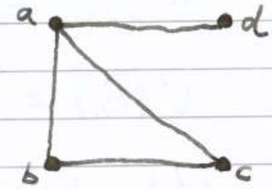
Graph Theory ①

Modern Graph Theory, Bollobás



Formally, a graph is a pair $G = (V, E)$

where $E \subset V^{(2)} = \{ \{x, y\} : x, y \in V \}$

e.g. $V = \{a, b, c, d\}$, $E = \{ \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\} \}$

The set $V(G)$ is the <u>vertex set</u> and $E(G)$ is the <u>edge set</u>.

The <u>order</u> of G is $|V(G)|$, often written $|G|$. The <u>size</u> of G is $|E(G)|$, often written $e(G)$.

Most graphs are finite in this course.

Note that the definition precludes more than one edge between two vertices. (Some authors allow graphs to have 'multiple edges' and call our graphs '<u>simple</u>'.)

It is easier to write the edge $\{a, b\}$ as ab. If $ab \in E(G)$ then we say that a is <u>adjacent</u> to b, sometimes written $a \sim b$. We might write $ab \in G$.

Edges containing vertex x are said to be <u>incident</u> with each other, and also with x.

Two graphs G, H are <u>isomorphic</u> if there is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G) \iff f(u)f(v) \in E(H)$

<u>Examples</u>

The empty graph, E_n, order n (n vertices), size 0 (0 edges)

The complete graph, K_n, order n, size $\binom{n}{2}$, with each vertex adjacent to all others.

A path P_n, order n, size $n-1$

$V(P_n) = \{x_1, \ldots, x_n\}$, $E(P_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$

The circuit C_n, order n, size n.

$V(C_n) = V(P_n)$, $E(C_n) = E(P_n) \cup \{x_nx_1\}$

Some authors refer to circuits as 'cycles', using the term circuit to mean that vertices can be repeated

A sub-graph of G is a graph H with $V(H) \subset V(G)$ and $E(H) \subset E(G)$

The sub-graph of G induced by $W \subset V(G)$, denoted $G[W]$, is the graph $(W, E(G) \cap W^{(2)})$ i.e. taking all edges of G lying within W. Hence, every graph of order $\leq n$ is a subgraph of K_n. However, the only induced subgraphs of K_n are K_k, $k \leq n$.

A graph is connected if there is a uv-path (path from vertex u to vertex v) for all $u, v \in V(G)$, $u \neq v$.

The components of a graph are the maximal connected subgraphs (necessarily induced, by the equivalence classes under the relation "there is a uv-path or $u = v$").

e.g. $G = \triangle \; | \; \top \; \bullet$ has 4 components

A forest is a graph with no circuit.

A tree is a connected forest (so that the components of a forest are trees)

Graph Theory (1)

Theorem 1.1

The following are equivalent:

a) G is a tree

b) G is minimal connected (G is connected but removing any edge disconnects G)

c) G is maximal circuit-free (G is circuit-free but adding any edge gives a circuit)

Proof

a)⇒b) Let G be a tree. If $G \setminus uv$ is connected, then there is a uv-path P in $G \setminus uv$, so $P + uv$ is a circuit ※

So G is minimally connected.

b)⇒a) Suppose that G has a circuit C, and let $xy \in E(C)$. Then $G \setminus xy$ is connected, because if P is some uv-path in G, then $P \cup C \setminus xy$ contains a uv-path in $G \setminus xy$. Hence, a minimal connected graph is circuit free.

a)⇒c) If G is a tree, and $xy \notin E(G)$, then G has an xy-path P, and $P + xy$ is a circuit in $G + xy$. So G is maximal circuit-free.

c)⇒a) If G is circuit free, but $G + xy$ is not, there is a circuit C in $G + xy$, containing xy, and $C \setminus xy$ is an xy-path in G, so G is connected.

