

Lets consider operators on $\mathbb{C}[[q]]$.

Define (for $m \geq 1$) $u_m : \sum_{n \geq 0} a_n q^n \mapsto \sum_{n \geq 0} a_n q^{n/m}$,

0 if $m \nmid n$ (convention)

$V_m : \sum_{n \geq 0} a_n q^n \mapsto \sum_{n \geq 0} a_n q^{mn}$

The above proposition says that $T_p = U_p + \chi(p) p^{k-1} V_p$.

It is easy to check that $U_m \circ V_m = id$

However, $V_m \circ U_m : \sum_{n \geq 0} a_n q^n \mapsto \sum_{\substack{n \geq 0 \\ m | n}} a_n q^n$

We have the identity:

$$(1 - T_p X + \chi(p) p^{k-1} X^2) = (1 - U_p X)(1 - \chi(p) p^{k-1} V_p X)$$

(in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[[q]])[X]$)

<u>Proposition</u>

$$T_n = \underbrace{\sum_{0 < d | n} \chi(d) d^{k-1} V_d \circ U_{n/d}}_{\widetilde{T}_n} \quad \text{(equality in } \mathrm{End}_{\mathbb{C}} \mathbb{C}[[q]])$$

(N.B. T_n is in the algebra generated by T_p, $\langle p \rangle$ for $p | n$)

<u>Proof</u>

$(a, b) = 1$, $a, b \geq 1$. We know that $T_{ab} = T_a \circ T_b$.

We can check that $\widetilde{T}_{ab} = \widetilde{T}_a \circ \widetilde{T}_b$.

(It follows from $U_a \circ V_b = V_b \circ U_a$ for coprime (a, b)).

$\Rightarrow$ it suffices to prove that $T_{p^r} = \widetilde{T}_{p^r}$ for prime powers.

This is left as an exercise.

See Sheet 3

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Alternatively, we have a formal identity

$$\sum_{n=1}^{\infty} T_n n^{-s} \underset{\uparrow}{=} \prod_{p \mid N} \frac{1}{1 - T_p p^{-s}} \prod_{p \nmid N} \frac{1}{1 - T_p p^{-s} + T_{p,p} p^{1-2s}}$$

relations for $T_{p^{r+1}} = \ldots$

$$= \prod_p (1 - \chi(p) p^{k-1} V_p p^{-s})^{-1} (1 - u_p p^{-s})^{-1}$$

Exercise : Verify these formal identities and show that we can obtain our proposition, and that these identities make sense in some ring. $\square$

Corollary

$f \in M_k(N, \chi)$, $f(\tau) = \sum_{n \geq 0} a_n q^n$, $T_m f(\tau) = \sum_{n \geq 0} b_n q^n$

$$b_n = \sum_{0 < d \mid (m,n)} \chi(d) d^{k-1} a_{\frac{mn}{d^2}}$$

Proof

Follows from the proposition. See Sheet 3 $\square$

Definition

i) $f \in M_k(N, \chi)$ is an eigenform if $T_n f = \lambda_n f$ for some $\lambda_n \in \mathbb{C}$, $\forall n \geq 1$.

ii) f is normalised if $a_1(f) = 1$.

Lemma

Let f be a non-constant eigenform. Then $a_1(f) \neq 0$,

$$\lambda_n = \frac{a_n(f)}{a_1(f)}$$

Moreover, if $a_0(f) \neq 0$, then $\overset{\lambda_n = a_n(f)}{\frac{a_n(f)}{a_0(f)}} = \sum_{0 < d \mid n} \chi(d) d^{k-1}$ for all $n \geq 1$.

Proof next time

Lemma

Suppose that $f \in M_k(N, \chi)$, (χ a character of $(\mathbb{Z}/N\mathbb{Z})^*$) is a non-constant eigenform: for $n \geq 1$, $T_n f = \lambda_n f$

Let $f = \sum_{n \geq 0} a_n q^n$. Then, $a_1 \neq 0 \Rightarrow \lambda_n = \frac{a_n}{a_1}$.

If $a_0 \neq 0$, then $\lambda_n = \sum_{0<d|n} \chi(d) d^{k-1}$

Proof

Look at the formula for the q-expansion of $T_n f$:

$$\underset{\substack{\| \\ \lambda_n a_1}}{a_1(T_n f)} = a_n$$

So $a_1 = 0 \Rightarrow a_n = 0$ for all $n \geq 1 \Rightarrow f$ constant.

Otherwise, $\lambda_n = \frac{a_n}{a_1}$.

The final statement follows from

$$a_0(T_n f) = \sum_{0<d|n} \chi(d) d^{k-1} a_0. \qquad \square$$

Examples

1. $E_k(\tau)$ are eigenforms for $k \geq 3$ (k even)

$$\lambda_n = \sum_{d|n} d^{k-1} = \sigma_{k-1}(n)$$

More generally, we can write down Eisenstein series $E_k^\chi \in M_k(\Gamma_1(N), \chi)$ such that $\lambda_n = \sum_{d|n} \chi(d) d^{k-1}$ if $\chi(-1) = (-1)^k$, and $k \geq 2$, with $k > 2$ if χ is trivial.

Remark

$f \in M_k(\Gamma_1(N), \chi)$, $f \neq 0$. Then $\chi(-1) = (-1)^k$ (exercise)

In particular, if χ is trivial, $M_k(N, \text{trivial}) = M_k(\Gamma_0(N))$ $\ni I$

So $M_k(\Gamma_0(N)) \neq 0 \Rightarrow k$ even.

2. $\Delta(\tau) \in S_{12}(\Gamma(1))$ is a Hecke eigenform.

$\left(\dim S_{12}(\Gamma(1)) = 1\right) \longrightarrow$ because $\Delta(\tau)$ generates all $S_{12}(\Gamma(1))$ and $T_n \Delta(\tau) \in S_{12}(\Gamma(1))$

3. $\Theta(\tau)^2 = \sum_{n \geq 0} r_2(n) q^n \in M_1(\Gamma_1(4), \chi)$

$\chi : (\mathbb{Z}/4\mathbb{Z})^* \cong \{\pm 1\}$, "$\chi(-1) = (-1)^k$"

$\Rightarrow M_1(\Gamma_1(4)) = M_1(\Gamma_1(4), \chi)$. χ the non-trivial character of $(\mathbb{Z}/4\mathbb{Z})$

Exercise : $\dim M_1(\Gamma_1(4), \chi) = 1$.

So $f = \Theta(\tau)^2$ is a Hecke eigenform.

$a_0(f) = 1$, $a_1(f) = 4$. So the Hecke eigenvalues are

$$\lambda_n = \sum_{d \mid n} \chi(d) d^{k-1} = \sum_{\substack{d \mid n \\ \text{odd}}} \left(\frac{-1}{d}\right) d^{k-1}$$

$$\text{So } a_n(f) = 4 \sum_{\substack{d \mid n \\ \text{odd}}} \left(\frac{-1}{d}\right) d^{k-1} \Longleftarrow k = 1$$

So p an odd prime $\Rightarrow a_p(f) = 4\left(1 + \left(\frac{-1}{p}\right)\right)$

$$\text{So } r_2(p) = \begin{cases} 0 & p \equiv 3 \ (4) \\ 8 & p \equiv 1 \ (4) \end{cases}$$

$\lambda_p = 1 + \left(\frac{-1}{p}\right)$ for odd primes p.

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \arrow[r] & \mathbb{C}^* \\ (\mathbb{Z}/4\mathbb{Z})^* \arrow[u, phantom, "\cong"] \arrow[ur, "\chi"'] & \end{tikzcd}$$

acted on by $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$

$\lambda_p =$ trace of Frob_p on $(\mathrm{triv}) \oplus \tilde{\chi}$

Theorem (Deligne, Serre)

$f \in M_1(N, \chi)$ an eigenform, $T_n f = \lambda_n f$.

Then $\exists$ a numberfield $K/\mathbb{Q}$ and a representation

$\rho : \mathrm{Gal}(K/\mathbb{Q}) \hookrightarrow GL_2(\mathbb{C})$ such that

for $p \nmid N$, p is unramified in K and

$\rho(\mathrm{Frob}_p)$ $(p \nmid N)$ has characteristic polynomial

$X^2 - \lambda_p X + \chi(p)$.

$(\chi(-1) = (-1)^k \Rightarrow (\det \rho)(\text{complex conj.}) = -1$ "ρ is odd" $)$

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Proposition

$f \in M_k(N, \chi)$. f is a normalised eigenform if and only if:

i) $a_1(f) = 1$

ii) $a_{mn}(f) = a_m(f) a_n(f)$ for m, n coprime.

$(T_{mn} = T_m \circ T_n)$

iii) p prime, $r \geq 1$: $a_{p^r}(f) a_p(f) = a_{p^{r+1}}(f) + \chi(p) p^{k-1} a_{p^{r-1}}(f)$

(the same as for $T_{p^r} \circ T_p = \dots$)

Proof

If f is a normalised eigenform, then i) is obvious, and ii), iii) follow from the corresponding relations between the Hecke operators.

Assume that i), ii), iii) hold.

It is enough to show that f is an eigenvector for each T_p with p prime (the T_p generate the T_n).

If $p \nmid n$, $a_n(T_p f) = a_{np}(f) \overset{(ii)}{=} a_p(f) a_n(f)$

If $p \mid n$, $n = p^r m$, p, m coprime, then

$$a_n(T_p f) = \underset{p^{r+1}m}{a_{np}}(f) + \chi(p) p^{k-1} a_{n/p}(f) \qquad \frac{n}{p} = p^{r-1}m$$

formula for T_p on q-expansions.

$$a_n(T_p f) = a_m(f) \left(a_{p^{r+1}}(f) + \chi(p) p^{k-1} a_{p^{r-1}}(f) \right) \quad \text{by ii)}$$

$$= a_m(f) \left(a_{p^r}(f) a_p(f) \right) \quad \text{by iii)}$$

$$= a_p(f) a_n(f)$$

So $a_n(T_p f) = a_p(f) a_n(f)$ for all n and primes p.

Assume that $a_0(f) = 0$. Then $T_p f = a_p(f) f$.

We want to show that $M_k(\Gamma(1))$ has a basis of Hecke eigenforms. We will define an inner-product on $S_k(N, \chi)$ with respect to which T_n is essentially self adjoint (at least for n coprime to N).

Petersson Inner Product

We define a measure on $\mathbb{H}$, $\tau = x+iy$, $d\mu(\tau) = \frac{dx\,dy}{y^2}$

$d\mu(\tau)$ is invariant under the action of $SL_2(\mathbb{Z})$.

For $f, g \in M_k(\Gamma)$, we define

$$\langle f, g \rangle = \underbrace{(\ldots)}_{\substack{\text{normalisation}\\ \text{constant}}} \int_{\underbrace{\mathcal{F}(\Gamma)}_{\substack{\text{fundamental}\\ \text{domain for } \Gamma}}} f(\tau)\overline{g(\tau)}(\operatorname{Im}\tau)^k \, d\mu(\tau)$$

really the integral over $Y(\Gamma)$ or $X(\Gamma)$.

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We can integrate continuous functions on $Y(\Gamma)$ or $X(\Gamma)$ with respect to our measure $\frac{dx\,dy}{y^2} = d\mu(\tau)$ on $\mathbb{H}$.

Definition

Let f be continuous on $Y(\Gamma)$. Then

$$\int_{Y(\Gamma)} f\,d\mu = \int_{F(\Gamma)} f(\Gamma\tau)\,d\mu(\tau)$$ with $F(\Gamma)$ a fundamental domain for Γ. For simplicity, we restrict to

$$F(\Gamma) = \coprod_{\alpha \in \bar{\Gamma} \backslash PSL_2(\mathbb{Z})} \alpha F(1)$$

$\uparrow$ Standard fundamental domain for $SL_2(\mathbb{Z})$

Remark

This is well defined and independent of the choice of $F(\Gamma)$.

Lemma

$$\mu(\Gamma) := \int_{Y(\Gamma)} 1\,d\mu < \infty$$

and $$\mu(\Gamma)/\mu(\Gamma(1)) = [PSL_2(\mathbb{Z}) : \bar{\Gamma}]$$

use this

Proof

First, check the second part, then show explicitly that

$\mu(\Gamma(1)) < \infty$ Easy $\left|\int_{\mathcal{F}} \frac{dx\,dy}{y^2}\right| \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{\frac{\sqrt{3}}{2}}^{\infty} \frac{dy}{y^2}$ $\square$

Definition

$f, g \in M_k(\Gamma)$, with one of f, g a cusp form, define

$$\langle f, g\rangle_\Gamma := \frac{\mu(\Gamma(1))}{\mu(\Gamma)} \int_{Y(\Gamma)} \underbrace{f(\tau)\,\overline{g(\tau)}\,(\mathrm{Im}\,\tau)^k}_{\Gamma\text{-invariant}}\,d\mu$$

We will show that $\langle f, g\rangle_\Gamma$ converges absolutely. This needs the assumption that one of f, g is a cusp form.

Lemma

Let $f \in S_k(\Gamma)$. Then $|f(x+iy)| \leq \frac{c}{y^{k/2}}$, $x, y \in \mathbb{R}$, $y > 0$

c independent of x, y.

Proof

Just need to check Γ-invariance. $\gamma \in \Gamma$. Use $f(\gamma\tau) = (c\tau+d)^k f(\tau)$, $\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$

Define $\phi(x+iy) = |f(x+iy)|\, y^{k/2}$

Check that ϕ descends to a continuous function on $Y(\Gamma)$.

$q = e^{2\pi i\tau} = e^{2\pi i x - 2\pi y}$, $y^{k/2} q \to 0$ as $y \to \infty$, $\phi(i\infty) := 0$ $\Rightarrow$ finite continuous at each cusp as $SL_2(\mathbb{Z})$ transitive on C_Γ

Then check that this extends to a continuous function of $X(\Gamma)$.

$X(\Gamma)$ compact $\Rightarrow$ ϕ bounded $\square$

Corollary

If $f \in S_k(\Gamma)$, with $f(\tau) = \sum_{n \geq 1} a_n q_h^n$, then

$|a_n| \leq c n^{k/2}$ "the Trivial Bound"

Proof

Exercise. Sheet 3

Remark

A Theorem of Deligne: If $f \in S_k(N, \chi)$ then for n coprime to N, $\forall \varepsilon > 0$, $|a_n| \leq C_\varepsilon n^{\frac{k-1}{2} + \varepsilon}$

This follows from the Weil Conjectures.

Corollary 2

$\langle f, g \rangle_\Gamma$ converges absolutely (if one of $f, g \in S_k(\Gamma)$).

Proof

product of f and g. NOT composition

$|f(\tau)\overline{g(\tau)}| = |fg(\tau)|$, $fg \in S_k(\Gamma)$

$\leq \frac{c}{(\mathrm{Im}\,\tau)^k}$ $\square$

Reason for $\mu(\Gamma(1))/\mu(\Gamma)$ normalising factor

Exercise: If $\Gamma' \subset \Gamma$, $f, g \in M_k(\Gamma)$, can also think of $f, g \in M_k(\Gamma')$. Check $\langle f, g \rangle_\Gamma = \langle f, g \rangle_{\Gamma'}$

Therefore we can just write $\langle f, g \rangle$ for $\langle f, g \rangle_\Gamma$.

Proposition

$f, g \in M_k(N, \chi)$. Suppose n, N are coprime.

Then $\langle T_n f, g \rangle = \chi(n) \langle f, T_n g \rangle$

First we need two lemmas.

Lemma 1

If $\alpha \in GL_2^+(\mathbb{Q})$, $\Gamma \subset SL_2(\mathbb{Z})$ a congruence subgroup, then $\underbrace{(\alpha \Gamma \alpha^{-1}) \cap SL_2(\mathbb{Z})}_{= G} = $ (a congruence subgroup)

Proof Must show $\Gamma(N) \subseteq G$, some N

Exercise. $\square$

Remark

If $\alpha \in GL_2^+(\mathbb{Q})$, $f \in M_k(\Gamma)$, define

$f|_{\alpha,k}(\tau) = \frac{1}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right)$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

This is not always the convention used in the literature.

$f|_{\alpha,k} \in M_k((\alpha^{-1}\Gamma\alpha) \cap SL_2(\mathbb{Z}))$

Lemma 2

For $\alpha \in GL_2^+(\mathbb{Q})$, $f, g \in M_k(\Gamma)$.

$\langle f, g \rangle = (\det \alpha)^k \langle f|_{\alpha,k}, g|_{\alpha,k} \rangle$

Proof

$\Gamma' = \Gamma \cap (\alpha \Gamma \alpha^{-1})$. $f, g \in M_k(\Gamma')$.

$f|_{\alpha,k}, g|_{\alpha,k} \in M_k(\alpha^{-1}\Gamma'\alpha)$

If $\mathcal{F}(\Gamma')$ is a fundamental domain for Γ', then

$\alpha^{-1}\mathcal{F}(\Gamma')$ is a fundamental domain for $\alpha^{-1}\Gamma'\alpha$.

We compute the right hand side as

$$(\det\alpha)^k \frac{1}{\underset{[PSL_2(\mathbb{Z}):\overline{\Gamma'}]}{\overset{\shortparallel}{[PSL_2(\mathbb{Z}):\overline{\alpha^{-1}\Gamma'\alpha}]}}} \int_{\alpha^{-1}\mathcal{F}(\Gamma')} f|_{\alpha,k}\ \overline{g|_{\alpha,k}}\ (\operatorname{Im}\tau)^k\, d\mu(\tau)$$

Substitute τ by $\alpha^{-1}\tau$. Then the integral becomes

$$\int_{\mathcal{F}(\Gamma')} f(\tau)\,\overline{g(\tau)} \left(\frac{\operatorname{Im}\tau}{\det\alpha}\right)^k d\mu(\tau)$$

This gives RHS $= \langle f, g\rangle$ $\square$

<u>Proof of Proposition</u>

Recall $S_n^N \ni \sigma = \begin{pmatrix} a & b \\ 0 & d\end{pmatrix}$

$$T_n f = n^{k-1}\sum_{\sigma\in S_n^N} \chi(a)\, f|_{\sigma,k}$$

So $\langle T_n f, g\rangle = n^{k-1}\sum_{\sigma\in S_n^N} \chi(a) \langle f|_{\sigma,k}, g\rangle$

(by Lemma 2) $$= \frac{1}{n}\sum_{\sigma\in S_n^N} \underset{\substack{\uparrow \\ \text{a root of unity}}}{\chi(a)} \langle f, g|_{\sigma^{-1},k}\rangle$$

$$= \frac{1}{n}\sum \langle f, \overline{\chi(a)}^{-1} g|_{\sigma^{-1},k}\rangle$$

$$= \frac{\chi(n)}{n}\sum \langle f, \chi(d)\, f|_{\sigma^{-1},k}\rangle$$

$$= \chi(n)\, n^{k-1} \langle f, \underbrace{n^{k-1}\sum_{\sigma\in S_n^N} \chi(d)\, g|_{\begin{pmatrix} d & -b \\ 0 & a\end{pmatrix},k}}_{= T_n g}\rangle$$ $\square$

If we restrict $\langle\cdot,\cdot\rangle$ to $S_k(N,\chi)$, we get a Hermitian inner product. If we fix square roots c_n of $\chi(n)^{-1}$ for $(n,N)=1$ then the Proposition $\Rightarrow$ $\langle c_n T_n f, g\rangle = \langle f, c_n T_n g\rangle$

So $S_k(N,\chi)$ has an orthonormal basis consisting of eigenvectors for T_n with $(n,N)=1$.

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Recall

$S_k(\Gamma_1(N))^{old}$ = span of $f(\tau), f(p\tau)$ for $p|N$ prime

for $f \in S_k(\Gamma_1(N/p))$

$S_k(\Gamma_1(N))^{new} :=$ orthogonal complement of $S_k(\Gamma_1(N))^{old}$

under $\langle \cdot , \cdot \rangle$

Definition

$S_k(N, \chi)^{new} := S_k(\Gamma_1(N))^{new} \cap S_k(N, \chi)$

Theorem

$S_k(N, \chi)^{new}$ is stable under T_n for all $n \geq 1$.

If $f \in S_k(N, \chi)^{new}$ is an eigenvector for T_n with $(n, N) = 1$

then f is an eigenform (i.e. eigenvector for all T_n).

In fact, if $\exists L \in \mathbb{Z}$ such that $f \in S_k(N, \chi)^{new}$ is an

eigenvector for all T_n with $(n, L) = 1$, then f is an

eigenform.

Proof

Diamond and Shurman, Chapter 4.

If $f \in S_k(N, \chi)^{new}$ is an eigenform, it is sometimes

referred to as a "newform". $S_k(..)^{new}$ is also referred to

as the space of primitive forms.

The theorem has a more natural interpretation in the theory

of automorphic representations.

Functions

$\in M_k(\Gamma)$, $f = \sum_{n\geq 0} a_n q_h^n \quad (q_h = e^{\frac{2\pi i\tau}{h}})$

finition

$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $s \in \mathbb{C}$.

ecall that if $f \in S_k(\Gamma)$, then $|a_n| \leq C n^{k/2}$

This implies that $L(f,s)$ is absolutely convergent and converges uniformly on compact subsets of $\{s \in \mathbb{C} \mid Re(s) > \frac{k}{2} + 1\}$

Remark

For $f \in M_k(\Gamma)$, $|a_n| \leq C n^{k-1}$ $(k \geq 2)$

To see this, observe that $M_k(\Gamma) = S_k(\Gamma) \oplus \mathcal{E}_k(\Gamma)$ where $\mathcal{E}_k(\Gamma)$ is given by explicit Eisenstein series.

So in general, $L(f,s)$ converges for $Re(s) > k$.

Functional Equation

Analogue of the equation relating $\zeta(s)$ and $\zeta(1-s)$.

Restrict to $\Gamma = \Gamma_1(N)$.

Remark

If Γ is a congruence subgroup, $\exists N$ such that $\alpha\Gamma\alpha^{-1} \supset \Gamma_1(N)$, so if $f \in M_k(\Gamma)$, then $f|_{\alpha^{-1},k} \in M_k(\Gamma_1(N))$.

$w_N := \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$, $w_N^{-1}\Gamma_1(N)w_N = \Gamma_1(N)$

So if $f \in M_k(\Gamma_1(N))$, then $f|_{w_N,k} \in M_k(\Gamma_1(N))$

$f|_{w_N,k}(\tau) = N^{-k/2}\tau^{-k} f(-\frac{1}{N\tau})$

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Theorem

Let $f \in S_k(\Gamma_1(N))$ and set $g = i^k f|_{W_N,k}$.

Then $L(f,s)$ extends to a holomorphic function on $s \in \mathbb{C}$.

Set $\Lambda(f,s) := N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s)$

Then $\Lambda(f,s) = \Lambda(g,k-s)$ "functional equation"

Recall that $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt$ (*) for $\mathrm{Re}(s) > 0$, and has meromorphic continuation to $s \in \mathbb{C}$, with poles at $0,-1,-2,\dots$

$\Gamma(s)$ has no zeroes in $\mathbb{C}$ and $\Gamma(n) = (n-1)!$

Proof

Set $\phi(y) = f(iy)$ for $y \in \mathbb{R}_{>0}$.

Mellin Transform: $F(s) := \int_0^\infty \phi(y)y^{s-1}\,dy$

Recall that $|\phi(y)| \le Cy^{-k/2}$ so the integral converges at the lower limit for $\mathrm{Re}(s) > \frac{k}{2}+1$ and that $\phi(y) \approx e^{-y}$ as $y \to \infty$, so the integral also converges at the upper limit for all $s \in \mathbb{C}$.

$\phi\left(\frac{1}{Nu}\right) = N^{k/2}u^k g(iu)$

$f = \sum a_n q^n$, $\phi(y) = \sum_{n\geq1} a_n e^{-2\pi ny}$, so

$F(s) = \sum_{n\geq1} a_n \int_0^\infty e^{-2\pi ny}y^{s-1}\,dy$

(*) $= \sum_{n\geq1} \underbrace{a_n(2\pi n)^{-s}}_{}\Gamma(s)$ for $\mathrm{Re}(s) > \frac{k}{2}+1$.

So $F(s) = N^{-s/2}\Lambda(f,s)$ — $= (2\pi)^{-s}L(f,s)$

Choose $A > 0$.

$F(s) = \underbrace{\int_0^A \phi(y)y^{s-1}\,dy}_{} + \int_A^\infty \phi(y)y^{s-1}\,dy$

$y = \frac{1}{Nu}$

$\int_{1/NA}^\infty g(iu)N^{\frac{k}{2}-s}u^{k-1-s}\,du$ which converges for all s because even if u is raised to a large power, $g(iu) \to 0$ as $u \to \infty$, faster

This gives a holomorphic function extending $F(s)$ to $s \in \mathbb{C}$.

$\Rightarrow \Lambda(f,s)$ extends to $s \in \mathbb{C}$.

$\Gamma(s)$ has no zeroes $\Rightarrow L(f,s)$ extends to $s \in \mathbb{C}$.

We can check that $N^{-\frac{s}{2}} \Lambda(f,s) = N^{\frac{k}{2}-s} \underbrace{N^{-\frac{k-s}{2}} \Lambda(g,k-s)}_{\text{Mellin Transform of } g(iu)}$

This implies the functional equation. $\square$

<u>Remark</u>

$N=1$: $\Lambda(f,s) = i^k \Lambda(f,k-s)$

Next time:

Hecke's Converse Theorem says that this functional equation characterises $f \in S_k(\Gamma(1))$.

More convincing: from online notes

$$
\begin{aligned}
N^{-\frac{s}{2}} \Lambda(f,s) &= F(s) \\
&= N^{-s} \int_0^\infty \phi\left(\frac{1}{\sqrt{N}u}\right) u^{-1-s}\, du \\
&= N^{-s} \int_0^\infty f\left(-\frac{1}{Niu}\right) u^{-1-s}\, du \\
&= N^{\frac{k}{2}-s} \int_0^\infty g(iu)\, u^{k-1-s}\, du \\
&= N^{\frac{k}{2}-s} N^{-\frac{k-s}{2}} \Lambda(g,k-s) \\
&= N^{-\frac{s}{2}} \Lambda(g,k-s)
\end{aligned}
$$

Last time

$f \in S_k(\Gamma_1(N))$, $\phi(y) = iy$

$F(s) = \int_0^\infty \phi(y)\, y^{s-1}\, dy$

$y \to \infty$, $\phi(y) = O(e^{-2\pi y})$

$y \to 0$, $\phi(y) = O(y^{-k/2})$

So the integral converges for $\mathrm{Re}(s) > 1 + \frac{k}{2}$

In fact, $f(-\frac{1}{\tau}) = \underbrace{\tau^k f|_{\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right),k}(\tau)}_{\text{a cusp form}}$

$\Rightarrow \phi(\frac{1}{y}) = O(y^k e^{-2\pi y})$ as $y \to \infty$

so the integral converges for all s.

Then substitute $u = \frac{1}{Ny}$ to get a functional equation.

Hecke Eigenforms and L-functions

Theorem

$f \in S_k(N, \chi)$, $f = \sum_{n \geq 1} a_n q^n$

f is a normalised Eigenform

$\Leftrightarrow$ $L(f, s) = \prod_{p \text{ prime}} (1 - a_p p^{-s} + \chi(p) p^{k-1-2s})^{-1}$

$\chi(p)$: 0 if $p | N$, $\chi(p \bmod N)$ if $p \nmid N$

Proof conditional on Lemma

Lemma

Suppose we have two sequences $(a_n)_{n\geq 1}$, $(b_n)_{n \geq 1} \subset \mathbb{C}$,

such that $F(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$, $G(s) = \sum_{n \geq 1} \frac{b_n}{n^s}$

both converge absolutely, and uniformly on compact subsets of $\{\mathrm{Re}(s) > \sigma\}$, and $F(s) = G(s)$ for $\mathrm{Re}(s) > \sigma$

Then $a_n = b_n \; \forall n$.

Proof

See Sheet 3, Q9

Proof (of Theorem)

Suppose that $L(f,s) = \prod_{p \text{ prime}} (1 - a_p p^{-s} + \chi(p) p^{k-1-2s})^{-1}$

Then the previous Lemma and some formal computation shows that the a_n satisfy $a_1 = 1$,

i.e. formally manipulate the product given

$a_{mn} = a_m a_n$ for $(m,n) = 1$

$a_{p^r} a_p = a_{p^{r+1}} + p^{k-1} \chi(p) a_{p^{r-1}}$

$\Rightarrow f$ a normalised Hecke eigenform.

Conversely, if f is a normalised Hecke eigenform, then the product given is equal to $\sum_{n \geq 1} a_n/n^s = L(f,s)$ $\square$

by the relations between coefficients

Remark

1. For $k=2$, compare the L-function with the Hasse-Weil L-function of an elliptic curve.

Theorem (Shimura)

$f \in S_2(\Gamma_0(N))$, a normalised Hecke eigenform, with all $a_n \in \mathbb{Q}$. Then there is an elliptic curve $E_f/\mathbb{Q}$ ~~where~~ with $L(E_f, s) = L(f,s)$.

($f \rightsquigarrow$ a differential on $X(\Gamma_0(N))$, so we can find E_f as a quotient of Jacobian $(X_0(N))$ algebraic curve over $\mathbb{Q}$).

2. On Sheet 3, we will show that for $f \in S_k(\Gamma_0(N))$ a normalised Hecke eigenform, then $\exists\, F/\mathbb{Q}$ a number field with $a_n \in F$ for all $n \geq 1$, F totally real.

10/03/14

Modular Forms (23)

Shimura also proved that for such f $\in S_2(\Gamma_0(N))$, there exists an abelian variety $A_f/_{\mathbb{Q}}$, with $F \hookrightarrow \mathrm{End}(A_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L(A_f, s) = \prod L(f_i, s)$, f_i twists of f.

Theorem (Wiles, Taylor et al)

If $E/_{\mathbb{Q}}$ is an elliptic curve, then $\exists f \in S_2(\Gamma_0(N))$ such that $L(f, s) = L(E, s)$ ($\underset{\text{deep}}{\Rightarrow}$ E is isogenous to E_f) ← Faltings

Converse Theorems (Hecke, Weil)

Hecke ← $N=1$, Weil ← general N

Deitmar Automorphic Forms

Miyake Modular Forms

Mellin Inversions

Proposition

Suppose that $g : \mathbb{R}_{>0} \to \mathbb{C}$, twice continuously differentiable, and suppose $c \in \mathbb{R}_{>0}$ such that $x^{c-1} g(x)$, $x^c g'(x)$, $x^{c+1} g''(x)$ are absolutely integrable on $\mathbb{R}_{>0}$. Then

$G(s) = \int_0^\infty x^{s-1} g(x)\, dx$ exists for $\mathrm{Re}(s) \geq c$ and (or $=$?)

$G(c+it) = O(1 + |t|^{-2})$

Moreover, $\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} G(s)\, ds = g(x)$

Proof (Sketch)

Substitute $x = e^t$. We get $G(s) = \int_{\mathbb{R}} e^{st} g(e^t)\, dt$

Set $s = c - 2\pi i y$ and we get

$G(s) = \int_{\mathbb{R}} \underbrace{e^{ct} g(e^t) e^{-2\pi i y t}}_{\text{fourier transform}} dt$

Everything follows from Fourier inversion for $e^{ct} g(e^t)$

Hecke's Converse Theorem

$\{a_n \in \mathbb{C}\}_{n\geq 1}$ with $|a_n| \leq C n^\sigma$ for some $\sigma \in \mathbb{R}$.

Set $Z(s) := \sum_{n\geq 1} a_n/n^s$ and $\Lambda(s) := (2\pi)^{-s}\Gamma(s)Z(s)$

Suppose that $\Lambda(s)$ extends to a holomorphic function on $\mathbb{C}$ and that $\Lambda(s) = i^k \Lambda(k-s)$ for some $k \in \mathbb{Z}$. Then:

functional equation

$f(\tau) = \sum_{n\geq 1} a_n q^n$ is a ~~modular~~ cusp form of weight k, level 1

Proof

Let $f = \sum_{n\geq 1} a_n q^n$. This gives a holomorphic function on $\mathbb{H}$ (since $|a_n| \leq Cn^\sigma$) with $f(\tau) = f(\tau+1)$. Therefore it is sufficient to show that

(*) $f(-1/\tau) = \tau^k f(\tau)$ (since $SL_2(\mathbb{Z}) = \langle \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix} \rangle$)

and enough to show that $f(i/y) = (iy)^k f(iy)$, $y \in \mathbb{R}_{>0}$, by uniqueness of analytic continuation.

this is (*) on the imaginary axis

Set $\phi(y) = f(iy)$, and check that

$\Lambda(s) = F(s) := \int_0^\infty y^{s-1}\phi(y)\,dy$ for $Re(s) \gg 0$.

we had e.g. $F(s) = N^{-\frac{s}{2}}\Lambda(s)$ for $f \in S_k(\Gamma_1(N))$

Mellin Inversion:

online notes justify this

$\phi(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Lambda(s)y^{-s}\,ds.$

We can show that this is equal to $\frac{1}{2\pi i}\int_{k-c-i\infty}^{k-c+i\infty}\Lambda(s)y^{-s}\,ds$

Substitute $s = k-s$ and use the functional equation ($\Lambda(s) = i^k\Lambda(k-s)$) to get $\phi(y) = \underbrace{i^k y^{-k}\phi(1/y)}_{\text{as required}}$, since if we do $\tau \mapsto (-1/\tau)$ twice, we get that identity $\Rightarrow$ k even if f is non-zero. $\square$

$$\underbrace{(2\pi)^{-s}\Gamma(s)Z(s)}_{\Lambda(s)} = \underbrace{i^k(2\pi)^{s-k}\Gamma(k-s)Z(k-s)}_{i^k\Lambda(k-s)}$$

$$\phi(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Lambda(s)y^{-s}\,ds = \frac{1}{2\pi i}\int_{k-c-i\infty}^{k-c+i\infty}\Lambda(k-t)y^{t-k}\,dt = \frac{1}{2\pi i}\,i^k y^{-k}\int_{k-c-i\infty}^{k-c+i\infty}\Lambda(t)\left(\frac{1}{y}\right)^{t}dt$$
$$= i^k y^{-k}\phi\left(\frac{1}{y}\right)$$

12/03/14 Modular Forms (24)

<u>Weil's Converse Theorem</u> (1967)

(like the previous converse theorem, but for level $\Gamma_0(N)$ or $\Gamma_1(N)$)

<u>Twists of Modular Forms</u>

$m \in \mathbb{Z}_{\geq 1}$, $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$ <u>primitive</u>

(i.e. not induced by $(\mathbb{Z}/m\mathbb{Z})^* \underset{d|m,\, d\neq m}{\longrightarrow} (\mathbb{Z}/d\mathbb{Z})^* \xrightarrow{\chi} \mathbb{C}^*$)

Let $f(\tau) = \sum_{n\geq 1} a_n q^n \in S_k(\Gamma_0(N))$ with m, N coprime

<u>Definition</u>

$f_\chi(\tau) := \sum_{n\geq 1} a_n \chi(n) q^n$

<u>Proposition</u> $(\mathbb{Z}/Nm^2\mathbb{Z})^* \to (\mathbb{Z}/m\mathbb{Z})^* \xrightarrow{\chi^2} \mathbb{C}^*$

$f_\chi \in S_k(\Gamma_1(Nm^2), \chi^2)$

<u>Proof</u>

$g(\chi) = \sum_{n=1}^{m} \chi(n) e^{-2\pi i n/m}$ (extend χ to $\mathbb{Z}$ in the usual way)

Property of Gauss Sums:

$\chi(n) = \frac{1}{m} g(\chi) \sum_{a=1}^{m} \overline{\chi(a)} e^{2\pi i a n/m}$

So $f_\chi = \frac{1}{m} g(\chi) \sum_{a=1}^{m} \overline{\chi(a)} f(\tau + a/m)$

$= \frac{1}{m} g(\chi) \sum_{a=1}^{m} \overline{\chi(a)} f|_{\begin{pmatrix} 1 & a/m \\ 0 & 1 \end{pmatrix}} (\tau)$

<u>Check carefully</u> that this implies the proposition. □

Suppose that $f \in S_k(\Gamma_0(N))$ satisfies

$\Lambda(f, s) = \varepsilon \Lambda(f, k-s)$ with $\varepsilon = \pm 1$

[i.e. $f|_{\begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}, k} = \varepsilon i^{-k} f$]

<u>Lemma</u>

Assume that $\Lambda(f, s) = \varepsilon \Lambda(f, k-s)$ with $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$ as above

Then $f_\chi \Big|_{\left(\begin{smallmatrix} 0 & -\frac{1}{m\sqrt{N}} \\ m\sqrt{N} & 0 \end{smallmatrix}\right), k} = \underbrace{\varepsilon i^{-k} \frac{g(\chi)}{g(\bar{\chi})} \chi(-N)}_{\varepsilon_\chi \cdot i^{-k}} f_{\bar{\chi}}$

So $\Lambda(f_\chi, s) = \varepsilon_\chi \Lambda(f_{\bar{\chi}}, k-s)$

<u>Definition</u>

$\mathcal{M} = \{$a set whose elements are odd primes, or 4$\}$

We say that $\mathcal{M}$ is <u>big</u> if $\mathcal{M} \cap \{a + nb : n \in \mathbb{Z}\} \neq \emptyset$ for a, b coprime.

<u>Example</u>

Dirichlet $\Rightarrow$ If $N \in \mathbb{Z}_{\geq 1}$, $\{$odd primes : $p \nmid N\}$ is big.

<u>Theorem</u> (Weil)

Let $\{a_n : n \geq 1\}$, with $|a_n| = O(n^\sigma)$ for some $\sigma \in \mathbb{R}_{>0}$. Set $f = \sum_{n \geq 1} a_n q^n$, $Z(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$.

Fix $N \in \mathbb{Z}_{>0}$ and set $\Lambda(s) = (N^{\frac{s}{2}}) \Gamma(s)(2\pi)^{-s} Z(s)$

Suppose that

i) $\Lambda(s)$ has holomorphic continuation to all of $\mathbb{C}$ and is bounded on $\{Re(s) \in [a, b]\}$ for any $a, b \in \mathbb{R}$

"Entire and bounded on vertical strips"

and $\Lambda(f, s) = \varepsilon \Lambda(f, k-s)$ for $\varepsilon = \pm 1$, $k \in \mathbb{Z}$

ii) $\exists$ a big set $\mathcal{M}$ such that For any primitive character χ of $(\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$ with $m \in \mathcal{M}$, f_χ satisfies part (i), with $\Lambda(f_\chi, s) = \varepsilon_\chi \Lambda(f_{\bar{\chi}}, k-s)$ where $\varepsilon_\chi = \varepsilon \frac{g(\chi)}{g(\bar{\chi})} \chi(-N)$

Then f is a modular form of level $\Gamma_0(N)$, weight k

$f \in M_k(\Gamma_0(N))$

Moreover, if $Z(s)$ converges absolutely for $\mathrm{Re}(s) > k-\delta$, for some $\delta \in \mathbb{R}_{>0}$, then $f \in S_k(\Gamma_0(N))$.

Remark

$$M_k(\Gamma_0(N)) = S_k(\Gamma_0(N)) \oplus \underbrace{\mathcal{E}_k(\Gamma_0(N))}$$

explicit Eisenstein series with q-expansions that have coefficients like $\sigma_{k-1}(n)$

($\Rightarrow \sum a_n/n^s$ will not converge absolutely for $\mathrm{Re}(s) < k$)

Proof (Idea)

See Miyake 4·3. $\Lambda(f_\chi, s) = \varepsilon_\chi \Lambda(f_{\bar\chi}, k-s)$

$$\Rightarrow f_\chi \big|_{\left(\begin{smallmatrix} 0 & -\frac{1}{m\sqrt{N}} \\ m\sqrt{N} & 0 \end{smallmatrix}\right), k} = \varepsilon_k\, i^{-k} f_{\bar\chi}$$

Assume this for all χ of conductors a, b. Then for $\gamma = \left(\begin{smallmatrix} a & v \\ uN & b \end{smallmatrix}\right)$, $f|_{\gamma,k} = f$.

If this holds for enough a, b then we get $f|_{\gamma,k} = f \ \forall \gamma \in \Gamma_0(N)$.

CM Forms

$K/\mathbb{Q}$ an imaginary quadratic extension, discriminant $-d$.

Algebraic Hecke-Characters of K:

$$\psi : \left\{ \begin{matrix} \text{fractional ideals of } K \\ \text{coprime to } \underline{c} \end{matrix} \right\} \to \mathbb{C}^*$$

$\underline{c}$ some $\mathcal{O}_K$-ideal

$\psi((\alpha)) = \alpha^t$ for $\alpha \in K^*$, $\alpha \equiv 1 \bmod^* \underline{c}$

$v_{\underline{p}}(\alpha - 1) \geq v_{\underline{p}}(\underline{c}) \ \forall \underline{p} \mid \underline{c}$

Assume $\underline{c}$ is minimal for ψ (ψ primitive).

$$\omega_\psi : \mathbb{Z} \to \mathbb{C}^*, \quad a \mapsto \frac{\psi((a))}{a^t}$$ (for a coprime to $\underline{c}$, 0 otherwise)

For Ψ as above, define

$$f_\Psi = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \text{ ideals} \\ \text{prime to } \mathfrak{f}}} \Psi(\mathfrak{a})\, q^{N\mathfrak{a}}$$

<u>Theorem</u> (Hecke)

$f_\Psi \in S_{k+1}(d\, N\mathfrak{f}, \omega_\Psi \chi_K)$

Here χ_K is the quadratic character of $(\mathbb{Z}/d\mathbb{Z})^*$ corresponding to $K \subset \mathbb{Q}(\mu_d)$ (in fact f_Ψ is a newform, and a Hecke Eigenform)

<u>Proof</u>

Use Θ-series, or use Weil's Converse Theorem:

$$L(f_\Psi, s) = \underbrace{L(\Psi, s)}_{\text{has property (i) in Weil's Theorem}}$$

$$L(f_{\Psi,\chi}, s) = L(\Psi \cdot (\chi \circ N_{K/\mathbb{Q}}), s)$$

$\Rightarrow$ property (ii) in Weils Theorem

$\Rightarrow f_\Psi$ modular. $\square$

$E/\mathbb{Q}$ an elliptic curve, $\mathcal{O}_K \hookrightarrow \mathrm{End}(E_{\mathbb{C}})$, then

$\exists \Psi$ with $L(E, s) = L(\Psi, s) \rightsquigarrow f_\Psi$.

MODULAR FORMS

JAMES NEWTON

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1. Introduction

1.1. **Basic notation.** The *modular group*, sometimes denoted $\Gamma(1)$, is

$$\mathrm{SL}_2(\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1\}.$$

The *upper half plane* is $\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. We can define an action of $\Gamma(1)$ on $\mathcal{H}$ as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

Exercise 1. Check that this action preserves $\mathcal{H}$ and is a group action. Hint: first show that

$$\mathrm{Im}(\gamma \cdot \tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}.$$

Definition 1.1. Let k be an integer and Γ a finite index subgroup of $\Gamma(1)$. A meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is *weakly modular of weight k and level Γ* if

$$f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathcal{H}$.

Remark 1.2. Made more precise later: A function f being weakly modular of weight 0 and level Γ means it gives a meromorphic function on $\Gamma\backslash\mathcal{H}$. A function f being weakly modular of weight 2 means $f(\tau)d\tau$ gives a meromorphic differential on $\Gamma\backslash\mathcal{H}$.

Modular forms will be defined precisely in the next couple of lectures, but for now I will say that a weakly modular function of weight k and level Γ is a *modular form* (of weight k and level Γ) if it is holomorphic on $\mathcal{H}$ and satisfies some other condition.

When $\Gamma = \Gamma(1)$, this 'other condition' is that there exist constants $C, Y \in \mathbb{R}_{>0}$ with

$$|f(\tau)| \leq C$$

for all τ with $\mathrm{Im}(\tau) > Y$.

1.2. **Some motivating examples.**

- **Representation numbers for quadratic forms**

  For an integer $k \geq 1$ and $n \geq 0$ write $r_k(n)$ for the number of distinct ways of writing n as a sum of k squares, allowing zero and counting signs and orderings. For example, we have $r_2(1) = 4$ since $1 = 0^2 + 1^2 = 0^2 + (-1)^2 = 1^2 + 0^2 = (-1)^2 + 0^2$.

  Define a function θ on $\mathcal{H}$ by taking

$$\theta(\tau) = \sum_{-\infty}^{\infty} e^{2\pi i n^2 \tau}.$$

We write q for the variable $e^{2\pi i\tau}$. Then for a positive integer $k \geq 1$

$$\theta(\tau)^k = \sum_{n\geq 0} r_k(n) q^n.$$

It turns out that for even integers k, $\theta(\tau)^k$ is a modular form, and we will see later that one can obtain information about the function $r_k(n)$ using this. It allows you to write $r_k(n)$ as 'nice formula'+'error term'. For example,

$$r_4(n) = 8 \sum_{\substack{0<d|n \\ 4\nmid d}} d$$

(in this case there's no error term!).

- **Complex uniformisation of elliptic curves**
  If we have a lattice (rank two discrete subgroup) $\Lambda \subset \mathbb{C}$ the Weierstrass $\wp$-function $\wp(z, \lambda)$ is a holomorphic function $\mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$ and the map

$$z \mapsto (\wp(z, \Lambda), \wp'(z, \Lambda))$$

gives an isomorphism between $\mathbb{C}/\Lambda$ and complex points $E_\Lambda(\mathbb{C})$ of an elliptic curve E_Λ over $\mathbb{C}$ with equation

$$y^2 = 4x^3 - 60G_4(\Lambda) - 140G_6(\Lambda)$$

where

$$G_k(\Lambda) = \sum_{\omega\in\Lambda} \frac{1}{\omega^k}$$

— these are examples of modular forms (if we consider the function $\tau \mapsto G_k(\mathbb{Z}\tau \oplus \mathbb{Z})$). We write Λ_τ for the lattice $\mathbb{Z}\tau \oplus \mathbb{Z}$. Similarly, any homogeneous polynomial in G_4, G_6 is a modular form, for example the discriminant function

$$\tau \mapsto (60G_4(\Lambda_\tau))^3 - 27(140G_6(\Lambda_\tau))^2.$$

- **Dirichlet series and L-functions**
  Recall the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

(this is the definition for $\mathrm{Re}(s) > 2$). It has a meromorphic continuation to all of $\mathbb{C}$ and satisfies a functional equation relative $\zeta(s)$ and $\zeta(1-s)$. In the course we will prove Hecke's converse theorem: if we are given a set of complex numbers $\{a_n\}_{n\geq 1}$ such that the Dirichlet series

$$Z(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

is absolutely convergent for $\mathrm{Re}(s) >> 0$, with suitable analytic continuation and functional equation, then the function

$$f(\tau) = \sum_{n\geq 1} a_n e^{2\pi i n\tau}$$

is a modular form.

Given $E/\mathbb{Q}$ an elliptic curve, the Hasse-Weil L-function of E, $L(E,s)$ is given by

$$\prod_p L_p(E,s) = \sum \frac{a_n}{n^s}$$

where for p a prime of good reduction (with E reducing to $\widetilde{E}_p$) $L_p(E,s) = (1 - a_p p^{-s} + p^{1-2s})^{-1}$, and $a_p = p + 1 - |\widetilde{E}_p(\mathbb{F}_p)|$ (and one can also write down the local factors at the primes of bad reduction). A very deep theorem (due to Wiles, Breuil, Conrad, Diamond and Taylor) is that $f(\tau) = \sum_{n\geq 1} a_n e^{2\pi i n\tau}$ is also a modular form. This is not proved using the converse theorem! Indeed, the only proof that $L(E,s)$ has analytic continuation and functional equation is to first show that $L(E,s)$ comes from a modular form in this way.

2. Modular forms of level one

In this section we will be interested in weakly modular functions of weight k and level $\Gamma(1)$.

2.1. **Fourier expansions.** Note that the definition of a weakly modular function of level $\Gamma(1)$ implies that $f(\tau+1) = f(\tau)$ for all $\tau \in \mathcal{H}$. Suppose f is holomorphic on the region $\{\mathrm{Im}(\tau) > Y\}$ for some $Y \in \mathbb{R}$. Denote by D^* the punctured unit disc $\{q \in \mathbb{C} : 0 < |q| < 1\}$.

The map $\tau \mapsto e^{2\pi i\tau}$ defines a holomorphic, surjective, map from $\mathcal{H}$ to D^* and we can define a function F on D^* by $F(q) = f(\tau)$ where $\tau \in \mathcal{H}$ is something satisfying $q = e^{2\pi i\tau}$. The function F is well-defined since the value of $f(\tau)$ is independent of the choice of τ. Moreover, F is holomorphic on the region $\{0 < |q| < e^{-2\pi Y}\}$, since f is holomorphic on the corresponding region in $\mathcal{H}$ and we can define $F(q) = f(\frac{1}{2\pi i}\log(q))$ for appropriate branches of log on open subsets of D^*.

Therefore we obtain a Laurent series expansion $F(q) = \sum_{n\in\mathbb{Z}} a_n(f)q^n$, with $a_n(f) \in \mathbb{C}$. This is called the Fourier expansion, or q-expansion, of f.

2.2. **Modular forms.**

Definition 2.1. Suppose f is a weakly modular function of weight k and level $\Gamma(1)$.

- We say that f is *meromorphic* (resp. *holomorphic*) at ∞ if f is holomorphic for $\mathrm{Im}\tau >> 0$ and $a_n(f) = 0$ for all $n << 0$ (resp. for all $n < 0$). Equivalently, F extends to a meromorphic (resp. holomorphic) function on an open neighbourhood of 0 in the unit disc D.
- If f is meromorphic at ∞ we say that f is a *meromorphic form of weight k* (non-standard, but will need them later).
- If f is holomorphic on $\mathcal{H}$ and at ∞ we say that it is a *modular form of weight k*, and if moreover $a_0(f) = 0$ we say that it is a *cusp form*.