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Graph Theory (2)

Corollary 1·2

A graph G is connected $\Leftrightarrow$ it contains a spanning tree (a tree subgraph T with $V(T) = V(G)$)

Proof

($\Leftarrow$) T is connected, G contains T, so G is connected.

($\Rightarrow$) Suppose G is connected, and let T be a minimal connected spanning subgraph of G. T is then minimal connected, so by Theorem 1·1, T is a tree. $\square$

Definition

The set of neighbours of $v \in V(G)$ is

$\Gamma(v) = \{w \in V(G) : vw \in E(G)\}$

The degree of v is $d(v) = |\Gamma(v)|$

The minimum and maximum degrees of a graph are denoted by $\delta(G)$ and $\Delta(G)$.

If $\delta(G) = \Delta(G) = k$, then G is k-regular. For example, in a 5-regular graph every vertex has degree 5.

A regular graph is k-regular for some $k \in \mathbb{N}_0$.

A 3-regular graph is sometimes called cubic. The degrees of G, written in some order (often monotonically) is a degree sequence of G.

Clearly $\sum_{v \in G} d(v) = 2e(G)$ (Handshaking Lemma)

A leaf is a vertex of degree 1.

Theorem 1·3

Every tree T with $|T| > 2$ has at least 2 leaves.

Proof

T is connected, so $\delta(T) \geq 1$. Let x_1 be a vertex (a leaf if there is one) and let $x_1, x_2, \ldots, x_k$ be a maximal path beginning at x_1 (exists since T is finite). By maximality, the only neighbour of x_k is x_{k-1}, i.e. x_k is a leaf. Then, start with x_1 a leaf and do the same to show that there is another. □

Corollary 1·4

A tree of order n has size $n-1$.

Proof (by induction on n)

For $n \leq 2$, the result is trivial.

Given T, then by 1·3 there is a leaf $v \in V(T)$. $T-v$ is circuit free, and is connected, since any xy-path in T lies in $T-v$ unless $x=v$ or $y=v$, because v is a leaf. Thus $T-v$ is a tree, and by the induction hypothesis, $e(T-v) = (n-1)-1$. Since v is a leaf, we add one more edge and $e(T) = n-1$.

Corollary 1·5

Let G be a graph of order n. The following are equivalent:

a) G is a tree   b) G is connected and $e(G) = n-1$

c) G is circuit free and $e(G) = n-1$

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Proofs

$a \Rightarrow b$ and $a \Rightarrow c$ by definition and Corollary 1·4.

($b \Rightarrow a$) By Corollary 1·2, G contains a spanning tree T. By Corollary 1·4, $e(T) = n-1$, but $e(T) = e(G)$ so $G = T$ i.e. G is a tree.

($c \Rightarrow a$) Add edges to G to get G', which is maximal circuit free. By Theorem 1·1, G' is a tree, and by Corollary 1·4, $e(G') = n-1 = e(G)$; hence $G = G'$. So G is a tree $\square$

How many graphs of order n are there? Let the vertex set be $[n] = \{1, \dots, n\}$

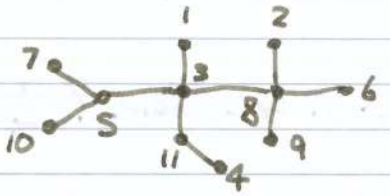
By inspection, there are 8 graphs of order 3.

In general, there are either $2^{\binom{n}{2}}$ labelled graphs (since each edge is either in the graph or not).

However, there are 4 unlabelled graphs (isomorphism classes) of order 3. To count these, we need Burnside's Lemma (to count the number of orbits under a group action which permutes the vertices.

How many labelled trees of order n are there?