Lemma 2.2. *Suppose f is weakly modular of level $\Gamma(1)$. Then f is holomorphic at ∞ if and only if there exists $C, Y \in \mathbb{R}$ such that $|f(\tau)| \leq C$ for all τ with $\mathrm{Im}\tau > Y$.*

Proof. Suppose f is holomorphic at ∞. Then the function F extends to a holomorphic function on an open neighbourhood of 0 in D. Therefore F is bounded on some sufficiently small disc in D with centre 0. This implies the desired boundedness statement for f.

Conversely, suppose there exists $C, Y \in \mathbb{R}$ such that $|f(\tau)| \leq C$ for all τ with $\mathrm{Im}\tau > Y$. This implies that f is holomorphic for $\mathrm{Im}\tau > Y$ (since it is bounded and meromorphic) and so we get a holomorphic function F on the region $\{0 < |q| < e^{-2\pi Y}\}$ satisfying $F(e^{2\pi i\tau}) = f(\tau)$. The boundedness condition on f implies that $qF(q)$ tends to zero as q tends to 0, so F has a removable singularity at 0 and we are done. $\square$

Definition 2.3. We denote the set of modular forms of weight k by $M_k(\Gamma(1))$, and denote the set of cusp forms of weight k by $S_k(\Gamma(1))$ (sometimes M_k and S_k for short).

Exercise 2.
(1) M_k and S_k are $\mathbb{C}$-vector spaces (obvious addition and scalar multiplication)
(2) $f \in M_k$, $g \in M_l$, then $fg \in M_{k+l}$
(3) $f \in M_k \implies f(-\tau) = (-1)^k f(\tau)$, so k odd $\implies M_k = \{0\}$.

We will later show that M_k and S_k are finite dimensional and compute their dimensions (main goal of the first half of the course). One of the reasons for imposing the 'holomorphic at ∞' condition is to ensure these spaces are finite dimensional.

2.3. First examples of modular forms.

Definition 2.4. Let $k > 2$ be an even integer. Then the Eisenstein series of weight k is a function on $\mathcal{H}$, defined, for $\tau \in \mathcal{H}$, by

$$G_k(\tau) = \sum_{(c,d)\in\mathbb{Z}^2}{}' \frac{1}{(c\tau + d)^k}.$$

Here the $'$ on the summation sign tells us to omit the $(0, 0)$ term.

Lemma 2.5. *This sum is absolutely convergent for $\tau \in \mathcal{H}$, and converges uniformly on compact subsets of $\mathcal{H}$, hence G_k is a holomorphic function on $\mathcal{H}$.*

Proof. Let's fix a compact subset C of $\mathcal{H}$ and think of τ varying over this compact set. Consider the parallelogram P_1 in $\mathbb{C}$ with vertices $1 + \tau$, $1 - \tau$, $-1 - \tau$ and $-1 + \tau$. Denote by $D(\tau)$ the minimum absolute value of a point in the boundary of P_1 (i.e. the length of the shortest line joining the origin and the boundary of P_1). As τ varies over the compact set C, $D(\tau)$ attains a minimum, which we denote by r.

For $m \in \mathbb{Z}_{\geq 1}$ denote by P_m the parallelogram whose vertices are m times the vertices of P_1. It is clear that as τ varies over C the minimum absolute value of a point in the boundary of P_m is mr.

Now let's consider how many points of the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$ lie in each of the P_m. The parallelogram P_m contains a $(2m+1) \times (2m+1)$ grid of these lattice points, so the boundary of P_m contains $(2m + 1)^2 - (2m - 1)^2 = 8m$ lattice points.

For $M \in \mathbb{Z}_{\geq 1}$ let's consider the sum

$$\sum_{(c,d)\in S_M} \frac{1}{|c\tau + d|^k}$$

where S_M is the set of pairs of integers (c, d) such that $c\tau + d$ is in the boundary of P_m for some $m \geq M$.

For $\tau \in C$ we have

$$\sum_{(c,d)\in S_M} \frac{1}{|c\tau + d|^k} \leq \sum_{m=M}^{\infty} \frac{8m}{(mr)^k} = \frac{8}{r^k} \sum_{m=M}^{\infty} \frac{1}{m^{k-1}}$$

and if $k > 2$ the final expression tends to 0 as M tends to ∞. Therefore the Eisenstein series is uniformly absolutely convergent for $\tau \in C$.

A slightly different proof of this Lemma is suggested in Exercise 1.1.4 of the book by Diamond and Shurman. □

Proposition 2.6. *The holomorphic function G_k is weakly modular of weight k.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. Now

$$G_k(\gamma\tau) = \sum_{(m,n)\in\mathbb{Z}^2}' \frac{(c\tau + d)^k}{(m(a\tau + b) + n(c\tau + d))^k} = (c\tau+d)^k \sum_{(m,n)\in\mathbb{Z}^2}' \frac{1}{((am + cn)\tau + (bm + dn))^k}.$$

Right multiplication by γ gives a bijection from $\mathbb{Z}^2 - \{0, 0\}$ to itself. Therefore the last term in the displayed equation is equal to $(c\tau + d)^k G_k(\tau)$ as required. □

Proposition 2.7. *The q-expansion of G_k is*

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

where $\zeta(k) = \sum_{n\geq 1} \frac{1}{n^k}$ is the Riemann zeta function and

$$\sigma_{k-1}(n) = \sum_{\substack{m|n \\ m>0}} m^{k-1}.$$

In particular, G_k is a modular form of weight k (we just had holomorphy at ∞ left to show).

The proof of this is postponed to the end of this section.

Definition 2.8. A normalisation: $E_k(\tau) = G_k(\tau)/2\zeta(k)$.

Fact 2.9. Proved later: $\dim(M_8(\Gamma(1))) = 1$.

Application of this fact: E_4^2 and E_8 are both in M_8 and their q-exapnsions have the same constant term (namely 1), so they are equal.

Corollary 2.10. *We deduce from $E_4^2 = E_8$ that*

- $\zeta(4) = \pi^4/90$, $\zeta(8) = \pi^8/2 \cdot 3^3 \cdot 5^2 \cdot 7$
- $\sigma_7(n) = \sigma_3(n) + 120 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_3(n - j)$.

Proof. For first part, compare a_1 and a_2 terms of q-expansions. Then compare general term to get the second part. $\square$

Some more interesting examples. A cusp form:

Definition 2.11. The Ramanujan delta function

$$\Delta(\tau) = \frac{E_4^3 - E_6^2}{1728} = q - 24q^2 + \ldots = \sum_{n\geq 1} \tau(n)q^n.$$

We will later see that $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$. Ay any rate, from its definition we have $\Delta \in S_{12}(\Gamma(1))$.

Now I'll give the proof of Proposition 2.7. We will use Poisson summation:

Fact 2.12. Let $h : \mathbb{R} \to \mathbb{C}$ be a continuous function such that

- h is L^1, i.e. $\int\limits_{-\infty}^{\infty} |h(x)|dx < \infty$.
- the sum $S(x) = \sum_{d\in\mathbb{Z}} h(x+d)$ converges absolutely and uniformly as x varies in a compact subset of $\mathbb{R}$, and $S(x)$ is an infinitely differentiable function in x.

Then, if we denote by $\hat{h}$ the Fourier transform

$$\hat{h}(t) = \int\limits_{-\infty}^{\infty} h(x)e^{-2\pi i x t}dx$$

we have

$$\sum_{d\in\mathbb{Z}} h(x+d) = \sum_{m\in\mathbb{Z}} \hat{h}(m)e^{2\pi i m x}.$$

Idea of the proof: the sum $S(x)$ satisfies $S(x) = S(x+1)$ and the right hand side of the final equality is the Fourier expansion for S.

Now let's apply this to the case we're interested in. We have

$$G_k(\tau) = 2\sum_{d=1}^{\infty}\frac{1}{d^k} + 2\sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z}}\frac{1}{(c\tau+d)^k}.$$

For c and τ fixed, let's define

$$h_c(x) = \frac{1}{(c\tau+x)^k}.$$

Now we can compute $\sum_{d\in\mathbb{Z}}\frac{1}{(c\tau+d)^k}$ by applying Poisson summation to h_c. Exercise: check that h_c satisfies the conditions for Poisson summation.

We have

$$\hat{h}_c(m) = \int\limits_{-\infty}^{\infty}\frac{e^{-2\pi i m x}}{(c\tau+x)^k}dx = \frac{1}{c^{k-1}}\int\limits_{-\infty}^{\infty}\frac{e^{-2\pi i m c u}}{(\tau+u)^k}du$$

For the last equality we substitute $x = cu$.

To compute this integral we use Cauchy's residue theorem applied to the complex function $f_n(z) = \frac{e^{-2\pi i n z}}{z^k}$. For $n \in \mathbb{Z}$, denote by I_n the integral of $f_n(z)$ along the horizontal line from $\tau - \infty$ to $\tau + \infty$. Then we have

$$\hat{h}_c(m) = \frac{e^{2\pi i m c \tau}}{c^{k-1}} I_{cm}.$$

Lemma 2.13. *We have $I_n = 0$ for $n \leq 0$. For $n > 0$, we have $I_n = \frac{(-2\pi i)^k n^{k-1}}{(k-1)!}$.*

Proof. The residue of f_n at $z = 0$ is $\frac{1}{(k-1)!}(-2\pi i n)^{k-1}$ (consider the Taylor expansion of $e^{-2\pi i n z}$). So it follows that $-2\pi i \mathrm{Res}_{z=0} f_n(z) = \frac{(-2\pi i)^k n^{k-1}}{(k-1)!}$.

For $n \leq 0$ we integrate over rectangles with vertices at $\tau - C, \tau + C, \tau + C + iC, \tau - C + iC$, for $C \in \mathbb{R}$ tending to ∞. These integrals are all equal to zero, and it's easy to see that the integrals over the upper horizontal and vertical sides tend to zero.

For $n > 0$ integrate (clockwise) over rectangles with vertices at $\tau - C, \tau + C, \tau + C - iC, \tau - C - iC$. Now the integrals (for C large enough that the rectangle contains $z = 0$) are all equal to $\frac{(-2\pi i)^k n^{k-1}}{(k-1)!}$ and the integrals over three of the sides tend to zero. $\square$

So now we conclude that $\hat{h}_c(m) = 0$ for $m \leq 0$ and

$$\hat{h}_c(m) = \frac{(-2\pi i)^k m^{k-1}}{(k-1)!} e^{2\pi i m c \tau}$$

for $m > 0$. Now applying Poisson summation, we can deduce Proposition 2.7 (recall that k is assumed even, so $(-2\pi i)^k = (2\pi i)^k$).

2.4. **Fundamental domain for $\Gamma(1)$.** We want to study some of the properties of the action of $\Gamma(1)$ on $\mathcal{H}$.

Definition 2.14. Suppose a group G acts continuously on a topological space X. Then a *fundamental domain* for G is an open subset $\mathscr{F} \subset X$ such that no two distinct points of $\mathscr{F}$ are equivalent under the action of G and every point $x \in X$ is equivalent (under G) to a point in the closure $\overline{\mathscr{F}}$.

Proposition 2.15. *The set*

$$\mathscr{F} = \{\tau \in \mathcal{H} : |\tau| > 1, |\mathrm{Re}(\tau)| < \frac{1}{2}\}$$

is a fundamental domain for the action of $\Gamma(1)$ on $\mathcal{H}$.

More precisely, if we set

$$\widetilde{\mathscr{F}} = \mathscr{F} \cup \{\tau \in \mathcal{H} : |\tau| \geq 1, \mathrm{Re}(\tau) = -\frac{1}{2}\} \cup \{\tau \in \mathcal{H} : |\tau| = 1, -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq 0\}$$

then $\widetilde{\mathscr{F}}$ contains a unique representative for every $\Gamma(1)$-orbit.

The group $\Gamma(1)$ is generated by the elements

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. We let G be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by S and T. Fix $\tau \in \mathcal{H}$. For $\gamma \in G$ have $\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$. Since c and d are integers, $|c\tau + d|^2$ attains a minimum as c and d vary over possible bottom rows of matrices in G. Therefore $\mathrm{Im}(\gamma\tau)$ attains a maximum as γ varies over G. So there is a $\gamma_0 \in G$ with $\mathrm{Im}(\gamma_0\tau) \geq \mathrm{Im}(\gamma\tau)$ for all $\gamma \in G$.

In particular

$$\frac{\mathrm{Im}(\gamma_0\tau)}{|\gamma_0\tau|^2} = \mathrm{Im}(-\frac{1}{\gamma_0\tau}) = \mathrm{Im}(S\gamma_0\tau) \leq \mathrm{Im}(\gamma_0\tau)$$

which implies that $|\gamma_0\tau| \geq 1$. Since applying T does not change the imaginary part we have $|T^n\gamma_0\tau| \geq 1$ for all n, and for some n we have $\mathrm{Re}(T^n\gamma_0\tau)| \in [-1/2, 1/2)$.

If $T^n\gamma_0\tau \in \overline{\mathscr{F}}\backslash\widetilde{\mathscr{F}}$ then $ST^n\gamma_0\tau \in \widetilde{\mathscr{F}}$ so we have proven that every G-orbit has a representative in $\widetilde{\mathscr{F}}$. This immediately implies that every $\Gamma(1)$-orbit has a representative in $\widetilde{\mathscr{F}}$.

It remains to prove that every $\Gamma(1)$-orbit has a unique representative in $\widetilde{\mathscr{F}}$.

Suppose that we have two distinct but $\Gamma(1)$-equivalent points $\tau_1 \neq \tau_2 = \gamma\tau_1$ in $\widetilde{\mathscr{F}}$. Since both τ_i's have real part $< 1/2$ we have $\gamma \neq \pm T^n$, so $c \neq 0$. Moreover, $\mathrm{Im}\tau \geq \sqrt{3}/2$ for all $\tau \in \mathscr{F}$, so

$$\frac{\sqrt{3}}{2} \leq \mathrm{Im}(\tau_2) = \frac{\mathrm{Im}(\tau_1)}{|c\tau_1 + d|^2} \leq \frac{\mathrm{Im}(\tau_1)}{c^2\mathrm{Im}(\tau_1)^2} \leq \frac{2}{c^2\sqrt{3}},$$

which implies that $c = \pm 1$. So we have

$$\mathrm{Im}(\tau_2) = \frac{\mathrm{Im}(\tau_1)}{|\pm\tau_1 + d|^2}$$

but $|\pm\tau_1 + d| \geq |\tau_1| \geq 1$ which implies that $\mathrm{Im}(\tau_2) \leq \mathrm{Im}(\tau_1)$. But everything was symmetric between τ_1 and τ_2, so we have $\mathrm{Im}(\tau_2) \leq \mathrm{Im}(\tau_1)$ and $|\tau_1| = |\tau_2| = 1$. Since $\tau_1, \tau_2 \in \widetilde{\mathscr{F}}$ this implies that $\tau_1 = \tau_2$. So there are no distinct but $\Gamma(1)$-equivalent points in $\widetilde{\mathscr{F}}$ and $\widetilde{\mathscr{F}}$ indeed contains a unique representative of every $\Gamma(1)$-orbit.

Now we can deduce that $\Gamma(1) = G$. Let $\gamma \in \Gamma(1)$ and consider the action on $2i \in \mathscr{F}$. There exists a $g \in G$ such that $g\gamma(2i) \in \overline{\mathscr{F}}$. We write $g\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and observe that

$$\mathrm{Im}(g\gamma(2i)) = \frac{2}{4c^2 + d^2} \geq \frac{\sqrt{3}}{2}.$$

This implies that $c = 0$ and $d = \pm 1$, hence $g\gamma = \pm T^n$ for some integer n and therefore $\gamma \in G$. $\square$

Exercise 3. Suppose $\tau \in \mathcal{H}$ is such that $\gamma\tau = \gamma$ for $\gamma \in \Gamma(1)$ with $\gamma \neq \pm I$. Then τ is in the $\Gamma(1)$ orbit of i or $\omega = -1/2 + \sqrt{3}/2$. If we define $n_\tau = |\mathrm{Stab}_{\Gamma(1)/\pm I}(\tau)|$ then $n_\tau = 2, 3$ if τ is in the orbit of i, ω respectively.

Note that to do the above exercise, it is enough to compute the stabilisers for $\tau \in \widetilde{\mathscr{F}}$.

Definition 2.16. If τ is a point of $\mathcal{H}$ such that $\mathrm{Stab}_{\Gamma(1)/\pm I}(\tau)$ is non-trivial, we say that τ is an elliptic point, of order $n_\tau = |\mathrm{Stab}_{\Gamma(1)/\pm I}(\tau)|$.

Note that the value of n_τ only depends on the orbit $\Gamma(1)\tau$.

2.5. **Zeros and poles of meromorphic forms.**

Definition 2.17. If f is a weight k meromorphic form and $\tau \in \mathcal{H}$ we write $\mathrm{ord}_\tau(f)$ for the order of vanishing of f at τ (i.e. it is the order of the zero if f vanishes at τ, 0 if f is holomorphic and non-vanishing at τ and it is the negative of the order of the pole if f has a pole at τ).

We write $\mathrm{ord}_\infty(f)$ for the smallest n such that $a_n(f) \neq 0$, where $f(\tau) = \sum_{n\in\mathbb{Z}} a_n(f)q^n$.

Since f is weakly modular, the integer $\mathrm{ord}_\tau(f)$ depends only on the $\Gamma(1)$-orbit $\Gamma(1)\tau$.

Proposition 2.18. *Let f be a non-zero meromorphic form of weight k. Then*

$$\mathrm{ord}_\infty(f) + \sum_{\Gamma(1)\tau\in\Gamma(1)\backslash\mathcal{H}} \frac{1}{n_\tau}\mathrm{ord}_\tau(f) = \frac{k}{12}.$$

Note that the sum in the above has finitely many non-zero terms — fixing $Y \in \mathbb{R}_{>0}$, f has finitely many zeros and poles in the compact region $\tau \in \overline{\mathscr{F}} \cap \{\mathrm{Im}(\tau) \leq Y\}$, and by meromorphy at ∞ it has finitely many zeros and poles in the region $\tau \in \overline{\mathscr{F}} \cap \mathrm{Im}(\tau) \geq Y$.

Proof. We drew a complicated contour and integrated $f'(\tau)/f(\tau)$ around it. See Serre, 'A Course in Arithmetic', Chapter VII, Theorem 3. □

Here are some immediate consequences of Proposition 2.18. Recall the weight 12 cusp form

$$\Delta = \frac{E_4^3 - E_6^2}{1728}.$$

We can also define a meromorphic form of weight 0:

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta}.$$

Corollary 2.19. *Δ is non-vanishing on $\mathcal{H}$. The weight 0 meromorphic form $j(\tau)$ is holomorphic on $\mathcal{H}$ and induces a bijection*

$$\Gamma(1)\backslash\mathcal{H} \to \mathbb{C}.$$

Proof. The q-expansion of Δ is $q - 24q^2 + \cdots$. In particular, we have $\mathrm{ord}_\infty(\Delta) = 1$. It follows immediately from Proposition 2.18 that Δ has no zeros in $\mathcal{H}$. This implies that j is indeed holomorphic on $\mathcal{H}$. To show that it induces a bijection

$$\Gamma(1)\backslash\mathcal{H} \to \mathbb{C},$$

fix $z \in \mathbb{C}$ and consider the weight 12 modular form $f_z(\tau) = E_4(\tau)^3 - z\Delta(\tau)$. By definition, we have $f_z(\tau) = 0$ if and only if $j(\tau) = z$. So to show that j induces a bijection it suffices to show that the zeros of f_z are given by a single $\Gamma(1)$-orbit.

By considering the q-expansion of f_z, we see that $\mathrm{ord}_\infty(f_z) = 0$. So Proposition 2.18 implies that we have an equality

$$\sum_{\Gamma(1)\tau\in\Gamma(1)\backslash\mathcal{H}} \frac{1}{n_\tau}\mathrm{ord}_\tau(f) = 1.$$

Now we see that the possibilities for the zeros of f_z are that there is a simple zero at a single non-elliptic orbit, a double zero at $\Gamma(1)i$ or a triple zero at $\Gamma(1)\omega$. In any case, the zeros form a single $\Gamma(1)$-orbit, as required. □

2.6. Dimension formula.

Lemma 2.20. *$M_k = \{0\}$ for $k < 0$. $M_0 \cong \mathbb{C}$, and is given by constant functions.*

Proof. Suppose $k < 0$. Then if $f \in M_k$ is non-zero, Proposition 2.18 implies that f cannot be holomorphic on $\mathcal{H}$ and at ∞. So f is zero.

Suppose $f \in M_0$. Then the constant term in the q-expansion of f, $a_0(f)$, is also in M_0, so $g = f - a_0(f) \in S_0$. Applying Proposition 2.18 to g tells us that g is zero, so f is constant. □

Lemma 2.21. *For even $k \geq 0$ we have*

$$\dim M_k(\Gamma(1)) \leq \lfloor k/12 \rfloor$$

if $k \equiv 2 \bmod 12$ and

$$\dim M_k(\Gamma(1)) \leq \lfloor k/12 \rfloor + 1$$

otherwise.

Proof. For general even k, we set $m = \lfloor k/12 \rfloor + 1$, and fix m distinct non-elliptic orbits $P_1, ..., P_m$ in $\Gamma(1)\backslash\mathcal{H}$. Suppose $f_1, ..., f_{m+1} \in M_k(\Gamma(1))$. Then we can find a linear combination of the f_i, denoted f, such that f has a zero at each of the m points P_i. Applying Proposition 2.18 implies that $f = 0$, so $\dim M_k \leq m$.

Now we suppose we are in the special case $k = 12l + 2$, $l \in \mathbb{Z}_{\geq 0}$. We now set $m = l$, choose l non-elliptic points as before, and suppose we have $l + 1$ elements $f_1, ..., f_{l+1}$ of $M_k(\Gamma(1))$. We denote by f a linear combination of the f_i with a zero at the l chosen points, therefore if f is non-zero we now have an equation

$$\mathrm{ord}_\infty(f) + l + \sum_{P \neq P_i} \frac{\mathrm{ord}_P(f)}{n_P} = l + \frac{1}{6}$$

which is impossible. So $f = 0$ and we conclude that $\dim M_k \leq l = \lfloor k/12 \rfloor$. □

Exercise 4. Consider the graded ring $\bigoplus_{k \geq 0} M_k$. Show that this direct sum injects into the ring of holomorphic functions on $\mathcal{H}$. In other words, there are no non-trivial linear dependence relations between modular forms of different weights.

Theorem 2.22. *Let $R : \mathbb{C}[X, Y] \to M$ be the map given by sending X to E_4 and Y to E_6. Then R is an isomorphism of rings (and respects the graded, if we give X degree 4 and Y degree 6).*

Corollary 2.23. *The set*

$$\{E_4^a E_6^b : a, b \geq 0, 4a + 6b = k\}$$

is a basis for M_k.

$$\dim M_k = \lfloor k/12 \rfloor$$

if $k \equiv 2$ mod 12 and

$$\dim M_k = \lfloor k/12 \rfloor + 1$$

otherwise.

Proof. This is an exercise. □

Proof of Theorem 2.22. The proof of the above Corollary, together with Lemma 2.21 tells us that, appropriately graded, the degree k part of $\mathbb{C}[X,Y]$ has dimension $\geq$ the dimension of M_k. So it's enough to show that R is an injection.

In other words we must show that E_4 and E_6 are algebraically independent, considered as elements of the field of meromorphic functions on $\mathcal{H}$. It is enough to show that E_4^3 and E_6^2 are algebraically independent. Suppose there is a dependence relation

$$\sum_{a,b} \lambda_{a,b} E_4^{3a} E_6^{2b} = 0.$$

By considering parts of fixed degree we can assume that the sum is only over a, b with $3a + 2b$ fixed. Dividing by a suitable power of E_6, we see that E_4^3/E_6^2 is the root of a polynomial with coefficients in $\mathbb{C}$, which implies that E_4^3/E_6^2 is a constant function. This implies that $(E_6/E_4)^2$ is a constant multiple of E_4, but E_4 is holomorphic so this would imply that $E_6/E_4 \in M_2 = \{0\}$, which gives a contradiction. □

3. Modular forms for congruence subgroups

3.1. Definitions.

Definition 3.1. Suppose $N \in \mathbb{Z}_{\geq 1}$, then we define the *principal congruence subgroup* of level N

$$\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv Id \mod N\}.$$

Definition 3.2. A subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a *congruence* subgroup if $\Gamma(N) \subset \Gamma$ for some N.

It follows immediately that congruence subgroups have finite index in $\mathrm{SL}_2(\mathbb{Z})$ (the converse is false — c.f. congruence subgroup problem).

Definition 3.3. $\Gamma_0(N)$: upper triangular mod N $\Gamma_1(N)$: upper triangular with 1, 1 on diagonal mod N

Recall that we already defined weak modularity with respect to a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. For this course, we will only consider weak modularity with respect to congruence subgroups, although much of the theory goes through for any finite index subgroup.

Definition 3.4. Let k be an integer and Γ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is *weakly modular of weight k and level Γ* if

$$f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathcal{H}$.

A fundamental domain for Γ acting on $\mathcal{H}$ can be obtained by taking a union of translates of $\mathscr{F}$ (the level one fundamental domain) by coset representatives for $\mathrm{SL}_2(\mathbb{Z})/\Gamma$. If you look at a picture of such a fundamental domain (e.g. use the applet at `http://www.math.lsu.edu/~verrill/fundomain/` written by Helena Verrill), then you'll see that there are (a finite number of) boundary points, known as cusps, on the real line (actually rational numbers). To get finite dimensional spaces of modular forms, we will need to impose conditions on the behaviour of weakly modular functions as τ approaches each of these limit points, as well as when τ has imaginary part going to ∞.

Definition 3.5. Suppose Γ is a congruence subgroup. We define the *period* of the cusp ∞ by

$$h(\Gamma) = \min\{h \in \mathbb{Z}_{>0} : \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma\}.$$

Definition 3.6. Suppose $f : \mathcal{H} \to \mathbb{C}$ weakly modular of weight k and level Γ, and that f is holomorphic for $\mathrm{Im}(\tau) >> 0$. Set $q_h = e^{2\pi i\tau/h}$, and define a function F on the punctured unit disc by

$$F(q_h) = f(\tau).$$

As before, F is holomorphic and has a Laurent series expansion

$$F(q_h) = \sum_{n\in\mathbb{Z}} a_n q_h^n.$$

We say that f is meromorphic (resp. holomorphic) at ∞ if F extends to a meromorphic (resp. holomorphic) function around 0 (i.e. if the appropriate condition holds on vanishing of the negative coeffients in the Laurent series for F).

Definition 3.7. The *slash operator* of weight k is defined as follows: for $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$, $f : \mathcal{H} \to \mathbb{C}$ and $k \in \mathbb{Z}$, define $f|_{\gamma,k} : \mathcal{H} \to \mathbb{C}$ by $f|_{\gamma,k}(\tau) = (c\tau + d)^{-k} f(\gamma \cdot \tau)$.

Remark 3.8. If $f : \mathcal{H} \to \mathbb{C}$ is meromorphic and Γ is a congruence subgroup, then f is weakly modular of weight k and level Γ if and only if

$$f|_{\gamma,k} = f$$

for all $\gamma \in \Gamma$.

To show that f is weakly modular of weight k and level Γ, it suffices to show that $f|_{\gamma_i,k} = f$ for a set of generators $\gamma_1, ..., \gamma_n$ of Γ.

If $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and f is weakly modular of weight k and level Γ, then $f|_{\alpha,k}$ is weakly modular of weight k and level $\alpha^{-1}\Gamma\alpha$. Moreover, the function $f|_{\alpha,k}$ only depends on the coset $\Gamma\alpha \in \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$.

Definition 3.9. Let Γ be a congruence subgroup, $k \in \mathbb{Z}$ and f weakly modular of weight k and level Γ. We say that f is *meromorphic at the cusps* (resp. *holomorphic at the cusps*) if $f|_{\alpha,k}$ is meromorphic at ∞ (resp. holomorphic at ∞) for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 3.10. If f is weakly modular of weight k and level Γ and meromorphic at the cusps, we say that f is a *meromorphic form* of weight k and level Γ.

If f is weakly modular of weight k and level Γ, holomorphic on $\mathcal{H}$ and holomorphic at the cusps, we say that f is a *modular form* of weight k and level Γ. If moreover $a_0(f|_{\alpha,k}) = 0$ for all α we say that f is a cusp form.

We denote the space of modular forms of weight k and level Γ by $M_k(\Gamma)$, and denote the subspace of cusp forms by $S_k(\Gamma)$.

Here is a usual condition in practice for checking that a weakly modular function is a modular form:

Proposition 3.11. *If $\Gamma(N) \subset \Gamma$, f holomorphic on $\mathcal{H}$ and weakly modular of weight k and level Γ, with $f(\tau) = \sum_{n=0}^{\infty} a_n(f) e^{2\pi i \tau n/N}$ and $|a_n(f)| \leq Cn^r$ for some constants $C, r \in \mathbb{R}_{>0}$, then f is holomorphic at the cusps. Therefore f is a modular form of weight k and level Γ.*

Proof. See Diamond and Shurman Exercise 1.2.6 □

3.2. **Examples: θ-functions.** Recall the definition $\theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}$. It is straightforward to show that this series is absolutely and uniformly convergent on compact subsets of $\mathcal{H}$. We have

$$\theta(\tau, k) = \theta(\tau)^k = \sum_{n=0}^{\infty} r(n,k) q^n$$

where $r(n,k)$ is the number of ways of writing n as the sum of k squares.

Proposition 3.12. *We have $\theta(\tau + 1) = \theta(\tau)$ and $\theta(-1/4\tau) = \sqrt{2\tau/i}\,\theta(\tau)$. Here by $\sqrt{\ }$ we mean the branch on $\mathrm{Re}(z) > 0$ extending the positive square root on the positive real axis.*

Proof. The first equality is clear. For the second we use Poisson summation. Set $h(x) = e^{-\pi t x^2}$ with $t \in \mathbb{R}_{>0}$. We have

$$\hat{h}(y) = \int_{-\infty}^{\infty} e^{-\pi t x^2 - 2\pi i x y} dx = e^{-\pi y^2/t} \int_{-\infty}^{\infty} e^{-\pi(\sqrt{t}x + iy/\sqrt{t})^2} dx.$$

We substitute $u = \sqrt{t}x + iy/\sqrt{t}$, use $\int_{-\infty}^{\infty} e^{-\pi u^2} du = 1$, and conclude that $\hat{h}(y) = e^{-\pi y^2/t}/\sqrt{t}$.

So Poisson summation tells us that

$$\sum_{d \in \mathbb{Z}} e^{-\pi t d^2} = \sum_{m \in \mathbb{Z}} e^{-\pi m^2/t}/\sqrt{t}$$

whence $\theta(it/2) = \frac{1}{\sqrt{t}}\theta(i/2t)$ for $t \in \mathbb{R} > 0$. Now uniqueness of analytic continuation implies that the conclusion of the Proposition. □

Corollary 3.13. $\theta(\tau/4\tau + 1)^2 = (4\tau + 1)\theta(\tau)^2$

Proof. Easy computation. □

We conclude that for even k $\theta(\tau/4\tau+1,k) = (4\tau+1)^{k/2}\theta(\tau,k)$. In particular, for positive integers k, $\theta(\tau,4k)$ is weakly modular of weight $2k$ and level Γ, where Γ is the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by

$$\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}.$$

In the first example sheet, it is shown that $\Gamma = \Gamma_0(4)$, so $\theta(\tau,4k) \in M_{2k}(\Gamma_0(4))$.

3.3. **Examples: old forms.** It's worth noting that if $\Gamma \subset \Gamma'$ then if a function f is a modular form of weight k and level Γ' it is also a modular form of weight k and level Γ.

Lemma 3.14. *Suppose $f \in M_k(\Gamma_0(N))$ and $M \in \mathbb{Z}_{\geq 1}$. Then*

$$f_M : \tau \mapsto f(M\tau)$$

is in $M_k(\Gamma_0(MN))$.

Proof. First we check that f_M is weakly modular. We can write $f_M(\tau) = f|_{\gamma_M}(\tau)$, where

$$\gamma_M = \begin{pmatrix}M & 0\\ 0 & 1\end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}).$$

So f_M is weakly modular of weight k and level $(\gamma_M^{-1}\Gamma_0(N)\gamma_M) \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(MN)$.
To check holomorphy at the cusps, let

$$\alpha = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and observe that

$$f_M|_{\alpha,k}(\tau) = \left(\frac{cM\tau+d}{c\tau+d}\right)^k f|_{alpha,k}(M\tau)$$

so $f|_{alpha,k}$ and $f_M|_{\alpha,k}$ have the same behaviour as $\mathrm{Im}(\tau)$ tends to ∞. □

Similarly, we have

Lemma 3.15. *Suppose $f \in M_k(\Gamma_1(N))$ and $M \in \mathbb{Z}_{\geq 1}$. Then*

$$f_M : \tau \mapsto f(M\tau)$$

is in $M_k(\Gamma_1(MN))$.

Definition 3.16. Fix $N \in \mathbb{Z}_{\geq 1}$. For each divisor $M \mid N$, let i_M be the map

$$\begin{aligned} i_M : M_k(\Gamma_1(N/M)) \oplus M_k(\Gamma_1(N/M)) &\to M_k(\Gamma_1(N)) \\ (f,g) &\mapsto f + g_M. \end{aligned}$$

We define the space of *oldforms* at level N, $M_k(\Gamma_1(N))^{\mathrm{old}}$ to be the span of the union of the images of i_M as $M > 1$ varies over divisors of N.

3.4. **Examples: weight 2 Eisenstein series.** We would like to define an Eisenstein series of weight 2 by

$$G_2(\tau) = \sum_{d\neq 0} \frac{1}{d^2} + \sum_{c\neq 0}\sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^2}.$$

The above sum is *not* absolutely convergent, so we cannot interchange the order of summation over c and d (recall that this was used to prove the G_k was weakly modular for $k > 2$). However, the Poisson summation argument still allows us to compute the sums over d and conclude that this sum converges, and

$$G_2(\tau) = 2\zeta(2) + 2(2\pi i)^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

where this latter series is absolutely and uniformly convergent on compact subsets.

We use something known as 'Hecke's trick' to determine how G_2 transforms under the action of $\Gamma(1)$. This will then allow us to define some weight 2 modular forms of higher levels.

Definition 3.17. For $\epsilon \in \mathbb{R}_{>0}$ we define

$$G(\tau,\epsilon) = \sum_{(c,d)\in\mathbb{Z}^2}{}' \frac{1}{(c\tau+d)^2|c\tau+d|^{2\epsilon}}.$$

The point is that we have perturbed $G_2(\tau)$ a little, to obtain a double sum which is now absolutely convergent. Now in exactly the same way as for the higher weight Eisenstein series we deduce

$$G(\gamma\tau,\epsilon) = (c\tau+d)^2|c\tau+d|^{2\epsilon}G(\tau,\epsilon)$$

for $\gamma \in \Gamma(1)$.

Theorem 3.18 (non-examinable). *For any $\tau \in \mathcal{H}$, the limit $\varinjlim_{\epsilon\to 0} G(\tau,\epsilon)$ exists and is equal to $G_2^*(\tau) = G_2(\tau) - \frac{\pi}{\mathrm{Im}(\tau)}$.*

As a consequence, we have $G_2^*(\gamma\tau) = (c\tau+d)^2 G_2^*(\tau)$, but note that G_2^* is not holomorphic. However, we can use G_2^* to get higher level modular forms.

Corollary 3.19. *For N a positive integer we let $G_2^{(N)}(\tau) = G_2(\tau) - NG_2(N\tau)$. Then $G_2^{(N)} \in M_2(\Gamma_0(N))$, and its q-expansion is given by*

$$2(1-N)\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty}\Big(\sum_{\substack{0<d\mid n\\ N\nmid d}} d\Big)q^n.$$

Proof. We deduce that $G_2^{(N)}$ is weakly modular of weight 2 from the equality $G_{2,N} = G_2^* - N\iota_N(G_2^*)$ and our discussion of oldforms. To show that $G_2^{(N)}$ is holomorphic at the cusps we either show it directly or apply Proposition 3.11. The computation of the q-expansion is easily deduced from the q-expansion of G_2. □

Fact 3.20. The space $M_2(\Gamma_0(4))$ has dimension 2.

It follows from this fact that $G_2^{(2)}, G_2^{(4)}$ is a basis for $M_2(\Gamma_0(4))$, and by comparing q-expansions we see that

$$\theta(\tau,4) = -\frac{1}{\pi^2}G_{2,4}(\tau)$$

and as a consequence we obtain:

Theorem 3.21. *For integers $n \geq 1$*

$$r(n,4) = 8\sum_{\substack{0<d|n\\ 4\nmid d}} d.$$

Finally, I should sketch the proof of Theorem 3.18. Unfortunately, it's a little painful... Similarly to the higher weight case, we apply Poisson summation to the sums

$$\sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^2|c\tau+d|^{2\epsilon}}.$$

We write $h_{c,\epsilon}(x) = (c\tau+x)^{-2}|c\tau+x|^{-2\epsilon}$ and then we have Fourier coefficients

$$\hat{h}_{c,\epsilon}(m) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i m x}}{(c\tau+x)^2|c\tau+x|^{2\epsilon}}dx = \frac{1}{c^{1+2\epsilon}}\int_{-\infty}^{\infty}\frac{e^{-2\pi i c m x}}{(\tau+x)^2|\tau+x|^{2\epsilon}}dx$$

so we can write

$$G_2(\epsilon,\tau) = 2\sum_{d=1}^{\infty}\frac{1}{d^{2+2\epsilon}} + 2\sum_{c=1}^{\infty}\sum_{m\neq 0}\hat{h}_{c,\epsilon}(m) + 2\sum_{c=1}^{\infty}\hat{h}_{c,\epsilon}(0).$$

The following Lemma tells us that the second of these sums is nice enough that we can compute its limit as $\epsilon \to 0$ by exchanging the limit and the summation:

Lemma 3.22. *Suppose $m \neq 0$ and $\epsilon < 1$. Then there exists a constant $K \in \mathbb{R}_{>0}$ (independent of ϵ, c, depending on τ) such that*

$$|\hat{h}_{c,\epsilon}(m)| \leq \frac{K}{c^{3+2\epsilon}m^2}.$$

Proof. It's enough to show that there exists K with

$$\left|\int_{-\infty}^{\infty}\frac{e^{-2\pi i c m x}}{(\tau+x)^2|\tau+x|^{2\epsilon}}dx\right| \leq \frac{K}{c^2m^2}.$$

This follows from observing that $|\tau+x|^{2\epsilon} \geq |\mathrm{Im}(\tau)|^{2\epsilon} \geq \min\{1, |\mathrm{Im}(\tau)|^2\}$ and then showing that

$$\left|\int_{-\infty}^{\infty}\frac{e^{-2\pi i c m x}}{(\tau+x)^2}dx\right| \leq \frac{K}{c^2m^2}.$$

This last estimate is derived by integrating by parts twice — it comes down to the fact that

$$\int_{-\infty}^{\infty}\frac{1}{|\tau+x|^4}dx$$

is finite. □

So for the $m \neq 0$ terms we take the limit inside the sum and we can also interchange the limit with the integral defining $\hat{h}_c(\epsilon, m)$. To prove the theorem, it is now sufficient to show that

$$\lim_{\epsilon \to 0} 2 \sum_{c=1}^{\infty} \hat{h}_c(\epsilon, 0) = -\frac{\pi}{\mathrm{Im}\tau}.$$

We do this as follows: first, translating the variable x and using $|x-i|^{2\epsilon} = (x+i)^{\epsilon}(x-i)^{\epsilon}$ (these powers are defined using the principal branch of the logarithm) we obtain

$$\hat{h}_c(\epsilon, 0) = \frac{1}{(c\mathrm{Im}\tau)^{1+2\epsilon}} \int_{-\infty}^{\infty} \frac{1}{(x+i)^{2+\epsilon}(x-i)^{\epsilon}} dx.$$

Integration by parts tells us that this integral is equal to

$$-\frac{\epsilon}{1+\epsilon} \int_{-\infty}^{\infty} {\frac{1}{1+x^2}}^{1+\epsilon} dx$$

so we have

$$\sum_{c=1}^{\infty} \hat{h}_c(\epsilon, 0) = -\frac{\zeta(1+2\epsilon)\epsilon}{1+\epsilon} \frac{1}{(\mathrm{Im}\tau)^{1+2\epsilon}} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{1+\epsilon}} dx.$$

Since $\zeta(s)$ has a simple pole with residue 1 at $s = 1$ the first fraction tends to $1/2$ as $\epsilon \to 0$, whilst the integral tends to π. This gives us the desired result.

3.5. **Finite dimensionality.** Now we can give a cheap proof that $M_k(\Gamma)$ is finite dimensional for all congruence subgroups Γ. We won't determine the dimensions for a while, however!

Suppose Γ' is a normal subgroup of Γ, and denote the quotient group $\Gamma' \backslash \Gamma$ by G (I'm thinking of the elements as right cosets, hence the notation). We define a right action of G on $M_k(\Gamma')$ by setting $f^g = f|_{\gamma,k}$ for $g = \Gamma'\gamma \in G$. The action of g is well-defined (i.e. independent of the choice of coset representative γ).

Now we can see that $M_k(\Gamma) = M_k(\Gamma')^G$, where by ^{G} we mean the invariants under the action of G (i.e. a function in $M_k(\Gamma')$ is Γ-invariant under the slash operator if and only if it is G-invariant).

Lemma 3.23. *Suppose Γ' is a normal subgroup of Γ and $f \in M_k(\Gamma')$. Then there exist modular forms $h_i \in M_{ik}(\Gamma)$ for $i = 1, \ldots, [\Gamma : \Gamma']$ with*

$$f^n + h_1 f^{n-1} + \cdots + h_n = 0.$$

Proof. Consider the identity

$$\prod_{g \in G} (f - f^g) = 0.$$

Expanding out the product we get a monic polynomial in f with the coefficient of f^{n-i} given by a symmetric polynomial of degree i in the f^g. This coefficient is therefore in $M_{ik}(\Gamma')^G = M_{ik}(\Gamma)$. □

Lemma 3.24. (1) *for* $k < 0$, $M_k(\Gamma) = 0$
(2) $M_0(\Gamma) = \mathbb{C}$ *(the constant functions)*

Proof. Since Γ is a congruence subgroup, we have $\Gamma(N) \subset \Gamma$ for some N. Note that $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$. To show the Lemma, it is enough to prove it for $\Gamma = \Gamma(N)$. Suppose $f \in M_k(\Gamma)$ and $k < 0$. Then Lemma 3.23 gives us some $h_i \in M_{ik}(\Gamma(1))$ which are all zero, since $ik < 0$. so we have $f^n = 0$ for some n. Hence $f = 0$.

If $k = 0$, then $h_i \in M_0(\Gamma(1)) = \mathbb{C}$ for all i, so f is a root of a polynomial with constant coefficients. Hence f is constant. □

To prove that the spaces $M_k(\Gamma)$ are finite dimensional we will use some commutative algebra. The key ingredient is the notion of integral extensions of rings.

Definition 3.25. Let A be a subring of B. An element $b \in B$ is said to be integral over A if b satisfies a monic polynomial $b^n + a_1 b^{n-1} + \cdots a_n = 0$ with coefficients in A.

The ring B is said to be integral over A if every element of B is integral over A.

The integral closure $\widetilde{A}$ of A in B is defined to be the set of elements of B which are integral over A.

Exercise 5. Show that b is integral over A if and only if there exists a ring C with $A \subset C \subset B$ and $b \in C$, such that C is a finitely generated A-module.

Deduce that $\widetilde{A}$ is a subring of B (i.e. the set $\widetilde{A}$ is closed under the ring operations).

Show that if we have an extension of rings $A \subset B$ with B integral over A and B a finitely generated A-algebra, then B is a finitely generated A-module.

Remark 3.26. It follows from Lemma 3.23 and the above exercise that if we set $A = \oplus_{k \geq 0} M_k(\Gamma)$ and $B = \oplus_{k \geq 0} M_k(\Gamma')$ then B is integral over A (we think of A as a subring of B via the natural inclusions $M_k(\Gamma) \subset M_k(\Gamma')$ for each k).

Here's an important general result in commutative algebra (due to E. Noether)

Theorem 3.27. *Let F be a field and let A be a finitely generated F-algebra. Suppose A is an integral domain and denote the field of fractions* Frac(A) *by K. Suppose L is a finite extension field of K and denote by $\widetilde{A}$ the integral closure of A in L. Then $\widetilde{A}$ is a finitely generated A-module (and is in particular a finitely generated F-algebra).*

Proof. See Corollary 13.13 in Eisenbud's book 'Commutative algebra...'. □

As a consequence, we obtain the following useful lemma:

Lemma 3.28. *Let F be a field, B a commutative F-algebra, and assume B is an integral domain. Let $A \subset B$ be a sub F-algebra and assume that B is integral over A and* Frac(B)/Frac(A) *is a finite extension of fields. Then B is a finitely generated F-algebra if and only if A is a finitely generated F-algebra.*

Proof. First we suppose that A is a finitely generated F-algebra. Then Theorem 3.27 implies that $\widetilde{A}$, the integral closure of A in Frac(B), is a finite generated A-module. We have $A \subset B \subset \widetilde{A}$.

Now A is Noetherian, so a submodule of a finitely generated A-module is finitely generated, hence B is a finitely generated A-module. Therefore B is a finitely generated F-algebra.

For the reverse implication, we now suppose that B is a finitely generated F-algebra. Pick generators $b_1, ..., b_n$ for B and pick monic polynomials with coefficients in A with b_i as a root. Let C be the finitely generated sub F-algebra of A generated by the coefficients of these polynomials. By construction B is integral over C and it is a finitely generated C-algebra (since it is a finitely generated F-algebra).

By the exercise above, we know B is a finitely generated C-module. So A is a submodule of a finitely generated C-module, and is hence itself a finitely generated C-module. Therefore A is a finitely generated F-algebra. □

Finally, we can give the desired result about finite dimensionality of spaces of modular forms.

Theorem 3.29. *Let Γ be a congruence subgroup. Then*
(1) *for $k < 0$, $M_k(\Gamma) = 0$*
(2) *$M_0(\Gamma) = \mathbb{C}$ (the constant functions)*
(3) *$M(\Gamma) := \oplus_{k\geq 0} M_k(\Gamma)$ is a finitely generated $\mathbb{C}$-algebra*

Proof. Since Γ is a congruence subgroup, we have $\Gamma(N) \subset \Gamma$ for some N. Note that $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$. Set $C = M(\Gamma)$, $B = M(\Gamma(N))$ and $A = M(\Gamma(1))$.

Denote by G the finite group $\Gamma(N)\backslash\Gamma(1)$ which acts on B, with invariants $B^G = A$ (we let G act on each graded piece $M_k(\Gamma(N))$ by the weight k slash operator we discussed earlier).

We can extend the G action to the fraction field $\mathrm{Frac}(B)$ by letting G act on the numerator and denominator of a fraction.

We first check that
$$\mathrm{Frac}(A) = (\mathrm{Frac}(B))^G.$$
This is because, if we write $x \in (\mathrm{Frac}(B))^G$ as $\frac{p}{q}$ with $p, q \in B$, then
$$x \prod_{g\in G} q^g \in B^G = A$$
so x is in $\mathrm{Frac}(A)$.

Now Artin's lemma (as in Galois theory) implies that $\mathrm{Frac}(B)/\mathrm{Frac}(A)$ is a finite extension of fields (indeed, it is Galois with Galois group G). This also implies that $\mathrm{Frac}(B)$ is a finite extension of $\mathrm{Frac}(C)$.

Now we can apply Lemma 3.28: we know that A is a finitely generated $\mathbb{C}$-algebra, so we deduce that B is a finitely generated $\mathbb{C}$-algebra. Applying Lemma 3.28 once more, we deduce that C is a finitely generated $\mathbb{C}$-algebra, as required. □

Corollary 3.30. *For $k \geq 0$, $M_k(\Gamma)$ is a finite dimensional $\mathbb{C}$-vector space.*

Proof. We have shown that $\oplus_{k\geq 0} M_k(\Gamma)$ is a finitely generated $\mathbb{C}$-algebra. By decomposing a generator into its components of fixed weight, we obtain a generating set whose elements are modular forms $f_1, \dots, f_n$ of weights $k_1 \dots, k_n$. This implies that $M_k(\Gamma)$ is spanned by the monomials

$$\{\prod_i f_i^{l_i} : l_i \geq 0, \sum_i k_i l_i = k\}$$

□

4. Modular curves as Riemann surfaces

4.1. Recap on Riemann surfaces.

Definition 4.1. Suppose X is a Hausdorff topological space (*topological space* for short). A *complex chart* on X is a homeomorphism $\phi : U \to V$ from $U \subset X$ open to $V \subset \mathbb{C}$ open.

Two charts $\phi_i : U_i \to V_i$ are *compatible* if

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is biholomorphic.

An *atlas* on X is a family

$$\mathcal{A} = \{\phi_i : U_i \to V_i : i \in I\}$$

of compatible charts, with $X = \cup_{i \in I} U_i$.

We define an equivalence relation on pairs of topological spaces and atlases by $(X, \mathcal{A}) \sim (X, \mathcal{A}')$ if every chart in $\mathcal{A}$ is compatible with every chart in $\mathcal{A}'$.