Theorem 1.6 (Cayley)
There are n^{n-2} labelled trees of order n.
Proof (Prüfer)
We construct a bijection between labelled trees and strings of length $n-2$ from the alphabet $[n]$. Start with a tree, obtain a string as follows:
Start with the lowest labelled leaf (there is one!), write down it's neighbour, and remove the leaf. Repeat until one edge is left.

e.g.  3, 8, 11, 8, 5, 8, 3, 5, 3

v appears $d(v)-1$ times in the string.
Notice that $\sum_{v \in G} (d(v)-1) = 2(n-1) - n = n-2$ (Sanity Check
To start with a string and obtain a tree:
Choose the smallest 'unused' vertex not in the ~~tree~~ string.
Mark it "used" and join to the first vertex in the string.
Remove the first vertex in the string and repeat.
Check that this gives a tree and that the process is a bijection.

{ Double up last digit of string?

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Graph Theory (3)

Last lecture: Start with all vertices unmarked. Mark 1 per step.

A graph is <u>r-partite</u> if its vertices can be partitioned into r parts so that no edge lies inside any part.

<u>Bipartite</u> means 2-partite, an important class of graphs.

Remarkably, we can characterise them.

<u>Theorem 1.7</u>

G is bipartite $\iff$ it has no odd-length circuits

<u>Proof</u>

"only if": Every circuit visits parts alternately so must have even length.

"if": We may assume that G is connected. Pick a vertex v_0 and let $V_i = \{v \in G : d(v_0, v) = i\}$, $i = 0, 1, 2, \ldots$ where $d(v_0, v)$ is the "distance" of v_0 to v, the length of the shortest $v_0 v$-path.

The sets V_i partition $V(G)$. Moreover, every edge of G lies inside V_i or between V_i and V_{i+1}, for some i, by definition of distance.

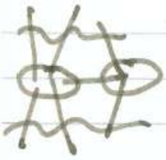
Suppose now there is some edge ab inside V_i for some i.

Then there is a $v_0 a$ path $v_0 = a_0, a_1, a_2, \ldots, a_i = a$ where $a_j \in V_j$. Likewise, there is a $v_0 b$ path $v_0 = b_0, b_1, \ldots, b_i =$

Let k be the maximum value of j such that $a_j = b_j$

$k = \max\{j : a_j = b_j\}$ exists because $a_0 = b_0$ and $k < i$.

But then $a_k, a_{k+1}, \ldots, a_i, b_i, b_{i-1}, \ldots, b_k$ is an odd circuit, a contradiction. Thus each V_i has no edges and so $X = \bigcup_{i \text{ even}} V_i$, $Y = \bigcup_{i \text{ odd}} V_i$, the required bipartition $\square$




An Euler-Cycle is a "walk" round the graph using each edge exactly once, and returning to the start (initial vertex).
A graph is Eulerian if it has an Euler cycle and no isolated vertices (vertices with degree 0), or $|G| = 1$.

Theorem 1.8

G Eulerian $\iff$ G is connected and all degrees are even.

Proof

($\Rightarrow$) This is clear.

($\Leftarrow$) By induction on $e(G)$.
Now, $\delta(G) \geq 2$, so by Theorem 1.3, G is not a tree, so has a circuit C. Observe that each component of $G - E(C)$ is Eulerian, by the induction hypothesis. We wander around C taking time out to traverse the cycle of a component whenever first encountered. This shows that G is Eulerian $\square$

A graph that can be drawn in the plane without edges crossing is called a planar graph. A plane graph is a graph so drawn in the plane. A face of a plane graph is a connected region of the plane, the region $\mathbb{R}^2 - G$.

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e.g. 

These are the same planar graphs but different plane graphs

Face sizes: 3, 3, 3, 5 and 3, 4, 3, 4

N.B. There are no analytical difficulties, because the edges can always be drawn piecewise linearly (in fact, each edge can be a straight line, not so obvious). All issues are purely combinatorial.

<u>Theorem 1.9</u> (Euler's ~~Theorem~~ Polyhedron Formula)

Let G be a connected plane graph with n vertices, m edges and f faces. Then $n - m + f = 2$.

<u>Proof</u> (By induction on m)

By Corollaries 1.2 and 1.4, $m \geq n-1$, and if $m = n-1$, G is a tree. Then $f = 1$ in this case, and all is well.