A Riemann surface is defined to be an equivalence class of pairs $(X, \mathcal{A}')$. We will usually work with connected Riemann surfaces.

For X a Riemann surface, a function $f : Y \to \mathbb{C}$ on an open subset $Y \subset X$ is defined to be *holomorphic* if for all charts (in some atlas) $\phi : U \to V$,

$$f \circ \phi^{-1} : \phi(U \cap Y) \to \mathbb{C}$$

is holomorphic. The set of holomorphic functions on Y is denoted by $\mathscr{O}_X(Y)$.

In fact the assignment $Y \mapsto \mathscr{O}_X(Y)$ determines the Riemann surface structure on the topological space X. We'll develop this viewpoint a little, as it's convenient for talking about quotients of Riemann surfaces.

Definition 4.2. Let X be a topological space. A presheaf (of Abelian groups) on X is a pair $(\mathscr{F}, \rho)$ comprising

- for any $U \subset X$ open, an Abelian group $\mathscr{F}(U)$
- for any $V \subset U \subset X$ open, a group homomorphism

  $$\rho_V^U : \mathscr{F}(U) \mathscr{F}(V)$$

  called 'restriction from U to V' such that $\rho_U^U = id$ and $\rho_W^V \circ \rho_V^U = \rho_W^U$ for $W \subset V \subset U$. For $f \in \mathscr{F}(U)$ we usually write $f|_V$ for $\rho_V^U(f) \in \mathscr{F}(V)$.

The group $\mathscr{F}(U)$ is called the sections of $\mathscr{F}$ on U. Examples of presheafs are given by $\mathscr{F}(U) =$ continuous functions from U to $\mathbb{C}$. Denote this presheaf by $\mathscr{O}_X^{cts}$. Also, for X a Riemann surface, we have the presheaf of holomorphic functions $\mathscr{F}(U) = \mathscr{O}_X(U)$.

Definition 4.3. A presheaf $\mathscr{F}$ on X is a *sheaf* if for every open $U \subset X$, and every covering family $\{U_i\}_{i \in I}$ of U (i.e. $U_i \subset U$ with $U = \cup_{i \in I} U_i$), we have

- if $f, g \in \mathscr{F}(U)$ and $f|_{U_i} = g|_{U_i}$ for all i, then $f = g$
- given $f_i \in \mathscr{F}(U_i), i \in I$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a section $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$ for every $i \in I$.

The first point says sections are determined by local data (i.e. restriction to covers by small open sets), the second says that we can define a section on U by 'gluing' sections defined on an open cover. Note that these statements are obviously satisfied by any reasonable presheaf of functions.

In particular, it's easy to check that the presheaves $\mathscr{O}_X^{cts}$ and $\mathscr{O}_X$ defined above are in fact sheaves.

Definition 4.4. A $\mathbb{C}$-space is a Hausdorff topological space X, equipped with a sheaf $\mathscr{F}$ such that $\mathscr{F}(U)$ is a sub $\mathbb{C}$-algebra of $\mathscr{O}_X^{cts}$ for all U, and the restriction maps ρ_V^U are given by restriction of functions.

A morphism of $\mathbb{C}$-spaces $(X, \mathscr{F}) \to (Y, \mathscr{G})$ is a continuous map $f : X \to Y$ such that for all opens $V \subset Y$, $g \in \mathscr{G}(V)$, we have $g \circ f \in \mathscr{F}(f^{-1}(V))$.

For example, if $\phi : U \to V$ is a chart of a Riemann surface X, then $\phi : (U, \mathscr{O}_X|_U) \cong (V, \mathscr{O}_V)$. Here $\mathscr{O}_X|_U$ denotes the sheaf on U given by $\mathscr{O}_X|_U(U') = \mathscr{O}_X(U')$ for $U' \subset U$.

Definition 4.5. A *sheafy Riemann surface* is a $\mathbb{C}$-space $(X, \mathscr{F})$ such that there is an open cover $\cup_{i \in I} U_i = X$ and isomorphisms of $\mathbb{C}-$spaces

$$\phi_i : (U_i, \mathscr{F}|_{U_i}) \cong (V_i, \mathscr{O}_{V_i})$$

with $V_i \subset \mathbb{C}$ open.

Proposition 4.6. *The map sending a Riemann surface X to the sheafy Riemann surface $(X, \mathscr{O}_X)$ identifies Riemann surfaces with sheafy Riemann surfaces, and identifies holomorphic maps between Riemann surfaces with $\mathbb{C}$-space morphisms between their associated sheafy Riemann surfaces.*

Proof. The inverse map takes $(X, \mathscr{O}_X)$ to an atlas provided by the definition of a sheafy Riemann surface. Now just check everything: for example to check compatibility of the charts, we need to show that $\tau = \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is holomorphic, but we know that τ identifies the holomorphic functions on these two open subsets of $\mathbb{C}$, and in particular it's composition with the identity map is holomorphic, so τ is holomorphic. $\square$

4.2. **Group actions on Riemann surfaces.** Let X be a Riemann surface, and G a group, with a homomorphism $r : G \to \mathrm{Aut}^{\mathrm{hol}}(X)$ (i.e. if $\gamma \in G$ then $r(\gamma)$ is a biholomorphic map from X to itself.

Definition 4.7. We say that the group G acts *properly* on X if for all compact subsets $A, B \subset X$ the set

$$\{\gamma \in G : r(\gamma)A \cap B \neq \emptyset\}$$

is finite.

In particular for each $x \in X$ the stabiliser G_x is finite.

Exercise 6. The group G acts properly if and only if the map $\alpha : G \times X \to X \times X$ taking (γ, x) to $x, r(\gamma)x$ is *proper*: i.e. when we give G the discrete topology and products the product topology, $\alpha^{-1}(K)$ is compact for any compact subset K of $X \times X$.

Lemma 4.8. *Suppose G acts properly on X. Then for each $x \in X$ there exists a connected open neighbourhood U_x of x with compact closure satisfying*

$$r(\gamma)U_x \cap U_x \neq \emptyset \iff r(\gamma)x = x.$$

Proof. First note that we can find a connected open neighbourhood U of x with compact closure such that $r(\gamma)U \cap U \neq \emptyset$ for only finitely many γ. We do this by taking $A = B$ to be (the pre-image under some chart of) a small closed ball around x in the definition of acting properly (U is then the interior of this closed ball). Let $g_1, ..., g_n$ enumerate the elements of G such that $r(\gamma)U \cap U \neq \emptyset$. We want to show that for each i with $g_i x \neq x$ we can find an open subset $x \in U_i \subset U$ such that $U_i \cap g_i U_i = \emptyset$. We will then set $U_x = \cap U_i$ (or the connected component of x in this intersection, if this intersection is disconnected). By the Hausdorff property of X (so U is also Hausdorff), if $g_i x \neq x$ we can find disjoint open neighbourhoods V_i, V_i' of $x, g_i x$ in U. Since G acts continuously on X we can find an open neighbourhood W_i of x in X such that $g_i W_i \subset V_i'$. We set $U_i = V_i \cap W_i$. Then U_i is disjoint from V_i', yet $g_i U_i \subset V_i'$, so $U_i \cap g_i U_i = \emptyset$. □

Lemma 4.9. *Suppose G acts properly on X. We topologise the set of orbits $G\backslash X$ by saying that a subset of $G\backslash X$ is open if and only if its preimage in X is open. With this definition, the maps $\pi : X \to G\backslash X$ is continuous and open, and the quotient topological space $G\backslash X$ is Hausdorff.*

In fact the above topology is the unique topology such that the quotient map $\pi : X \to G\backslash X$ is continuous and every continuous map of topological spaces $f : X \to Y$ satisfying $f(gx) = f(x)$ for all $g \in G$ factors uniquely (and continuously) through π.

Proof. First we show that π is open (it is obviously continuous). If $U \subset X$ is an open set, then $\pi^{-1}(\pi(U)) = \cup_{g \in G} gU$ is a union of open sets gU, hence it is open, so by definition $\pi(U)$ is open.

Now we show that $G\backslash X$ is Hausdorff. Let Gx, Gy be two distinct points of the quotient $G\backslash X$. Let K_x, K_y be two distinct compact neighbourhoods of x, y (say given by small closed balls with respect to some chart), and denote the interiors by U_x, U_y. We know that $gK_x \cap K_y \neq \emptyset$ only for g in a finite subset $G_0 \subset G$. By shrinking the neighbourhoods K_x, K_y we can assume that y is not in gK_x for any $g \in G_0$.

Let V_y be the open neighbourhood of y given by the intersection $U_y \cap (X \setminus \cup_{g \in G_0} gK_x)$. Now $gU_x \cap V_y = \emptyset$ for all $g \in G$, so $\pi(U_x)$ and $\pi(V_y)$ are disjoint open neighbourhoods of Gx and Gy. □

Note that we are just using the fact that X is a Hausdorff and *locally compact* topological space.

The next lemma tells everything we'll need to know about the structure of these quotient spaces.

Lemma 4.10. *Let G, X be as above, and let $x \in X$. Then there exists an open neighbourhood U_x of x (connected with compact closure) such that $gU_x = U_x$ for all $g \in G_x$ and satisfies*

$$\pi^{-1}(\pi(U_x)) = \coprod_{gG_x \in G/G_x} g(U_x).$$

Proof. First we take U a neighbourhood of x with compact closure such that $gU \cap U \neq \emptyset \iff g \in G_x$. Then we define U_x to be the connected component of x in $\cap_{g \in G_x} gU$ (each $g \in G_x$ maps a connected set containing x to a connected set containing x, so we have $gU_x = U_x$). □

Now for U an open subset in $G\backslash X$, consider the set of holomorphic functions $\mathscr{O}_X(\pi^{-1}(U))$. This set has a natural right action of G, given by

$$f^g : x \mapsto f(gx).$$

If we consider the invariants under the G-action, $\mathscr{O}_X(\pi^{-1}(U))^G$, then we have a set of G-invariant continuous functions on X. By the definition of the quotient topology on $G\backslash X$, this set naturally embeds in $\mathscr{O}^{cts}_{G\backslash X}(U)$. This means that we can make the following definition:

Definition 4.11. We given $G\backslash X$ the structure of a $\mathbb{C}$-space by setting $\mathscr{O}_{G\backslash X}(U) = \mathscr{O}_X(\pi^{-1}(U))^G$ for U an open subset of $G\backslash X$.

It's easy to see that $\mathscr{O}_{G\backslash X}$ is a sheaf on the topological space $G\backslash X$.

Theorem 4.12. *The pair $(G\backslash X, \mathscr{O}_{G\backslash X})$ defines a Riemann surface. The map π is holomorphic, and for $x \in X$, there exist charts around $x, \pi(x)$ such that π is locally given by*

$$z \mapsto z^{n_x}$$

where $n_x = |r(G_x)|$ (moreover, $r(G_x)$ is cyclic of order n_x).

The Riemann surface structure we have defined on $G\backslash X$ satisfies the universal property that every holomorphic map of Riemann surfaces $f : X \to Y$ which satisfies $f(gx) = f(x)$ for all $g \in G$ factors uniquely (and holomorphically) through π.

Proof. We can immediately assume that G is a subgroup of $\mathrm{Aut}(X)$. Let $x \in X$. We are going to define a chart on a neighbourhood of $\pi(x)$. Denote by U the open neighbourhood (connected with compact closure) of x provided by Lemma 4.10. Possibly shrinking U, we can assume that U is biholomorphic to an open subset of $\mathbb{C}$. Denote by V the image $\pi(U)$. Since π is open, this an open neighbourhood of $\pi(x)$. Also, since $\pi^{-1}(V) = \coprod_{gG_x \in G/G_x} g(U)$, we have $\mathscr{O}_{G\backslash X}(V) = \mathscr{O}_X(U)^{G_x}$ (the datum of a G_x invariant function on U is equivalent to a G-invariant function on the disjoint union of the gU). This tells us that it is enough to consider the case where G is a finite group fixing $0 \in X \subset \mathbb{C}$, with X a connected open subset of $\mathbb{C}$ (recall we are interested in the local structure of $G\backslash X$ in a neighbourhood of $\pi(x)$, and here we have mapped x to 0).

Now we claim that there is a biholomorphic map f from a neighbourhood U of 0 in X to the open unit disc D such that $f(0) = 0$, $gU = U$ for all $g \in G$ and for every g,

$f^{-1} \circ g \circ f$ is given by a rotation $z \mapsto \zeta(g)z$ ($\zeta(g)$ a root of unity). In particular, the group G is isomorphic to $\mathbb{Z}/n_x\mathbb{Z}$.

Let's assume this claim for the moment. Then we are reduced to the case where X is the open unit disc and $G \cong \mathbb{Z}/n_x\mathbb{Z}$ acts via $i \cdot z \mapsto \zeta^i z$ with ζ a primitive n_xth root of unity. Now the chart $G\backslash X \to X$ sending Gz to z^{n_x} gives an isomorphism of $\mathbb{C}$-spaces.

Finally we prove the claim. The key point is that for a sufficiently small open disc D_ϵ, centred at 0, in X, the set gD_ϵ is *convex* for every $g \in G$. See the Lemma below for a proof of this.

Then the intersection $U = \bigcap_{g\in G} gD_\epsilon$ is convex, hence simply connected, and moreover $gU = U$ for all $g \in G$. Since U is simply connected (with compact closure), it is biholomorphic to D via a map sending 0 to 0, and now we use the fact that biholomorphic maps from the unit disc to itself, fixing a point, are given by rotations (the Schwarz lemma). $\square$

Remark 4.13. The fact that this works for non-free group actions is special to one-dimensional complex manifolds. An alternative presentation of this material is given by Miranda III.3.

Lemma 4.14. *Let X be an open subset of $\mathbb{C}$, containing 0, and suppose that f is an automorphism of X with $f(0) = 0$. Then there is an $\epsilon \in \mathbb{R}_{>0}$ such that f maps every disc $D_r = \{|z| \le r\}$ with $r < \epsilon$ onto a convex region (of course, we take ϵ small enough so that all the D_r are contained in X).*

Proof. See Farkas-Kra, III.7.7. The region $f(D_r)$ is convex if and only if the curves $C_r = \{f(z) : |z| = r\}$ are all convex. Suppose $\arg(z) + \arg(f'(z))$ is an increasing function of $\arg(z)$ on $\{|z| = r\}$. Then we claim that the curve C_r is convex — this is because the tangent to the curve C_r at $f(z)$ has direction

$$\frac{d}{d\theta} f(z) = izf'(z),$$

where $z = re^{i\theta}$.

So we compute the derivative of $\theta + \arg(f'(re^{i\theta})) = \theta + \mathrm{Re}(\log(-if'(re^{i\theta})))$ with respect to θ, and get $1 + \mathrm{Re}(zf''(z)/f'(z))$. Since $f'(z) \neq 0$ (as f is biholomorphic), this derivative is positive for $|z|$ small. $\square$

Proposition 4.15. *The action of $\Gamma(1)$ on $\mathcal{H}$ is proper.*

Proof. Recall that $\mathrm{Im}(\gamma\tau) = \mathrm{Im}(\tau)/|c\tau + d|^2$. Suppose we have A, B compact subsets of $\mathcal{H}$. We are interested in the set G_0 of $\gamma \in \Gamma(1)$ such that $\gamma A \cap B \neq \emptyset$. Now, since B is compact, the set $\{\mathrm{Im}(\tau) : \tau \in B\}$ is contained in some compact interval $I = [c_1, c_2] \subset \mathbb{R}_{>0}$ (with $c_1 > 0$). So if $\gamma_\tau \in B$ then $\mathrm{Im}(\gamma\tau) \in I$ so we have inequalities

$$\mathrm{Im}(\tau)/c_2 \le |c\tau + d|^2 \le \mathrm{Im}(\tau)/c_1,$$

which imply that there are only finitely many possibilities for the integers c and d (since the real and imaginary parts of τ are also bounded). Suppose that two elements γ, δ of $\Gamma(1)$ have the same c, d. Then a computation shows that

$$\gamma\delta^{-1} = \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

for some $n \in \mathbb{Z}$. Since A and B are compact, there are only finitely many possibilities for n. So we have shown that G_0 is finite. $\square$

Definition 4.16. For Γ a congruence subgroup, we denote by $Y(\Gamma)$ the Riemann surface obtained from the quotient $\Gamma\backslash\mathcal{H}$.

Corollary 4.17. *The map $j : Y(\Gamma(1)) \to \mathbb{C}$ is a biholomorphic map.*

Proof. We showed before that j is a bijection. It is holomorphic since it is induced by a $\Gamma(1)$-invariant holomorphic function on $\mathcal{H}$, and holomorphic bijections are biholomorphisms. $\square$

4.3. Cusps and compactifications.

Definition 4.18. Suppose Γ is a congruence subgroup. Then the set of *cusps* of $Y(\Gamma)$ is defined to be the set of orbits $C_\Gamma := \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ where the action of an element of $\mathrm{SL}_2(\mathbb{Z})$ on $(x : y) \in \mathbb{P}^1(\mathbb{Q})$ is given by

$$\gamma(x : y) = (ax + by : cx + dy).$$

We denote the cusp $\Gamma(1 : 0)$ by ∞. As usual, we think of $\mathbb{P}^1(\mathbb{Q})$ bijecting with $\mathbb{Q} \cup \{\infty\}$ by sending $(x : y)$ to x/y (or ∞ if $y = 0$).

For $s \in C_\Gamma$, we define the *width* of the cusp $s = \Gamma x$ to be the index of the subgroup $\{\pm I\}\Gamma_x$ in the stabiliser $\Gamma(1)_x$. We denote this positive integer by h_s (Exercise: this is independent of the choice of representative x for s).

For example, when $\Gamma = \Gamma(1)$ we have a single cusp, since the action of $\mathrm{SL}_2(\mathbb{Z})$ on $P^1(\mathbb{Q})$ is transitive.

For $\Gamma = \Gamma_0(p)$ there are two cusps, one of width 1 and the other of width p.

If we denote by $\mathcal{H}^*$ the disjoint union $\mathcal{H} \coprod \mathbb{P}^1(\mathbb{Q})$ then we define (first as a set) $X(\Gamma) = \Gamma\backslash cH^* = Y(\Gamma) \coprod C_\Gamma$.

Now we make $\mathcal{H}^*$ into a topological space. We will list a bunch of open sets, and take the topology generated by them. First we let the usual open sets in $\mathcal{H}$ be open in $\mathcal{H}^*$. The sets $U_A = \{\tau \in \mathcal{H} : \mathrm{Im}(\tau) > A\} \cup \{\infty\}$are also declared to be open: they are the preimages of the open discs centred at 0 under the map $\tau \mapsto e^{2\pi i\tau}$).

Finally, we declare to be open sets of the form gU_A for $g \in \Gamma(1)$ — these will be open neighbourhoods of the point $g(1 : 0) \in \mathbb{P}^1(\mathbb{Q})$, and they are regions bounded by circles touching the real line at $g(1 : 0)$ (if g does not stabilise $(1 : 0)$).

We define a topology on the quotient $X(\Gamma)$ as usual, by saying an set is open if and only if its preimage in $\mathcal{H}^*$ is open.

Lemma 4.19. *Let x be an element of $\mathbb{P}^1(\mathbb{Q})$. Then there exists an open neighbourhood U of x in $\mathcal{H}^*$ such that*

$$\Gamma_x := \{g \in \gamma : gx = x\} = \{g \in \Gamma : gU \cap U \neq \emptyset\}.$$

Proof. First we do this for $x = \infty$. Let $A \in \mathbb{R}_{>0}$. We have

$$gU_A = \{\tau \in \mathcal{H} : \mathrm{Im}(g^{-1}\tau) > A\} = \{\tau \in \mathcal{H} : \frac{\mathrm{Im}(\tau)}{|-c\tau + d|^2} > A\}.$$

Since $|-c\tau+d|^2 \geq c^2\mathrm{Im}(\tau)^2$, if $c \neq 0$ we have

$$\tau \in gU_A \implies \mathrm{Im}(\tau) > A\mathrm{Im}(\tau)^2 \implies \mathrm{Im}(\tau) < 1/A$$

so, for large enough A, U and gU_A are disjoint for all g with $c \neq 0$. Now $c = 0$ if and only if $g\infty = \infty$, so we are done.

Now for general x we fix g_0 with $g_0\infty = x$, and take $U = g_0U_A$ for A large enough (as in the above paragraph). □

Proposition 4.20. *Let Γ be a congruence subgroup, then the topological space $X(\Gamma)$ is connected, Hausdorff and compact.*

Proof. First we check that $\mathcal{H}^*$ is connected, since $\mathcal{H}$ is connected, and each element of $\mathbb{P}^1(\mathbb{Q})$ has a base of open neighbourhoods having non-trivial intersection with $\mathcal{H}$. It follows that the continuous image $X(\Gamma)$ of $\mathcal{H}^*$ is connected.

To show that $X(\Gamma)$ is Hausdorff, first recall that $X(\Gamma(1))$ is homeomorphic to the Riemann sphere (elementary way to see this is to stare at the funamental domain). For general Γ we know we can separate points of $Y(\Gamma)$. Suppose we have a cusp s and a point $y \in Y(\Gamma)$. The image of s in $X(\Gamma(1))$ is ∞ and the image of y in $X(\Gamma(1))$ is in $Y(\Gamma(1))$. Since the image of these points can be separated by open neighbourhoods, s and y can be separated by the pre-images of these opens. The fact that two cusps can be separated by open neighbourhoods follows from Lemma 4.19.

For compactness, first note that the extended fundamental domain $\overline{\mathscr{F}}^* = \overline{\mathscr{F}} \cup \{\infty\}$ is a compact subset of $\mathcal{H}^*$. Now $X(\Gamma)$ is a continuous image of the finite union of compact sets $\cup_{\Gamma\gamma \in \Gamma\backslash\Gamma(1)} \gamma\overline{\mathscr{F}}^*$, so it is compact. □

Lemma 4.21. *Let Γ be a congruence subgroup. There exist open neighbourhoods U_s of each cusp s in $X(\Gamma)$ such that the U_s are pairwise disjoint, and are all homeomorphic to the unit disc D, via maps sending s to 0 which are biholomorphisms from $U_s\backslash\{s\} \subset Y(\Gamma)$ to the punctured unit disc D^*.*

Proof. Let π be the quotient map $\mathcal{H}^* \to X(\Gamma)$. It follows from Lemma 4.19 that for A large enough, if we choose $g_s \in \Gamma(1)$ with $g_s\infty = x$ and $\Gamma x = s$ for each cusp s, then $U(g_s, A) = \pi(g_sU_A)$ gives a pairwise disjoint set of open neighbourhoods of the cusps.

Set $U^* = g_sU_A$ and denote by U the intersection $U^* \cap \mathcal{H}$. The natural inclusion $U^* \to \mathcal{H}^*$ induces a map $\Gamma_x\backslash U^* \to X(\Gamma)$. For A large enough, Lemma 4.19 tells us that this map is injective. Its image is the open neighbourhood $U(g_s, A)$ of the cusp s. Recall that a holomorphic function on $V = U(g_s, A)\backslash s$ is by definition a Γ-invariant holomorphic function on $\pi^{-1}(V)$ which is the same thing as a Γ_x-invariant holomorphic function on U.

Let h_s be the width of the cusp s. Then

$$\{\pm I\}\Gamma_x = \{\pm I\}g_s \begin{pmatrix} 1 & h_s\mathbb{Z} \\ 0 & 1 \end{pmatrix} g_s^{-1}$$

and the map sending $\tau \mapsto e^{2\pi i(g_s^{-1}\tau)/h_s}$ for $\tau \in \Gamma_x\backslash U$ and x to 0 sends $\Gamma_x\backslash U^*$ homeomorphically to an open disc.

Rescaling gives a homeomorphism from U^*/Γ_x to the unit disc, and so we get a homeomorphism from $U(g_s, A)$ to the unit disc. Restricting this map to $U(g_s, A)\backslash s$ gives a biholomorphism to the punctured unit disc, since it is a homeomorphism induced by a Γ_x-invariant holomorphic function on U. □

Definition 4.22. We define a Riemann surface structure on $X(\Gamma)$ extending the Riemann surface structure on $Y(\Gamma)$ by adding the charts on the neighbourhoods of the cusps given by Lemma 4.21.

Since a continuous function on the unit disc which is holomorphic on the punctured unit disc is holomorphic everywhere, we can define the Riemann surface structure sheaf theoretically by saying that a continuous function on an open subset U of $X(\Gamma)$ is holomorphic if and only if its restriction to $U \cap Y(\Gamma)$ is holomorphic. Equivalently, we say such a function is holomorphic if and only if it defines a holomorphic function of D when we apply the homeomorphisms of Lemma 4.21.

Note that it follows from the proof of Lemma 4.21 that the map $X(\Gamma) \to X(\Gamma(1))$ has the form $z \mapsto z^{h_s}$ with respect to some charts around s and ∞.

5. Differentials and divisors on Riemann surfaces

5.1. Meromorphic differentials.

Definition 5.1. For $U \subset \mathbb{C}$ open, $n \in \mathbb{Z}_{>0}$ we define the space of meromorphic differentials of degree n on U by

$$\Omega^{\otimes n}(U) := \{f(z)dz^n : f \text{ meromorphic on } U\}.$$

For $\phi : U_1 \to U_2$ holomorphic, define

$$\phi^* : \Omega^{\otimes n}(U_2) \to \Omega^{\otimes n}(U_1)$$

by $\phi^*(f(z_2)(dz_2)^n) = f(\phi(z_1))(\phi'(z_1))^n(dz_1)^n$.

So $\Omega^{\otimes n}(U)$ is a $\mathbb{C}$-vector space, isomorphic to the space of meromorphic functions on U (but note that the pullback by ϕ^* of a differential is not the same as the pullback of a function).

Definition 5.2. Suppose X is a Riemann surface. Suppose we have two open subsets U_1, U_2 of X, with charts $\phi_i : U_i \cong D_i \subset \mathbb{C}$. Denote by τ_{ij} the transition functions $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \cong \phi_j(U_i \cap U_j)$. Then a meromorphic differential (of degree n) on X is a rule sending charts $\phi : U \to D$ on X to meromorphic differentials of $\omega(\phi)$ of degree n on D, such that for any two charts ϕ_1, ϕ_2 the differentials $\omega(\phi_1)$ and $\omega(\phi_2)$ are *compatible*: i.e. $\tau_{ij}^*(\omega(\phi_j)|_{\phi_j(U_i \cap U_j)}) = \omega(\phi_i)|_{\phi_i(U_i \cap U_j)}$ for $i, j \in \{1, 2\}$.

The set of differentials on X has an obvious structure of a $\mathbb{C}$-vector space.

Remark 5.3. By sending an open subset U of X to the $\mathbb{C}$-vector space of degree n meromorphic differentials on U, we can define a sheaf $\Omega_X^{\otimes n}$ on X. To check the sheaf property you need the following lemma:

Lemma 5.4. *Let X be a Riemann surface, and $\mathcal{A}$ an atlas on X. Suppose we have a collection of compatible meromorphic differentials for just the charts in $\mathcal{A}$. Then there exists a unique meromorphic differential on X extending the given meromorphic differentials on the charts.*

Proof. Given any chart $\phi : U \to D$ on X we can define a meromorphic differential on D by appropriate tranformations of the meromorphic differentials associated to charts in the atlas: if $\phi_i : U_i \to D_i$ is a chart then we get a biholomorphism $\phi(U \cap U_i) \cong \phi_i(U_i \cap U)$ and pulling back a differential on $\phi_i(U_i \cap U)$ gives a differential on $\phi(U \cap U_i)$. Doing this for all i gives a collection of compatible differentials on an open cover of D, which glue to the desired differential on D. $\square$

It is now straightforward, given a holomorphic map of Riemann surfaces $\phi : X \to Y$ to define the pullback map $\phi^* : \Omega_Y^{\otimes n}(V) \to \Omega_X^{\otimes n}(\phi^{-1}(V))$ for any open $V \subset Y$.

5.2. **Meromorphic differentials and meromorphic forms.** In this section we let Γ be a congruence subgroup. Recall that we have a holomorphic map $\pi : \mathcal{H} \to Y(\Gamma) \hookrightarrow X(\Gamma)$.

Definition 5.5. Suppose ω is an element of $\Omega^{\otimes}k(X(\Gamma))$. We denote by f_ω the meromorphic function on $\mathcal{H}$ given by $\pi^*\omega = f_\omega(\tau)(d\tau)^k$.

Theorem 5.6. *Suppose $\omega \in \Omega^{\otimes k}(X(\Gamma))$. Then f_ω is a meromorphic form of weight $2k$ and level Γ. Moreover, the map $\omega \mapsto f_\omega$ is an isomorphism of $\mathbb{C}$-vector spaces from $\Omega^{\otimes k}(X(\Gamma))$ to the space of meromorphic forms of weight $2k$ and level Γ.*

Proof. First we check that f_ω is weakly modular of weight $2k$ and level Γ. Let $\gamma \in \Gamma$. We have a biholomorphism γ from $\mathcal{H}$ to $\mathcal{H}$ which descends to the identity map on $Y(\Gamma)$. Consider the meromorphic differential on $\mathcal{H}$ given by $\gamma^*\pi^*\omega$. On the one hand, this is equal to $(\pi \circ \gamma)^*(\omega) = \pi^*\omega$, since $\pi \circ \gamma = \pi$. On the other hand, we have

$$\gamma^*(f_\omega(\tau)(d\tau)^k) = f_\omega(\gamma\tau)\left(\frac{d\gamma\tau}{d\tau}\right)^k (d\tau)^k = f_\omega(\gamma\tau)(c\tau+d)^{-2k}(d\tau)^k$$

so we deduce that $f_\omega(\gamma\tau)(c\tau+d)^{-2k} = f(\tau)$ and f_ω is indeed weakly modular of weight $2k$ and level Γ.

Next we check that f_ω is meromorphic at the cusps. Let $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. The map $\alpha : \mathcal{H}^* \to \mathcal{H}^*$ descends to a biholomorphism $\alpha : X(\alpha^{-1}\Gamma\alpha) \cong X(\Gamma)$. It follows from a calculation as above that $f_{\alpha^*\omega} = f_\omega|_{\alpha,k}$. So it suffices to show that f_ω is meromorphic at ∞. For large enough A, the image of $U_A \cup \{\infty\}$ in $X(\Gamma)$ is biholomorphic to an open disc via $\tau \mapsto e^{2\pi i\tau/h} = q_h$ so the fact that f_ω is meromorphic at ∞ follows immediately from the fact that ω is a meromorphic differential: if the differential on this chart in a neighbourhood of ∞ is $g(q_h)(dq_h)^k$ then f_ω satisfies

$$f_\omega(\tau) = \left(\frac{2\pi i q_h}{h}\right)^k g(q_h).$$

The map $\omega \mapsto f_\omega$ is clearly $\mathbb{C}$-linear. We show it is an isomorphism by writing down an inverse. For f a meromorphic form of weight $2k$ and level Γ we want to define a meromorphic differential $\omega(f)$ on $X(\Gamma)$ which pulls back to $f(\tau)(d\tau)^k$ on $\mathcal{H}$.

Let $x \in \mathcal{H}$. Then, from the proof of Theorem 4.12 we know that there are charts from neighbourhoods U of x and V of $\pi(x)$ to the unit disc such that π is the map $z \mapsto z^{n_x}$ in this coordinate and Γ_x acts via $z \mapsto \zeta^i z$, with ζ a primitive n_xth root of unity.

Since $f(\tau)(d\tau)^k$ defines a Γ_x-invariant meromorphic differential on U, in the new coordinate we have a meromorphic differential $g(z)(dz)^k$ on the open unit disc, such that $g(\zeta^i z)(d(\zeta^i z))^k = g(z)(dz)^k$ for all i. Therefore we have $g(\zeta^i z) = \zeta^{-ik} g(z)$ for all i, so the function $z^k g(z)$ is Γ_x-invariant and is equal to $h(z^{n_x})$ for a meromorphic function h on the open unit disc.

Now we define a meromorphic differential on the open unit disc by

$$\omega = (n_x z)^{-k} h(z)(dz)^k.$$

This pulls back under $z \mapsto z^{n_x}$ to

$$(n_x z^{n_x})^{-k} h(z^{n_x})(n_x z^{n_x - 1})^k (dz)^k = z^{-k} h(z^{n_x})(dz)^k = g(z)(dz)^k,$$

so we define $\omega(f)$ on the chart from V to the open unit disc to be given by ω — by construction, it pulls back to $f(\tau)(d\tau)^k$ on the neighbourhood U of x.

Finally, we need to define our meromorphic differential in neighbourhoods of the cusps. It is enough to consider the cusp $\Gamma\infty$, since for a general cusp $s = \Gamma x$ with $x = \alpha\infty$ we can define a meromorphic differential ω in a neighbourhood of s by taking the meromorphic differential $\omega(f|_{\alpha,2k})$ defined in a neighbourhood of $(\alpha^{-1}\Gamma\alpha)\infty$ in $X(\alpha^{-1}\Gamma\alpha)$ and pulling back by the biholomorphism $X(\Gamma) \cong X(\alpha^{-1}\Gamma\alpha)$.

Recall that associated to $f(\tau)$ we have a meromorphic function on the unit disc, extending the holomorphic function F on the punctured unit disc defined by $F(e^{2\pi i\tau/h}) = f(\tau)$. Here h is the width of the cusp ∞.

Recall that the chart of Lemma 4.21 is also given in terms of the biholomorphism $\tau \mapsto e^{2\pi} i\tau/h = q_h$ from U_A/Γ_∞ to an open disc. We define a meromorphic differential on this disc

$$\omega = \left(\frac{2\pi i q_h}{h}\right)^{-k} F(q_h)(dq_h)^k.$$

Then ω pulls back to the meromorphic differential $f(\tau)(d\tau)^k$ on U_A under the map $\tau \mapsto e^{2\pi} i\tau/h$.

We just have to check that the meromorphic differentials we have defined are all compatible. However, they all pull back to restrictions of the same meromorphic differential, $f(\tau)(d\tau)^k$ on $\mathcal{H}$, so this follows from the next lemma. □

Lemma 5.7. *Suppose $\pi : X \to Y$ is a non-constant morphism of Riemann surfaces, with Y connected. Then the map π^* is an injection from $\Omega^{\otimes n}(Y)$ to $\Omega^{\otimes n}(X)$.*

Proof. We can assume that X and Y are open subsets of $\mathbb{C}$, with Y connected. So we have $\omega = f(z)(dz)^{\otimes n}$ and $f(\pi(z))(\pi'(z))^n = 0$ for all $z \in X$. Since π is non-constant, the zeroes of π' are discrete, and so $f(\pi(z)) = 0$ on an open subset of X. Hence f is zero on an open subset of Y (since π is an open map by the open mapping theorem), and f is therefore zero. □

There is a natural definition of the order of vanishing of a meromorphic differential at a point:

Definition 5.8. Let X be a Riemann surface, and ω a meromorphic differential on X. Let $x \in X$ be a point. Then we define $v_x(\omega)$ to be the order of vanishing of $f(z)$ at $z = z_0$, where ω is given by $f(z)(dz)^n$ on a chart defined on a neighbourhood of x sending x to z_0.

It's an exercise to check that v_x is well-defined (i.e. it doesn't matter what chart we choose). Recall that for $f \in M_k(\Gamma(1))\backslash\{0\}$ we previously defined ord_x for $x \in X(\Gamma(1))$ — this was just the order of vanishing of f at $x \in Y(\Gamma(1))$, or the natural order of vanishing of f at ∞ defined in terms of its q-expansion.

Exercise 7. By considering the proof of Theorem 5.6, show that for $f \in M_{2k}(\Gamma(1))$ non-zero we have

$$\begin{aligned} v_\infty(\omega_f) &= \mathrm{ord}_\infty(f) - k \\ v_x(\omega_f) &= \frac{\mathrm{ord}_x(f) - k(n_x - 1)}{n_x} \end{aligned}$$

for $x \in Y(\Gamma)$.

Deduce that the equality

$$\mathrm{ord}_\infty(f) + \sum_{x \in Y(\Gamma)} \frac{1}{n_x} \mathrm{ord}_x(f) = \frac{2k}{12}$$

of Proposition 2.18 is equivalent to the equality

$$\sum_{x \in X(\Gamma)} v_x(\omega_f) = -2k.$$

The last equality is a statement that the degree of the divisor associated to ω_f is equal to $-2k = (2g-2)k$, where $g = 0$ is the genus of $X(\Gamma(1))$.

5.3. **Divisors.** We assume X is a compact connected Riemann surface throughout this section.

Definition 5.9. Recall the definition of the group of divisors on a Riemann surface X: it is the free Abelian group generated by the points of X, i.e. formal sums $\sum_{x \in X} a_x[x]$ with $a_x = 0$ for almost all x. A divisor D has a degree $\deg(D) = \sum a_x$, where $D = \sum_{x \in X} a_x[x]$.

We say a divisor is *effective* if $a_x \geq 0$ for all x, and write $D \geq 0$ if D is effective.

For f a non-zero meromorphic function on X we define the *divisor of* f to be

$$\mathrm{div}(f) = \sum_{x \in X} v_x(f)[x]$$

where v_x is the order of vanishing at x. Note that since X is compact this sum is finite.

Similarly, we define $\mathrm{div}(\omega)$ for a meromorphic differential ω.

For D a divisor on X, we also define a $\mathbb{C}$-vector space

$$L(D) = \{f \text{ a non-zero meromorphic function on } X : \mathrm{div}(f) + D \geq 0\} \cup \{0\}.$$

The vector space structure is given by scalar multiplication and addition of meromorphic functions.

In fact $L(D)$ is the group of sections of a sheaf on X.

Definition 5.10. For $D = \sum a_x[x]$ a divisor on X and U an open subset of X, denote by $D|_U$ the divisor on U given by $\sum_{x \in U} a_x[x]$ and define

$$\mathscr{O}_X(D)(U) = \{f \text{ a non-zero meromorphic function on } U : \operatorname{div}(f) + D|_U \geq 0\} \cup \{0\}.$$

Note that we're abusing notation a bit, since U is not necessarily compact, so $\operatorname{div}(f)$ might be an infinite formal sum.

We can also denote the sheaf of holomorphic differentials on X by Ω^1_X (it is the subsheaf of the meromorphic differentials of degree 1 given by demanding that in the local expressions $f(z)dz$, f is *holomorphic*).

Exercise 8. Supposing ω_0 is a non-zero meromorphic differential of degree 1 on X, then we have

$$\Omega^1_X(U) \cong \mathscr{O}_X(\operatorname{div}(\omega_0))(U)$$

via the map $\omega \mapsto \omega/\omega_0$.

Remark 5.11. Every compact Riemann surface has a non-constant meromorphic function. In fact, for the Riemann surfaces $X(\Gamma)$ this is easy to see: we have the function $X(\Gamma) \to X(\Gamma(1)) \cong \mathbb{P}^1$. We obtain a non-zero meromorphic differential by pulling back a non-zero meromorphic differential on $\mathbb{P}^1$.

The Riemann-Roch theorem:

Theorem 5.12. *Let X be a compact connected Riemann surface, and let D be a divisor on X. Denote by g the genus of X. Denote by K the divisor* $\operatorname{div}(\omega_0)$ *for a non-zero meromorphic differential of degree* 1 *on X. Then*

$$\dim L(D) - \dim L(K - D) = \deg(D) + 1 - g.$$

Remark 5.13. By Serre duality we can write the above equality as

$$\dim H^0(X, \mathscr{O}_X(D)) - \dim H^1(X, \mathscr{O}_X(D)) = \deg(D) + 1 - g$$

so this is an Euler characteristic formula for the cohomology of the sheaf $\mathscr{O}_X(D)$.

Corollary 5.14. *We have* $\dim H^0(X, \Omega^1_X) = g$ *and* $\deg(K) = 2g - 2$.

Proof. Set $D = 0$ and $D = K$. □

Lemma 5.15. *Suppose $\omega \in \Omega^{\otimes n}(X)$. Then* $\operatorname{div}(\omega)$ *has degree* $(2g-2)n$.

Proof. Let ω_0 be a non-zero meromorphic differential of degree one. If ω_0 is locally given by $f(z)dz$ it is easy to check that the local expressions $f(z)^n(dz)^n$ define a non-zero meromorphic differential of degree n, which we denote by ω_0^n. Now ω/ω_0^n defines a meromorphic function, whose associated divisor has degree 0, so the degree of $\operatorname{div}(\omega)$ is n times the degree of $\operatorname{div}(\omega_0)$ which is $(2g-2)n$. □

5.4. **The genus of modular curves.** We can compute the genus of modular curves using the Riemann-Hurwitz formula:

Theorem 5.16. *Suppose $f : X \to Y$ is a holomorphic map between compact connected Riemann surfaces. Then*

$$2 - 2g(X) = \deg(f)(2 - 2g(Y)) - \sum_{x \in X}(e_x - 1).$$

Here $g()$ denotes the genus and e_x denotes the ramification index of the map f at the point x (i.e. locally around x, f looks like $z \mapsto z^{e_x}$).

Our modular curves $X(\Gamma)$ come equipped with maps to $X(\Gamma(1)) \cong \mathbb{P}^1$, so we apply Riemann-Hurwitz to these maps.

Set $r_2 = |\{x \in Y(\Gamma) : n_x = 2\}|$ and $r_3 = |\{x \in Y(\Gamma) : n_x = 3\}|$. As before, we set r_∞ to be the number of cusps of $X(\Gamma)$.

Theorem 5.17. *We have*

$$g(X(\Gamma)) = 1 + \frac{[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]}{12} - \frac{r_2}{4} - \frac{r_3}{3} - \frac{r_\infty}{2}.$$

Proof. Let $f : X(\Gamma) \to X(\Gamma(1))$ be the natural map. First note that $\deg(f) = [\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]$. We denote this integer by d for the rest of the proof. Set $g = g(X(\Gamma))$.

Now we compute some ramification indexes. Let $x \in Y(\Gamma)$ with image $f(x) \in Y(\Gamma(1))$. Recall that the quotient map $\mathcal{H} \to Y(\Gamma)$ looks like $z \mapsto z^{n_x}$ around a pre-image of x, whilst the map $\mathcal{H} \to Y(\Gamma(1))$ looks like $z \mapsto z^{n_{f(x)}}$. This implies that the map f looks like $z \mapsto z^{n_{f(x)}/n_x}$, so we have $e_x = n_{f(x)}/n_x$.

For a cusp s a similar computation shows that we have $e_s = h_s$, the width of s.

Now Riemann-Hurwitz says that

$$2 - 2g = 2d - \sum_{x \in X(\Gamma)}(e_x - 1) = 2d - \sum_{f(x)=i}(e_x - 1) - \sum_{f(x)=\omega}(e_x - 1) - \sum_{f(x)=\infty}(e_x - 1).$$

For $P = i$ or ω we have $\sum_{f(x)=P}(e_x - 1) = (n_P - 1)(|f^{-1}(P)| - r_{n_P})$. On the other hand, we have $d = \sum_{f(x)=P} e_x = n_P(|f^{-1}(P)| - r_{n_P}) + r_{n_P}$. So we deduce that

$$\sum_{f(x)=P}(e_x - 1) = \frac{n_P - 1}{n_P}(d - r_{n_P}).$$

Therefore we have

$$2 - 2g = 2d - \frac{1}{2}(d - r_2) - \frac{2}{3}(d - r_{n_3}) - d + r_\infty$$

which rearranges to give the desired result. □

Note that it follows from this result that

$$2g - 2 + \frac{r_2}{2} + \frac{2r_3}{3} + r_\infty = \frac{d}{6} > 0.$$

5.5. **Riemann-Roch and dimension formulae.** Now we have everything we need to compute the dimensions of the $\mathbb{C}$-vector spaces $M_{2k}(\Gamma)$.

Recall that we have proved that meromorphic forms of weight $2k$ and level Γ correspond to meromorphic differentials of degree k on $X(\Gamma)$. We want to identify the image of the subspace of modular forms.

Suppose f is a meromorphic form and $\omega(f)$ its associated differential. It follows just as in Exercise 7 that for $x \in Y(\Gamma)$ we have

$$v_x(\omega(f)) = \frac{\mathrm{ord}_x(f) - k(n_x - 1)}{n_x}$$

and for $s = \Gamma x$ a cusp of $X(\Gamma)$ with $\alpha x = \infty$ ($\alpha \in \Gamma(1)$) we have

$$v_s(\omega(f)) = \mathrm{ord}_\infty(f|_{\alpha,2k}) - k.$$

Here the order of vanishing of $f|_{\alpha,2k}$ at ∞ is in terms of the variable q_{h_s}, where h_s is the width of the cusp s. Note that $f|_{\alpha,2k}$ has a Fourier expansion in the variable q_{h_s} because its weight is even.

We deduce the following

Proposition 5.18. *Suppose f is a meromorphic form of weight $2k$ and level Γ, with associated differential $\omega(f)$. Then f is a modular form if and only if $n_x v_x(\omega(f)) + k(n_x - 1) \geq 0$ for all $x \in Y(\Gamma)$ and $v_s(\omega(f)) + k \geq 0$ for all cusps s.*

The modular form f is cuspidal if and only if we moreover have $v_s(\omega(f)) + k - 1 \geq 0$ for all cusps s.

Note that the first condition is equivalent to

$$v_x(\omega(f)) + \lfloor \frac{k(n_x - 1)}{n_x} \rfloor \geq 0.$$

Definition 5.19. Let ω_0 be a non-zero differential on $X(\Gamma)$, set $K = \mathrm{div}(\omega_0)$ and define divisors

$$D(k) = kK + k \sum_{s \in X(\Gamma) \setminus Y(\Gamma)} [s] + \sum_{x \in Y(\Gamma)} \lfloor \frac{k(n_x - 1)}{n_x} [x]$$

and

$$D_c(k) = kK + (k-1) \sum_{s \in X(\Gamma) \setminus Y(\Gamma)} [s] + \sum_{x \in Y(\Gamma)} \lfloor \frac{k(n_x - 1)}{n_x} [x]$$

on $X(\Gamma)$.

Theorem 5.20. *We have isomorphisms $M_{2k}(\Gamma) \cong L(D(k))$ and $S_{2k}(\Gamma) \cong L(D_c(k))$ given by $f \mapsto \omega(f)/\omega_0^k$.*

Proof. This follows immediately from Proposition 5.18. $\square$

Corollary 5.21. *Denote by r_∞ the number of cusps of $X(\Gamma)$, and denote by g the genus of $X(\Gamma)$. Then we have, for $k \geq 1$*

$$\dim M_{2k}(\Gamma) = kr_\infty + \sum_{x \in Y(\Gamma)} \lfloor \frac{k(n_x - 1)}{n_x} \rfloor + (2k-1)(g-1)$$

and for $k \geq 2$

$$\dim S_{2k}(\Gamma) = (k-1)r_\infty + \sum_{x \in Y(\Gamma)} \lfloor \frac{k(n_x - 1)}{n_x} \rfloor + (2k-1)(g-1).$$

We have $\dim M_0(\Gamma) = 1$, $\dim S_0(\Gamma) = 0$ *and* $\dim S_2(\Gamma) = g$.

Proof. This all follows from Riemann-Roch and the fact that a holomorphic function on a compact connected Riemann surface is constant. We also use the observation that

$$2g - 2 + \frac{r_2}{2} + \frac{2r_3}{3} + r_\infty = \frac{d}{6} > 0.$$

□

Recall that we used the fact that $\dim(M_2(\Gamma_0(4))) = 2$ a few lectures ago. We can now prove this:

Corollary 5.22. *Let* $k \geq 0$. *We have* $\dim M_{2k}(\Gamma_0(4)) = k + 1$.

Proof. We have $[\Gamma(1) : \Gamma_0(4)] = 6$, so $g = 1 + \frac{1}{2} - \frac{r_2}{4} - \frac{r_3}{3} - \frac{r_\infty}{2}$. We also have two forms $G_2^{(2)}$ and $G_2^{(4)}$ which give linearly independent elements of the quotient vector space $M_2(\Gamma_0(4))/S_2(\Gamma_0(4))$ (see this by looking at the constant terms of their q-expansions at the cusps), so $\dim M_2(\Gamma) - \dim S_2(\Gamma) = r_\infty - 1 \geq 2$.

So $g \leq -\frac{r_2}{4} - \frac{r_3}{3}$, which implies that $g = r_2 = r_3 = 0$ and $r_\infty = 3$. Now apply the dimension formula.

In fact, it's probably easier just to determine the cusps of $X(\Gamma_0(4))$, which immediately gives $r_\infty = 3\ldots$ □

6. Hecke operators

In this section we will define Hecke operators on spaces of modular forms of level $\Gamma_1(N)$. To give the cleanest description, we begin by giving an alternative description of modular forms in terms of functions on lattices.

6.1. Modular forms and functions on lattices.

Definition 6.1. A *lattice* in $\mathbb{C}$ is a $\mathbb{Z}$-module $L \subset \mathbb{C}$ generated by two elements of $\mathbb{C}$ which are linearly independent over $\mathbb{R}$.

For $N > 1$ a $\Gamma_1(N)$-*level structure* on a lattice $L \subset \mathbb{C}$ is a point $t \in \mathbb{C}/L$ of exact order N (i.e. a point of the elliptic curve $\mathbb{C}/L$ of exact order N).

Denote the set $\{(L,t)\}$ comprising pairs of lattices with a $\Gamma_1(N)$-level structure by $\mathcal{L}_N$.

Suppose $k \in \mathbb{Z}$ and F is a function from $\mathcal{L}_N$ to $\mathbb{C}$. We say that F has *weight* k if $F(\lambda L, \lambda t) = \lambda^{-k} F(L,t)$ for all $(L,t) \in \mathcal{L}_N$ and $\lambda \in \mathbb{C}^\times$.