If $m > n-1$, then G is not a tree, so it has a circuit C. Pick an edge $e \in E(C)$. Then the faces on either side of e are distinct. Moreover, $G - e$ is connected and has $f-1$ faces. Then, $n - (m-1) + (f-1) = 2$ by the induction hypothesis. □

A <u>bridge</u> (or <u>isthmus</u>) is an edge whose removal increases the number of components. Equivalently, a bridge is an edge lying in no circuit. In a connected, bridgeless, plane graph, every edge separates two distinct faces. If there are f_i faces with i sides, $\sum i f_i = 2m$

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G planar and bridgeless $\Rightarrow \sum i f_i = 2m$

Note 

still true without these properties

could be replaced by planar

<u>Theorem 1.10</u>

Let G be a (connected bridgeless) plane graph of order n, size m, and <u>girth</u> $\geq g$ (length of shortest circuit). Then $m \leq \frac{g}{g-2}(n-2)$.

for ALL planar graphs.

In particular, a planar graph has size $\leq 3n-6$, so has a vertex of degree ≤ 5. $(n \geq 3)$

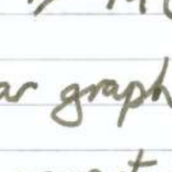
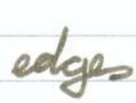
<u>Proof</u> N.B connected but no circuits $\Rightarrow$ bridge

$g \geq 3$ for connected, bridgeless graph as 3 is min circuit length.

$2m = \sum i f_i \geq g \sum f_i = gf$. By Theorem 1.9,

$n-2 = m-f \geq m(1-\frac{2}{g}) = m \cdot \frac{g-2}{g}$

For a general planar graph, we can if necessary add edges so that it becomes connected, bridgeless, and planar, and the result has size $\leq 3n-6$ (set $g=3$) $\square$

K_4 is planar  K_5 has a 5 circuit 

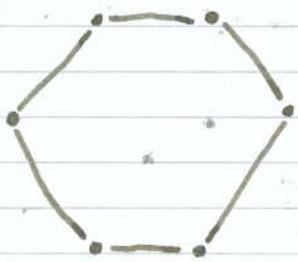
K_5 cannot be drawn in the plane: we can draw at most two edges outside, and two inside, the pentagon, but require 5.

Otherwise $e(G_5) = \binom{5}{2} = 10 > 3 \cdot (5-2)$ so non-planar

The <u>complete bipartite graph</u> $K_{m,n}$ is the bipartite graph with vertex class sizes m, n and every edge in between.

$|K_{m,n}| = m+n$, $e(K_{m,n}) = mn$, girth $= 4$ provided $m,n \geq$

$K_{3,3}$: contains C_6. Only 1 diagonal inside, and 1 outside, so that the graph is non-planar.



Alternatively, $e(K_{3,3}) = 9 > \frac{4}{2}(6-2)$

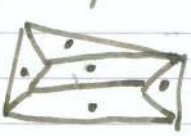
Graphs are not planar simply by having relatively few edges e.g. If we subdivide the edges of K_5, i.e. replace them by paths, then the result is sparser but still non-planar, and the same with $K_{3,3}$.

Kuratowski (1930) showed that these are the only obstructions to planarity. Pontryagin proved it some years earlier (unpublished).

<u>Theorem 1.11</u> (Kuratowski)

A graph is planar $\Leftrightarrow$ it contains no subdivision of K_5 or of $K_{3,3}$.

Given a plane graph, we can draw a new plane graph called the <u>dual</u> by placing inside each face a new vertex, and joining two new vertices by a new edge every time some old edge separates the two faces.

 $\Longleftrightarrow$ 

Graph : ~~17 edges~~ 7 vertices, 11 edges, 6 faces

Dual : 6 vertices, 11 edges, 7 faces

Note : The dual of the dual is the original.

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The dual of the graph need not be a graph. If the original is not 3-connected then the dual might have multiple edges.