Remark 6.2. For example, the function $G_k(L) = \sum_{0 \neq l \in L} l^{-k}$ for $k > 2$ even is a function of weight k on $\mathcal{L}_1$. Note that $G_k(\mathbb{Z}\tau \oplus \mathbb{Z}) = G_k(\tau)$ where $G_k(\tau)$ is previously defined usual Eisenstein series.

Denote by M the set of pairs $\omega = (\omega_1, \omega_2)$ of elements of $\mathbb{C}^\times$ such that $\omega_1/\omega_2 \in \mathcal{H}$. To such a pair we can associate a lattice $L(\omega_1, \omega_2) = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with $\Gamma_1(N)$-level structure $t(\omega_1, \omega_2) = \omega_2/N + L(\omega_1, \omega_2)$. This defines a surjective map from M to $\mathcal{L}_N$. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on M by sending (ω_1, ω_2) to $(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)$.

Lemma 6.3. *The map $M \to \mathcal{L}_N$ identifies $\mathcal{L}_N$ with the quotient of M by the action of $\Gamma_1(N)$.*

Proof. We first check that the map is surjective. Suppose $(L, t) \in \mathcal{L}_N$.

Let ω_1', ω_2' be any basis for L. We have

$$t = \frac{1}{N}(a\omega_1' + b\omega_2') + L$$

where $\gcd(a, b, N) = 1$. We can then find a', b', congruent to a, b mod N, such that a', b' are coprime. Now set $\omega_2 = a'\omega_1' + b'\omega_2'$. Since a' and b' are coprime, we have a basis ω_1, ω_2 for L, and $t = \frac{\omega_2}{N} + L$. If ω_1/ω_2 is not in $\mathcal{H}$, replace ω_1 with $-\omega_1$.

To show that this map identifies $\mathcal{L}_N$ with the quotient of M by $\Gamma_1(N)$, suppose we have two elements $(\omega_1, \omega_2), (\omega_1', \omega_2')$ of M with the same image in $\mathcal{L}_N$. In particular, we have $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with

$$\gamma \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}$$

the two elements of M span the same lattice L. Moreover, since $\omega_2'/N = \omega_2/N \bmod L$, the matrix γ lies in $\Gamma_1(N)$. □

We also have an action of $\lambda \in \mathbb{C}^\times$ on M by mapping (ω_1, ω_2) to $(\lambda\omega_1, \lambda\omega_2)$, so we can define a notion of weight k for complex functions on M. The quotient of M by this action can be identified with $\mathcal{H}$ via the map $(\omega_1, \omega_2) \mapsto \omega_1/\omega_2$. This identifies $\mathcal{L}_N/\mathbb{C}^\times$ with the quotient $\Gamma_1(N)\backslash\mathcal{H}$.

The left action of $\mathrm{GL}_2^+(\mathbb{R})$ on M induces a right action on functions on M, by setting $\tilde{F} \cdot \gamma(\omega) = \tilde{F}(\gamma\omega)$.

On making the above observations, there is a natural way to pass between functions on $M, \mathcal{L}_N$ and $\mathcal{H}$. Given $F : \mathcal{L}_N \to \mathbb{C}$ of weight k we first define $\tilde{F} : M \to \mathbb{C}$ by $\tilde{F}(\omega) = F(L(\omega), t(\omega))$. Then we define $f(\tau) = \tilde{F}(tau, 1)$ for $\tau \in \mathcal{H}$.

Proposition 6.4. *Let $k \in \mathbb{Z}$. The above association of F with $\tilde{F}$ and f gives a bijective correspondence between the following sets of complex-valued functions:*

(1) *functions $F : \mathcal{L}_N \to \mathbb{C}$ of weight k*
(2) *functions $\tilde{F} : M \to \mathbb{C}$ of weight k which are invariant under the action of $\Gamma_1(N)$*
(3) *functions $f : \mathcal{H} \to \mathbb{C}$ which are invariant under the slash operator $|_{\gamma,k}$ for $\gamma \in \Gamma_1(N)$*

Proof. Exercise. □

Now we say that a function F on $\mathcal{L}_N$ of weight k is weakly modular/a modular form/a cusp form if the associated function f on $\mathcal{H}$ is.

6.2. Hecke operators.

Definition 6.5. Suppose F is a function $\mathcal{L}_N \to \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 1}$. Then we define a function $T_n F$ by

$$T_n F(L,t) = \frac{1}{n}\sum_{L'} F(L',t)$$

where the sum is over lattices $L' \supset L$ with index $[L' : L] = n$ such that $t + L'$ is a point of exact order N in $\mathbb{C}/L'$.

For n coprime to N, we also define

$$T_{n,n}F(L,t) = \frac{1}{n^2}F(\frac{1}{n}L,t).$$

Proposition 6.6. *We have the following identities:*

(1) *if m and n are coprime, then $T_m \circ T_n = T_{mn}$*
(2) *if p is prime and divides N and $n \geq 1$ then $T_{p^n} = T_p^n$*
(3) *for p prime and coprime to N, $n \geq 1$, $T_{p^n} \circ T_p = T_{p^{n+1}} + pT_{p^{n-1}} \circ T_{p,p}$*
(4) $T_n \circ T_{m,m} = T_{m,m} \circ T_n$
(5) $T_{m,m} \circ T_{n,n} = T_{mn,mn}$.

Proof. The last two properties are easy to check, and are left to the reader. Now we consider the first claim. Let $(L,t) \in \mathcal{L}_N$. We observe that $T_{mn}(L,t)$ is a sum over lattices L'' containing L with index mn, such that t still has exact order N when reduced modulo L''. Since m and n are coprime, there is a unique lattice L' such that

$$L \subset L' \subset L''$$

and L has index n in L'. Indeed, L'/L is the unique subgroup of L''/L of order n. Clearly t has exact order N when reduced modulo L''.

Conversely, given

$$L \subset L' \subset L''$$

where L has index n in L' and L' has index m in L'', and $t \in \mathbb{C}/L$ such that t has exact order N modulo L'' we see that $L \subset L''$ has index mn. So we see that the elements of $\mathcal{L}_N$ occuring in $T_{mn}(L,t)$ and $T_m \circ T_n(L,t) = \frac{1}{m}\sum_{L'} T_n(L',t)$ are the same and we have $T_n = T_m \circ T_n$.

Now we consider the second item. By induction, it suffices to show that $T_{p^{n-1}}T_p = T_{p^n}$ for $n \geq 2$. Let $t' = (N/p)t$. Then $T_{p^n}(L,t) = p^{-n}\sum[(L',t)]$ where the summation is over $L' \supset L$ such that $L'/L \subset \frac{1}{p^n}L/L$ has order p^n and does not contain t'. Now we claim that L'/L is cyclic. This is because if it is not cyclic it contains $\frac{1}{p}L/L$ which contains t' (since $pt' = 0$). We may now argue as in the previous part, since a cyclic subgroup of order p^n contains a unique subgroup of order p.

For the third claim, first note that $T_{p^n} \circ T_p(L,t)$, $T_{p^{n+1}}(L,t)$ and $T_{p^{n-1}}T_{p,p}(L,t)$ are all given by linear combinations of lattices containing L with index p^{n+1}. Let L'' be such a lattice. Denote its coefficient in the three terms by a, b, c. Then we want to show that $a = b + pc$. We can immediately observe that $b = 1$. There are now two cases:

(1) $L'' \not\supset \frac{1}{p}L$: this implies that $c = 0$. Now a is the number of lattices L' contained in L'' with index p^n. Such an L' is contained in $L'' \cap \frac{1}{p}L$. Since $L'' \not\supset \frac{1}{p}L$ we actually have $L' = L'' \cap \frac{1}{p}L$ and so $a = 1$ and we are done.

(2) $L'' \supset \frac{1}{p}L$: in this case $c = 1$, and L' as above can be any sublattice of $\frac{1}{p}L$ of index p. So $a = 1 + p$ and we are again done.

□

Corollary 6.7. *The T_n are polynomials in the elements T_p and $T_{p,p}$, for $p|n$.*

Proof. This follows from induction on n. □

Corollary 6.8. *The $\mathbb{C}$-subalgebra $\mathbb{T}$ of $\mathrm{End}(\{F : \mathcal{L}_N \to \mathbb{C}\})$ generated by the T_p and $T_{p,p}$ for p prime is commutative and contains all the T_n and $T_{n,n}$.*

Proof. This follows from the above proposition and corollary. □

The above relations between the T_n and $T_{n,n}$ can be nicely summarised as indentities of formal power series with coefficients in $\mathbb{T}$. For $p|N$ we have (in the ring $\mathbb{T}[[X]]$) an identity

$$\sum_{n=0}^{\infty} T_{p^n} X^n = \frac{1}{1 - T_p X}.$$

For $p \nmid N$ we have

$$\sum_{n=0}^{\infty} T_{p^n} X^n = \frac{1}{1 - T_p X + p T_{p,p} X^2}.$$

If we replace X in these identities with p^{-s} we get

$$\sum_{n=1}^{\infty} T_n n^{-s} = \prod_p \sum_{n=0}^{\infty} T_{p^n} p^{-ns} = \prod_{p|N} \frac{1}{1 - T_p p^{-s}} \prod_{p\nmid N} \frac{1}{1 - T_p p^{-s} + T_{p,p} p^{1-2s}}.$$

Definition 6.9. For $d \in \mathbb{Z}$ coprime to N and $F : \mathcal{L}_N \to \mathbb{C}$ denote by $\langle d \rangle F$ the function defined by $\langle d \rangle F(L, t) = F(L, dt)$. Since t has order N this depends only on the class of d in $(\mathbb{Z}/N\mathbb{Z})^\times$.

Lemma 6.10. *The actions of T_n, $T_{n,n}$ and $\langle d \rangle$ take weight k functions on $\mathcal{L}_N$ to weight k functions on $\mathcal{L}_N$. If F is a weight k function on $\mathcal{L}_N$ then $T_{n,n}F = n^{k-2}\langle n \rangle F$.*

Proof. Left to the reader. □

As a consequence, we have actions of T_n, $T_{n,n}$ and $\langle d \rangle$ on the vector space of functions on $\mathcal{H}$ invariant under $\Gamma_1(N)$ acting by the weight k slash operator. There is a more matrix theoretic description of the Hecke operators, as follows.

Definition 6.11. Let S_n^N be the set of matrices (with integer entries) $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $ad = n$, $a \geq 1$, a coprime to N and $0 \leq b < d$.

Suppose $\sigma \in S_n^N$ and $(L, t) \in \mathcal{L}_N$.Write $L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with $\omega_1/\omega_2 \in \mathcal{H}$ and $t = \omega_2/N$. Then we denote by L_σ the lattice with basis $(\frac{a}{n}\omega_1 + \frac{b}{n}\omega_2, \frac{d}{n}\omega_2)$.

Remark. The choice of ω_1, ω_2 in the above definition is well-defined up to multiplication of the column vector $\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$ by an element of $\Gamma_1(N)$ (this is Lemma 6.3). The lattices L_σ depend on the choice of ω_1, ω_2 but the following lemma shows that the *set* of lattices $\{L_\sigma : \sigma \in S_n^N\}$ is independent of this choice.

Lemma 6.12. *The map $\sigma \mapsto L_\sigma$ is a bijection from S_n^N to the set of lattices $L' \supset L$ with $[L' : L] = n$ such that t has order N when reduced modulo L'.*

Proof. Since $\det(\sigma) = n$ we see that L has index n in L_σ. Since a is coprime to N, ω_2/N still has order N in $\mathbb{C}/L_\sigma$. Conversely, suppose L has index n in a lattice L' and t has order N modulo L'. Then we let a and d be the cardinality of $Y_1 = \frac{1}{n}L/(L' + \frac{1}{n}\mathbb{Z}\omega_2)$ and $Y_2 = \frac{1}{n}\mathbb{Z}\omega_2/L' \cap \frac{1}{n}\mathbb{Z}\omega_2$ respectively.

There is a short exact sequence of abelian groups

$$0 \to Y_2 \to \frac{1}{n}L/L' \to Y_1 \to 0$$

so $ad = n$. Since $t = \omega_2/N$ and $L' \cap \frac{1}{n}\mathbb{Z}\omega_2 = \frac{d}{n}\mathbb{Z}\omega_2 = \frac{1}{a}\mathbb{Z}\omega_2$, the condition that t has order N modulo L' is equivalent to the condition that a is coprime to N. Since $\frac{a}{n}\omega_1$ has image zero in Y_1, there exists $b \in \mathbb{Z}$ such that $\frac{a}{n}\omega_1 + \frac{b}{n}\omega_2 \in L'$. Since $\frac{d}{n}\omega_2 \in L'$, we can find a unique such b in the range $0 \le b < d$. We have now associated a, b, d to L' such that $\frac{a}{n}\omega_1 + \frac{b}{n}\omega_2$ and $\frac{d}{n}\omega_2$ are in L'. Since these elements span a lattice which contains L with index n, they span L'. Now we have constructed a map $L' \mapsto \begin{pmatrix} a & b\\ 0 & d\end{pmatrix}$ which is an inverse to the map $\sigma \mapsto L_\sigma$. □

Proposition 6.13. *The actions of T_n, $T_{n,n}$ and $\langle d\rangle$ preserve the spaces $M_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N))$.*

Proof. We saw above that $T_{n,n}F = n^{k-2}\langle n\rangle F$ for weight k functions F, so it is enough to consider the operators $\langle d\rangle$ and T_n.

First we consider $\langle n\rangle$, for n coprime to N. Let $f \in M_k(\Gamma_1(N))$, with associated weight k function F on $\mathcal{L}_N$. Let $\sigma_n = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma(1)$ be an element which is congruent mod N to $\begin{pmatrix} n^{-1} & 0\\ 0 & n\end{pmatrix}$. Such an element exists because $\Gamma(1)$ surjects onto $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Observe that

$$f|_{\sigma_n,k}(\tau) = (c\tau+d)^{-k}F(\mathbb{Z}\sigma_n\tau+\mathbb{Z}, 1/N) = F(\mathbb{Z}(a\tau+b)+\mathbb{Z}(c\tau+d), n/N) = F(\mathbb{Z}\tau+\mathbb{Z}, n/N).$$

Now we have

$$\langle n\rangle f(\tau) = \langle n\rangle F(\mathbb{Z}\tau + \mathbb{Z}, 1/N) = F(\mathbb{Z}\tau + \mathbb{Z}, n/N) = f|_{\sigma_n,k}(\tau),$$

and the statement of the proposition for $\langle n\rangle$ follows.

The case of T_n uses the set of matrices S_n^N. For $\tau \in \mathcal{H}$ we have the lattice $L_\tau = \mathbb{Z}\tau + \mathbb{Z}$. By definition, $f(\tau) = F(L_\tau, 1/N)$ and

$$T_n f(\tau) = T_n F(L_\tau, 1/N) = \frac{1}{n} \sum_{\sigma \in S_n^N} F(L_{\tau,\sigma}, 1/N).$$

For $\sigma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in S_n^N$ we have $(L_{\tau,\sigma}, 1/N) = \frac{1}{a}(L_{\sigma\tau}, a/N)$ so

$$F(L_{\tau,\sigma}, 1/N) = a^k F(L_{\sigma\tau}, a/N) = a^k \langle a \rangle F(L_{\sigma\tau}, 1/N) = a^k \langle a \rangle f(\sigma\tau).$$

We can write this as

$$T_n f(\tau) = n^{k-1} \sum_{\sigma \in S_n^N} (\langle a \rangle f)|_{\sigma,k}(\tau).$$

From here we can deduce the proposition. □

Remark 6.14. The above proof shows that the action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on $M_k(\Gamma_1(N))$ can be identified with the action of $\Gamma_0(N)/\Gamma_1(N)$ by the weight k slash operator, via the isomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \cong \Gamma_0(N)/\Gamma_1(N)$ given by

$$d + N\mathbb{Z} \mapsto \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \Gamma_1(N).$$

The finite Abelian group $(\mathbb{Z}/N\mathbb{Z})^\times$ now acts on the finite dimensional $\mathbb{C}$-vector space $M_k(\Gamma_1(N))$ by the $\langle \cdot \rangle$ action. So this vector space decomposes into a direct sum indexed by characters $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$:

$$M_k(\Gamma_1(N)) = \bigoplus_\chi M_k(\Gamma_1(N), \chi)$$

where $M_k(\Gamma_1(N), \chi)$ denotes the subspace of $M_k(\Gamma_1(N))$ on which $\langle d \rangle$ acts as multiplication by $\chi(d)$ for every $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. We write $M_k(N, \chi)$ to abbreviate $M_k(\Gamma_1(N), \chi)$.

We now consider the effect of the Hecke operator T_n on the q-expansion at ∞ of a modular form $f \in M_k(N, \chi)$.

Proposition 6.15. *Let $f \in M_k(N, \chi)$ with $f(\tau) = \sum_{i=0}^\infty a_n q^n$ and let $T_p f(\tau) = \sum_{i=0}^\infty b_n q^n$. Then*

$$b_n = a_{np} + \chi(p) p^{k-1} a_{n/p}$$

where we take $\chi(p) = 0$ if $p|N$ and $a_{n/p} = 0$ if $p \nmid n$.

Proof. The set of matrices S_p has a simple description. If $p|N$ then S_p consists of matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a = 1$, $d = p$ and $b = 0, 1, ..., p-1$. Therefore we have

$$T_p f(\tau) = \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{\tau + b}{p}\right).$$

Since the sum
$$\frac{1}{p}\sum_{b=0}^{p-1} e^{2\pi i n(\frac{\tau+b}{p})} = \frac{1}{p}q^{n/p}\sum_{b=0}^{p-1} e^{2\pi i n b/p}$$
is equal to zero if p does not divide n and equals $q^{n/p}$ if p does divide n, we see that $b_n = a_{np}$ as required.

Now suppose that $p \nmid N$. Then we have one additional element of S_p, given by $a = p$, $d = 1$ and $b = 0$. So we have
$$T_p f(\tau) = \frac{1}{p}\left(\sum_{b=0}^{p-1} f\left(\frac{\tau+b}{p}\right) + p^k\langle p\rangle f(p\tau)\right).$$
Since $\langle p\rangle f = \chi(p) f$ we obtain the desired result. □

We will write the above formula for the effect of T_p on q-expansions in terms of some operators on the ring of formal power series $\mathbb{C}[[q]]$. For $m \geq 1$ an integer we write U_m for the operator which takes $\sum_{n\geq 0} a_n q^n$ to $\sum_{n\geq 0} a_n q^{n/m}$ (where $q^{n/m}$ is zero if $m \nmid n$). We write V_m for the operator which takes $\sum_{n\geq 0} a_n q^n$ to $\sum_{n\geq 0} a_n q^{mn}$. We have $U_m \circ V_m$ equals the identity and $V_m \circ U_m$ equals the operator given by retaining only the q^n terms where $m|n$. The above Proposition just says that
$$T_p = U_p + \chi(p)p^{k-1}V_p$$
as operators on q-expansions. This allows us to write a formal factorisation
$$1 - T_p X + \chi(p)p^{k-1}X^2 = (1 - U_p X)(1 - \chi(p)p^{k-1}V_p X).$$
If you like, this equality takes place in the (non-commutative) ring of $\mathbb{C}$-linear endomorphisms of $\mathbb{C}[[q]]$.

Recall that we also have a formal identity
$$\sum_{n=1}^{\infty} T_n n^{-s} = \prod_{p|N}\frac{1}{1 - T_p p^{-s}}\prod_{p\nmid N}\frac{1}{1 - T_p p^{-s} + T_{p,p}p^{1-2s}}.$$
Here the right hand side is equal to
$$\prod_p [(1 - U_p p^{-s})(1 - \chi(p)p^{k-1}V_p p^{-s})]^{-1} = \prod_p (1 - \chi(p)p^{k-1}V_p p^{-s})^{-1}(1 - U_p p^{-s})^{-1}.$$
Note the change in order of the product when we compute the inverse, since U_p and V_p do not commute. Since for distinct p and p', U_p and $V_{p'}$ do commute, we can collect all the V_p and U_p terms. Doing the standard geometric series expansion, we obtain an equality
$$\sum_{n=1}^{\infty} T_n n^{-s} = \left(\sum_{n=1}^{\infty}\chi(n)n^{k-1}V_n n^{-s}\right)\left(\sum_{n=1}^{\infty} U_n n^{-s}\right).$$
From this we deduce

Proposition 6.16.
$$T_n = \sum_{0<d|n}\chi(d)d^{k-1}V_n \circ U_{n/d}.$$

Corollary 6.17. *Let $f \in M_k(N,\chi)$ with $f(\tau) = \sum_{i=0}^{\infty} a_n q^n$ and let $T_m f(\tau) = \sum_{i=0}^{\infty} b_n q^n$. Then*

$$b_n = \sum_{d|\gcd(m,n)} \chi(d) d^{k-1} a_{mn/d^2}.$$

Proof. We have

$$T_m f = \sum_{d|m} \chi(d) d^{k-1} V_d \circ U_{m/d} f = \sum_{d|m} \chi(d) d^{k-1} V_d \sum_{m/d|n} a_n q^{dn/m} = \sum_{d|m} \chi(d) d^{k-1} \sum_{m/d|n} a_n q^{d^2 n/m}.$$

Now we set $r = d^2 n/m$ so we have

$$T_m f = \sum_{d|m} \chi(d) d^{k-1} \sum_{d|r} a_{mr/d^2} q^r.$$

From here we can immediately obtain the statement of the corollary. □

Definition 6.18. We say that $f \in M_k(N,\chi)$ is an *eigenform* if $T_n f = \lambda_n f$ for some $\lambda_n \in \mathbb{C}$, for all $n \in \mathbb{Z}_{\geq 1}$. We also say thaat f is *normalised* if $a_1 = 1$.

Lemma 6.19. *Suppose f is a non-constant eigenform with Hecke eigenvalues λ_n. Then $a_1(f) \neq 0$ and $\lambda_n = a_n(f)/a_1(f)$. Moreover, if $a_0(f) \neq 0$ then $\lambda_n = \sum_{d|n} \chi(d) d^{k-1}$ for all $n \geq 1$.*

Proof. Since $\lambda_n a_1(f) = a_1(T_n f) = a_n(f)$ (by Corollary 6.17), if $a_1(f) = 0$ and f is an eigenform then $a_n(f) = 0$ for all $n \geq 1$, so f is constant.

Suppose that $a_0(f) \neq 0$. We have $\lambda_n a_0(f) = a_0(T_n f) = \sum_{d|n} \chi(d) d^{k-1} a_0(f)$, so $\lambda_n = \sum_{d|n} \chi(d) d^{k-1}$ as required. □

Here are some examples of eigenforms:

(1) The level one Eisenstein series $E_k(\tau)$ for $k > 2$ even. The Hecke eigenvalues λ_n are equal to $\sum_{d|n} \chi(d) d^{k-1}$.
(2) For characters χ with $\chi(-1) = (-1)^k$ and $k > 2$ there are Eisenstein series $E_k^{\chi} \in M_k(N,\chi)$ with Hecke eigenvalues $\sum_{d|n} \chi(d) d^{k-1}$.
(3) Whenever a space of modular forms (or cusp forms) is one-dimensional, an element of this space is automatically a eigenform. For example $\Delta \in S_{12}(\Gamma(1))$ and $\theta(\tau)^2 \in M_1(\Gamma_1(4))$.

Let's consider the final example of $f = \theta^2 \in M_1(4,\chi)$ a little more closely. Here χ is the unique non-trivial character of $(\mathbb{Z}/4\mathbb{Z})^\times$. We have $a_0(f) = 1$ and $a_1(f) = 4$, so $T_n(f) = (\sum_{d|n} \chi(d)) f$. In particular, for odd primes p the Hecke eigenvalue λ_p is equal to $1 + \left(\frac{-1}{p}\right)$ where $\left(\frac{-1}{p}\right)$ denotes the Legendre symbol (it is 1 if -1 is a square mod p and -1 otherwise).

This means we can interpret the Hecke eigenvalues as the traces of certain elements of $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ acting on a two-dimensional $\mathbb{C}$-vector space by the direct sum of characters $\mathbf{1} \oplus \tilde{\chi}$. Here $\tilde{\chi}$ is the unique non-trivial character of $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$.

These elements are Frobenius elements at places dividing p. More explicitly, if p splits in $\mathbb{Q}(i)$, i.e. if $\left(\frac{-1}{p}\right) = 1$, then we take the identity element. If p is inert in $\mathbb{Q}(i)$, i.e. $\left(\frac{-1}{p}\right) = -1$, then we take the non-trivial element (given by complex conjugation).

This is an example of a general theorem of Deligne and Serre which attaches two-dimensional Galois representations to all eigenforms $f \in M_k(N,\chi)$.

Proposition 6.20. *Let $f \in M_k(N,\chi)$. Then f is a normalised (i.e. $a_1(f) = 1$) eigenform if and only if:*

- $a_1(f) = 1$
- *for p prime, $n \geq 1$, $a_{p^n}(f)a_p(f) = a_{p^{n+1}}(f) + \chi(p)p^{k-1}a_{p^{n-1}}(f)$*
- *$a_{mn}(f) = a_m(f)a_n(f)$ when m and n are coprime.*

Proof. We already know that if f is a normalised eigenform then these properties are satisfied. For the other direction, we now suppose that f satisfies these properties. It is enough to show that f is an eigenform for every T_p, or indeed to show that $a_n(T_pf) = a_p(f)a_n(f)$ for every $n \in \mathbb{Z}_{\geq 1}$.

Let's suppose $n \geq 1$. We know that we have $a_n(T_pf) = a_{pn}(f)$ if $p \nmid n$ and $a_n(T_pf) = a_{pn} + \chi(p)p^{k-1}a_{n/p}(f)$ if $p|n$.

In the first case, we get $a_p(f)a_n(f)$ as desired. In the second case, we write $n = p^r m$ with $p \nmid m$ and then we have

$$\begin{aligned} a_n(T_pf) &= a_{p^{r+1}m} + \chi(p)p^{k-1}a_{p^{r-1}m}(f) = a_m(f)(a_{p^{r+1}}(f) + \chi(p)p^{k-1}a_{p^{r-1}}(f)) \\ &= a_m(f)a_{p^r}(f)a_p(f) = a_p(f)a_n(f). \end{aligned}$$

So we have proved that $T_pf - a_p(f)f$ is a constant. It is also an element of $M_k(N,\chi)$, so if $k > 0$ we have $T_pf = a_p(f)f$ as desired. If $k = 0$ then everything is constant and there are no normalised eigenforms! (Since $a_1 = 1$ is impossible). $\square$

6.3. **Petersson inner product.** We define a measure $d\mu$ on $\mathcal{H}$ by $d\mu(\tau) = \frac{dxdy}{y^2}$, where $\tau = x + iy$. This measure is actually $\mathrm{GL}_2^+(\mathbb{R})$-invariant, so defines a measure on $\Gamma\backslash\mathcal{H}$ for any congruence subgroup $\Gamma \subset \Gamma(1)$. Integrating over $Y(\Gamma)$ is the same as integrating over a fundamental domain for Γ in $\mathcal{H}$.

For simplicity we will only consider fundamental domains of the form $\coprod_{\alpha_i} \alpha_i\mathscr{F}(1)$, where $\mathscr{F}(1)$ is the standard fundamental domain for $\Gamma(1)$ and $\alpha_i \in \mathrm{PSL}_2(\mathbb{Z})$ are coset representatives for $\mathrm{PSL}_2(\mathbb{Z})/\overline{\Gamma}$.

Lemma 6.21. *Let $\mathscr{F}$ be a fundamental domain for Γ and define $\mu(\Gamma) = \int_{\mathscr{F}} \frac{dxdy}{y^2}$. Then*

1. *The integral $\mu(\Gamma)$ converges and is independent of the choice of $\mathscr{F}$.*
2. *$[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}] = \mu(\Gamma)/\mu(\Gamma(1))$.*

Proof. Write $[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}] = d$. Let $\mathrm{PSL}_2(\mathbb{Z}) = \coprod_{i=1}^{d} \overline{\Gamma}\alpha_i$ and take $\mathscr{F}(1)$ to be the standard fundamental domain for $\Gamma(1)$. Then we let $\mathscr{F} = \cup_i alpha_i\mathscr{F}(1)$. This is a fundamental domain for Γ. We can easily bound $\mu(\Gamma(1))$ by

$$\int_{\mathscr{F}(1)} \frac{dxdy}{y^2} < \int_{-1/2}^{1/2} \int_{\sqrt{3}/2}^{\infty} y^{-2} dydx = \frac{2}{\sqrt{3}}.$$

Under the change of variables $z \mapsto \alpha_i z$ the measure $\frac{dxdy}{y^2}$ is invariant so we get $\mu(\Gamma) = d\mu(\Gamma(1))$. The invariance of the measure under the action of $\Gamma(1)$ likewise enables us to easily show independence of the integral on the choice of a fundamental domain. □

Definition 6.22. Let $f, g \in M_k(\Gamma)$, with at least one of f, g a cusp form. We define

$$\langle f, g \rangle = \frac{\mu(\Gamma(1))}{\mu(\Gamma)} \int_{Y(\Gamma)} f(\tau)\overline{g(\tau)} y^k d\mu.$$

Lemma 6.23. *The integral in the above definition is absolutely convergent, and can be computed as an integral over a fundamental domain $\mathscr{F}$ for Γ. If $\Gamma' \subset \Gamma$ is another congruence subgroup then the definition of $\langle f, g \rangle$ is independent of whether f, g are considered in $M_k(\Gamma)$ or $M_k(\Gamma')$.*

Proof. The convergence follows from the following lemma, since $fg \in S_{2k}(\Gamma)$. We leave the rest of the lemma as an exercise. □

Lemma 6.24. *Suppose $f \in S_k(\Gamma)$. Then $|f(\tau)| \leq C(\mathrm{Im}(\tau))^{-k/2}$ for some constant C independent of τ.*

Proof. First we set $\phi(x + iy) = |f(x + iy)|y^{k/2}$. It is easy to check that ϕ is Γ-invariant, so we just need to show that it is bounded on $Y(\Gamma)$. Suppose s is a cusp with $\alpha\infty = s$. We have $f|_{\alpha,k}(\tau) = \sum_{n=1}^{\infty} b_n q_{h'}^n = q_{h'}\theta(q_{h'})$ for some h', and a holomorphic function θ on the open unit disc. So

$$\phi(\alpha\tau) = |f_{\alpha,k}(\tau)||y^{k/2} = |\theta(q_{h'})|e^{-2\pi y/h'} y^{k/2}$$

tends to zero uniformly in x as y tends to ∞. So in fact ϕ defines a continuous function on the compact topological space $X(\Gamma)$ (with value zero at the cusps), so it is in particular bounded on $Y(\Gamma)$. □

Here is another useful corollary of this lemma:

Corollary 6.25. *Suppose $f \in S_k(\Gamma)$, with $f(\tau) = \sum_{n=1}^{\infty} a_n q_h^n$. Then there is a constant C such that $|a_n| \leq Cn^{k/2}$ for all $n \geq 1$.*

Proof. Exercise. □

Lemma 6.26. *For $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$*

$$\langle f, g \rangle = (\det \alpha)^k \langle f|_{\alpha,k}, g|_{\alpha,k} \rangle.$$

Proof. Set $\Gamma' = \Gamma \cap \alpha\Gamma\alpha^{-1}$. We have $f, g \in M_k(\Gamma')$ and $f|_{\alpha,k}, g|_{\alpha,k} \in M_k(\alpha^{-1}\Gamma'\alpha)$.

Suppose $\mathscr{F}(\Gamma')$ is a fundamental domain for Γ'. Then $\alpha^{-1}\mathscr{F}(\Gamma')$ is a fundamental domain for $\alpha^{-1}\Gamma'\alpha$.

It follows from the $\mathrm{GL}_2^+(\mathbb{Q})$-invariance of the measure μ that $\mu(\alpha^{-1}\Gamma'\alpha) = \mu(\Gamma')$. We compute

$$\begin{aligned}(\det\alpha)^k\langle f|_{\alpha,k}, g|_{\alpha,k}\rangle &= \frac{\mu(\Gamma(1))}{\mu(\Gamma')}\int\limits_{\alpha^{-1}\mathscr{F}(\Gamma')} f|_{\alpha,k}\overline{g|_{\alpha,k}}(\det\alpha\ \mathrm{Im}\tau)^k d\mu \\ &= \frac{\mu(\Gamma(1))}{\mu(\Gamma')}\int\limits_{\mathscr{F}(\Gamma')} \frac{f\overline{g}}{|c\tau+d|^{2k}}(\det\alpha\ \mathrm{Im}\alpha^{-1}\tau)^k d\mu \\ &= \frac{\mu(\Gamma(1))}{\mu(\Gamma')}\int\limits_{\mathscr{F}(\Gamma')} f\overline{g}(\mathrm{Im}\tau)^k d\mu = \langle f, g\rangle,\end{aligned}$$

since $\det\alpha\ \mathrm{Im}\tau = |c\tau+d|^2\mathrm{Im}\alpha\tau$. □

I messed up the proof of the following corollary in lectures, sorry!

Corollary 6.27. *If $f, g \in M_k(N,\chi)$, with one of f, g a cusp form, and $n \in \mathbb{Z}_{\geq 1}$ coprime to N, then $\langle T_n f, g\rangle = \chi(n)\langle f, T_n g\rangle$.*

Proof. It suffices to prove the corollary when $n = p$ is prime. Recall that we have a matrix $\sigma_p \in \Gamma_0(N)$ with $\sigma_p = \begin{pmatrix}\alpha & \beta \\ N & p\end{pmatrix}$.

Consider the set of matrices

$$\Delta_p^N := \{\gamma = \begin{pmatrix}a & b \\ c & d\end{pmatrix} \in M_2(\mathbb{Z}) : N \mid c, N \mid (a-1), \det(\gamma) = p, \}.$$

We can check that

$$\begin{aligned}\Delta_p^N &= \Gamma_1(N)\begin{pmatrix}1 & 0 \\ 0 & p\end{pmatrix}\Gamma_1(N) = \{u\begin{pmatrix}1 & 0 \\ 0 & p\end{pmatrix}v : u, v \in \Gamma_1(N)\} \\ &= \coprod_{j=0,\dots,p-1}\Gamma_1(N)\begin{pmatrix}1 & j \\ 0 & p\end{pmatrix}\coprod\Gamma_1(N)\sigma_p\begin{pmatrix}p & 0 \\ 0 & 1\end{pmatrix}.\end{aligned}$$

Now suppose $\gamma \in \Delta_p^N$. We have $\langle f, g|_{\gamma^{-1},k}\rangle = p^k\langle f|_{\gamma,k}, g\rangle$ by the above Lemma. Moreover, the values of $\langle f|_{\gamma,k}, g\rangle$ and $\langle f, g|_{\gamma,k}\rangle$ are actually independent of the choice of $\gamma \in \Delta_p^N$: if $\gamma' = u\gamma v$ with $u, v \in \Gamma_1(N)$, then

$$\langle f|_{u\gamma v,k}, g\rangle = \langle f|_{\gamma,k}, g|_{v^{-1},k}\rangle = \langle f|_{\gamma,k}, g\rangle.$$

A similar argument applies to $\langle f, g|_{\gamma,k}\rangle$.

Since $T_p f = p^{k-1}\left(\sum_{j=0}^{p-1} f|_{\left(\begin{smallmatrix}1 & j\\ 0 & p\end{smallmatrix}\right),k} + f|_{\sigma_p\left(\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right),k}\right)$ we have

$$\begin{aligned}\langle T_p f, g\rangle &= p^{k-1}(p+1)\langle f|_{\left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right),k}, g\rangle = p^{-1}(p+1)\langle f, g|_{\left(\begin{smallmatrix}1 & 0\\ 0 & p^{-1}\end{smallmatrix}\right),k}\rangle \\ &= p^{k-1}(p+1)\langle f, g|_{\sigma_p^{-1}\sigma_p\left(\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right),k}\rangle = \chi(p)p^{k-1}(p+1)\langle f, g|_{\sigma_p\left(\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right),k}\rangle \\ &= \chi(p)p^{k-1}(p+1)\langle f, g|_{\left(\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right),k}\rangle = \chi(p)\langle f, T_p g\rangle.\end{aligned}$$

Here we use the observation that $\sigma_p\left(\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right) \in \Delta_p^N$.

□

Corollary 6.28. *The space $S_k(N,\chi)$ has a basis (orthonormal with respect to the Petersson inner product) consisting of simultaneous eigenvectors for the Hecke operators T_n with n coprime to N.*

Proof. For each n choose a square root c_n of $\overline{\chi(n)}$. Then for all $f, g \in S_k(N,\chi)$ we have

$$\langle c_n T_n f, g\rangle = \langle f, c_n T_n g\rangle.$$

So the operators $c_n T_n$ are Hermitian and so have an orthonormal basis of eigenvectors. Since all the Hecke operators T_n with n coprime to N commute, we have a basis of simultaneous eigenvectors. □

Remark 6.29. Note that the eigenvalues of $c_n T_n$ are real. In particular, if χ is a real valued character (i.e. it is either trivial or quadratic) then the eigenvalues of the T_n are real.

If we want to find a basis of eigenforms (i.e. eigenvectors for *all* the Hecke operators) then we have to restrict to certain subspaces of $S_k(N,\chi)$. Recalling definition 3.16, we can define $S_k(N,\chi)^{\mathrm{old}} = M_k(\Gamma_1(N))^{\mathrm{old}} \cap S_k(\Gamma_1(N))$.

Definition 6.30. Define $S_k(N,\chi)^{\mathrm{new}}$ to be the orthogonal complement of $S_k(N,\chi)^{\mathrm{old}}$ under the Petersson inner product.

An important property of the new subspace is that it has a basis of eigenforms:

Theorem 6.31. *The space $S_k(N,\chi)^{\mathrm{new}}$ is stable under the action of the Hecke operators. If $f \in S_k(N,\chi)^{\mathrm{new}}$ is an eigenvector for the T_n with n coprime to N, then f is an eigenform (i.e. an eigenvector for all then T_n).*

Corollary 6.32. *Suppose two non-zero elements f, g of $S_k(N,\chi)^{\mathrm{new}}$ are eigenvectors for the T_n with n coprime to N with the same eigenvalues. Then f and g are scalar multiples of each other.*

Proof. The Theorem implies that both f and g are eigenforms. By rescaling we can assume that they are both normalised eigenforms. Now $f - g$ is also an eigenform, but has first q-expansion coefficient $a_1 = 0$. Therefore $f - g = 0$. □

Remark 6.33. The above Corollary is a version of a 'multiplicity one' theorem. Various stronger forms of this theorem can be proven: for example, in Miyake's book 'Modular Forms' it is proven by fairly elementary arguments that if f, g have the same T_p eigenvalue for all but finitely many primes p, then f and g are scalar multiples of each other.

7. L–functions

Definition 7.1. For $f \in M_k(\Gamma)$ and $s \in \mathbb{C}$ set $L(f,s) = \sum_{n\geq 1} \frac{a_n}{n^s}$.

Lemma 7.2. *Suppose $f \in S_k(\Gamma)$ The series defining $L(f,s)$ converges absolutely and uniformly on compact subsets of $\{\mathrm{Re}(s) > k/2+1\}$.*

Proof. By Corollary 6.25, we have $|a_n| \leq Cn^{k/2}$. This suffices to prove the lemma. □

Remark 7.3. We can explicitly write down Eisenstein series which give the rest of the space $M_k(\Gamma)$. These have Fourier coefficents of order n^{k-1}, so for $f \in M_k(\Gamma)$ the L-function converges nicely for $\mathrm{Re}(s) > k$.

7.1. **Functional equation.** Now we are going to find a functional equation for $L(f,s)$. For simplicity we will assume that $\Gamma = \Gamma_1(N)$. Set $w_N = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$. Note that $w_N^{-1}\Gamma_1(N)w_N = \Gamma_1(N)$, so $f|_{w_N,k} \in S_k(\Gamma_1(N))$.

Explicitly, we have $f|_{w_N,k}(\tau) = N^{-k/2}\tau^{-k}f(-1/N\tau)$.

Theorem 7.4. *Let $f \in S_k(\Gamma_1(N))$ and set $g = i^k f|_{w_N,k}$. If $f = \sum_{n\geq 1} a_n q^n$ then the Dirichlet series $L(f,s) = \sum_{n\geq 1} a_n/n^s$ can be extended to a holomorphic function on $s \in \mathbb{C}$. Setting*

$$\Lambda(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s)$$

we have a functional equation

$$\Lambda(f,s) = \Lambda(g,k-s).$$

Here $\Gamma(s)$ is meromorphic continuation of the function defined by

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt.$$

Proof. We let ϕ be the function on $\mathbb{R}_{>0}$ given by $\phi(y) = f(iy)$. Consider the *Mellin transform*

$$F(s) = \int_0^\infty \phi(y)y^{s-1}dy.$$

Now $\phi(y)$ tends to zero exponentially fast as y tends to ∞. We also have $\phi(1/y) = f(-1/iy) = (iy)^k f|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),k}(iy)$. Since $f|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),k}$ is also a cusp form, $\phi(1/y)$ tends to zero like y^k times an exponential in $-y$ as y tends to infinity. So this integral converges absolutely (at both upper and lower limits) for all s.

We have $\phi(y) = \sum_{n\geq 1} a_n e^{-2\pi n y}$, and we can switch the sum and integral in $F(s)$ to get

$$F(s) = \sum_{n=1}^{\infty} a_n \int_0^{\infty} e^{-2\pi n y} y^{s-1} dy.$$

Substituting $t = 2\pi n y$ into the integral gives us

$$\int_0^{\infty} e^{-2\pi n y} y^{s-1} dy = (2\pi n)^{-s} \int_0^{\infty} e^{-t} t^{s-1} dt = (2\pi n)^{-s}\Gamma(s).$$

Therefore the switched expression also converges absolutely for $\mathrm{Re}(s) > k/2 + 1$, to $N^{-s/2}\Lambda(f, s)$, and we have $F(s) = N^{-s/2}\Lambda(f, s)$ (for $\mathrm{Re}(s) > k/2 + 1$). Since $F(s)$ extends to a holomorphic function for all $s \in \mathbb{C}$, $\Lambda(f, s)$ does. Moreover, $\Gamma(s)$ has no zeroes, so $L(f, s)$ also extends to a holomorphic function on the whole complex plane.

To prove the functional equation, let's substitute $u = 1/Ny$ in the integral defining $F(s)$. We get

$$\begin{aligned} N^{-s/2}\Lambda(f,s) = F(s) &= N^{-s}\int_0^{\infty} \phi(1/Nu)u^{-1-s}du = N^{-s}\int_0^{\infty} f(-1/Niu)u^{-1-s}du \\ &= N^{k/2-s}\int_0^{\infty} g(iu)u^{k-1-s}du = N^{k/2-s}N^{-(k-s)/2}\Lambda(g, k-s) = N^{-s/2}\Lambda(g, k-s). \end{aligned}$$

□

7.2. **Euler products.**

Theorem 7.5. *Suppose $f \in S_k(N, \chi)$. Then f is a normalised eigenform if and only if (for $\mathrm{Re}(s)$ sufficiently large)*

$$L(s, f) = \prod_p (1 - a_p p^{-s} + \chi(p)p^{k-1-2s})^{-1}.$$

Proof. By Proposition 6.20, it is enough to show that

$$L(s, f) = \prod_p (1 - a_p p^{-s} + \chi(p)p^{k-1-2s})^{-1}$$

if and only if

- $a_1(f) = 1$
- for p prime, $n \geq 1$, $a_{p^n}(f)a_p(f) = a_{p^{n+1}}(f) + \chi(p)p^{k-1}a_{p^{n-1}}(f)$
- $a_{mn}(f) = a_m(f)a_n(f)$ when m and n are coprime.

We leave it as an exercise to show this, using the following lemma. □

Lemma 7.6. *Suppose we have two Dirichlet series $\sum_{n\geq 1}\frac{a_n}{n^s}$ and $\sum_{n\geq 1}\frac{b_n}{n^s}$ which converge absolutely to the same function on $\mathrm{Re}(s) > \sigma$ for some positive real σ. Then $a_n = b_n$ for all n.*

Proof. Exercise. □

7.3. **Converse theorems.** Suppose $f \in S_k(N,\chi)$. We have shown $F(s) = N^{-s/2}\Lambda(f,s)$, where $F(s)$ is the Mellin transform

$$F(s) := \int_0^\infty f(iy)y^{s-1}dy.$$

The following Proposition establishes an inversion formula for the Mellin transform.

Proposition 7.7. *Suppose $g : \mathbb{R}_{>0} \to \mathbb{C}$ is twice continuously differentiable, and that c is a real number such that $y^{c-1}g(y)$, $y^c g'(y)$ and $y^{c+1}g''(y)$ are all in $L^1(\mathbb{R}_{>0})$. Then the integral*

$$G(s) := \int_0^\infty y^{s-1}g(y)dy$$

converges for $\mathrm{Re}(s) = c$ and satisfies $G(c+it) = \mathscr{O}((1+|t|)^{-2})$ (i.e. it is bounded and as t approaches ∞ it decays like $|t|^{-2}$).

Moreover, we have

$$g(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}G(s)ds$$

where the integral is up the vertical line $\mathrm{Re}(s) = c$.

Proof. If we set $s = c - 2\pi i x$ and substitute $y = e^u$ then we have

$$G(s) = F(x) = \int_{\mathbb{R}} e^{cu}g(e^u)e^{-2\pi i x u}du.$$

Now everything follows from standard properties of the Fourier transform applied to the function $f(u) = e^{cu}g(e^u)$. In particular, f is twice continuously differentible and f, f' and f'' are absolutely integrable. The Fourier transform of f'' is $(2\pi i x)^2 F(x)$, so $x^2F(x)$ is bounded, which gives the growth condition on $G(s)$. □

The following theorem, which is a converse to Theorem 7.4 (when the level $N = 1$), is now a simple consequence of Mellin inversion.

Theorem 7.8. *Let a_n be a sequence in $\mathbb{C}$ with $a_n \leq Cn^{sigma}$ for some $\sigma \in \mathbb{R}_{>0}$. Set*

$$Z(s) := \sum_{n\geq 0}\frac{a_n}{n^s}$$

and $\Lambda(s) := (2\pi)^{-s}\Gamma(s)Z(s)$. Suppose that $\Lambda(s)$ extends to a holomorphic function on $\mathbb{C}$ which is bounded on vertical strips (i.e. regions of the form $\mathrm{Re}(s) \in [a,b]$) and satisfies

$$\Lambda(s) = i^k\Lambda(k-s).$$

Then $f(\tau) = \sum_{n\geq 1} a_n q^n$ is in $S_k(\Gamma(1))$.

Proof. We define a holomorphic function on $\mathcal{H}$ by $f(\tau) := \sum_{n\geq 1} a_n q^n$. We need only to prove that $f(-1/\tau) = (\tau)^k f(\tau)$. By uniqueness of analytic continuation, it suffices to prove that $f(i/y) = (iy)^k f(iy)$ for $y \in \mathbb{R}_{>0}$.

Set $\phi(y) = f(iy)$. We have $|\phi(y)| \leq C\sum_{n\geq 1} n^\sigma e^{-2\pi n y}$, and (possibly increasing σ) we can assume that σ is a natural number. Since

$$\sum_{n\geq 0} e^{-2\pi n y} = \frac{1}{1-e^{-2\pi y}} = \frac{1}{2\pi y} + g(y)$$

with $g(y)$ holomorphic at $y = 0$, differentiating σ times gives

$$\sum_{n\geq 1} n^\sigma e^{-2\pi n y} = \mathscr{O}(y^{-(1+\sigma)})$$

as y approaches 0. A similar argument shows that $\phi(y)$ is $\mathscr{O}(e^{-2\pi y})$ as y approaches ∞, so the hypotheses of Proposition 7.7 are satisfied by ϕ (for any $c > \sigma + 1$). The Mellin transform of ϕ is given by $\Lambda(s)$. Therefore

$$\phi(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^{-s}\Lambda(s)ds$$

for $c > 1 + \sigma$. Fix such a c (which we also assume is $> k/2$) and consider the strip $\mathrm{Re}(s) \in [k-c, c]$. For $\mathrm{Re}(s) = c$ we have

$$|\Lambda(k-s)| = |\Lambda(s)| = \mathscr{O}((1+|\mathrm{Im}(s)|)^{-2}),$$

and by hypothesis $\Lambda(s)$ is bounded on the region $\mathrm{Re}(s) \in [k-c, c]$. So the Phragmén–Lindelöf principle (see Lemma 4.3.4 in Miyake) implies that we have $\Lambda(s) = \mathscr{O}((1+|\mathrm{Im}(s)|)^{-2})$ uniformly for $\mathrm{Re}(s) \in [k-c, c]$. This allows us to move the line of integration to get

$$\phi(y) = \frac{1}{2\pi i}\int_{k/2-i\infty}^{k/2+i\infty} y^{-s}\Lambda(s)ds = \frac{1}{2\pi i}\int_{k/2-i\infty}^{k/2+i\infty} y^{-s} i^k \Lambda(k-s)ds.$$

Substituting $t = k - s$ we have

$$\phi(y) = \frac{1}{2\pi i}\int_{k/2-i\infty}^{k/2+i\infty} y^{-(k-t)} i^k \Lambda(t)dt = i^k y^{-k}\phi(1/y).$$

This establishes that $f(-1/\tau) = (-1)^k(\tau)^k f(\tau)$. We didn't assume a priori that k was even, so we need to check this. We have

$$f(\tau) = f(-1/(-1/\tau)) = (-1)^k(-1/\tau)^k f(-1/\tau) = (-1)^k f(\tau)$$

so if f is non-zero then k is even. This completes the proof. $\square$