Chapter 2 : Connectivity and Matching

Let g be a bipartite graph, with vertex classes X, Y.

A matching from X to Y is a set of $|X|$ independent edges (pairwise non-incident).

If $|X| = |Y|$ a matching is also a 1-factor (1 regular spanning subgraph)

Imagine X = women, Y = men, $xy \in E$ means x happy to marry y, a matching is a marriage of X to Y.

For any $A \subset V(G)$, let $\Gamma(A) = \bigcup_{a \in A} \Gamma(a)$

Remarkably, a trivial necessary condition suffices.

Theorem 2.1 (Hall's Marriage Theorem)

Let G be a bipartite graph with bipartition X, Y. Then,

G has a matching from X to Y ("only if" is trivial)

$\iff |\Gamma(A)| \geq |A| \quad \forall A \subseteq X$ (Hall's Condition)

Proof 1 (By induction on X)

If $\forall A \subseteq X,\ A \neq \emptyset,\ A \neq X$, we have $|\Gamma(A)| > |A|$, choose any edge xy and let $G' = G - x - y$, then Hall's Condition holds in G', so G' has a matching which extends

By xy to a matching in G.
So we may assume that there is a 'critical set' $B \subseteq X$
where $|\Gamma(B)| = |B|$, $\emptyset \neq B \neq X$

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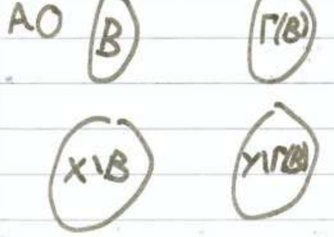
<u>Theorem 2.1</u> (Hall)

$\exists$ a matching $\iff |\Gamma(A)| \geq |A| \quad \forall A \subset X$

($\Rightarrow$) Trivial

① ($\Leftarrow$) If $\forall\ \emptyset \neq A \neq X$, $|\Gamma(A)| > |A|$ then $\forall xy$ $\exists$ a matching containing xy. If not, $\exists$ a "critical set" B of women, $\emptyset \neq B \neq X$, $|\Gamma(B)| = |B|$



Define $G_1 = G[B \cup \Gamma(B)]$,

$G_2 = G[(X \setminus B) \cup (Y \setminus \Gamma(B))]$

If $A \subset B$ then $\Gamma(A) \subset \Gamma(B)$ so Hall's Condition holds in G_1

If $A \subset X \setminus B$ then neighbours of A in $G_2 = \Gamma(A \cup B) \setminus \Gamma(B)$

so # neighbours of A in $G_2 = |\Gamma(A \cup B)| - |\Gamma(B)| \geq \underbrace{|A \cup B| - |B| = |A|}_{A, B \text{ disjoint}}$

so Hall's condition holds in G_2 also. Thus, by our induction hypothesis, there are matchings in both G_1 and G_2 giving a matching in G.

② ($\Leftarrow$) (Rado)

Removing edges from G as necessary, we may assume that G is a minimal graph in which Hall's Condition holds.

If $d(x) = 1\ \forall x \in X$ then G <u>is</u> a matching by Hall's Condition, so we are done.

If not, then $\exists\ a \in X$ and $b_1, b_2 \in \Gamma(a)$, and sets $A_1, A_2 \subseteq X \setminus \{a\}$ such that $\Gamma(A_i) \supset \Gamma(a) \setminus \{b_i\}$

and $|\Gamma(A_i)| = |A_i|$. So $\Gamma(A_1 \cup A_2 \cup \{a\})$

$= \Gamma(A_1 \cup A_2) = \Gamma(A_1) \cup \Gamma(A_2)$

Thus $|\Gamma(A_1 \cup A_2 \cup \{a\})| = |\Gamma(A_1) \cup \Gamma(A_2)|$

$$= |\Gamma(A_1)| + |\Gamma(A_2)| - |\Gamma(A_1) \cap \Gamma(A_2)|$$

$$\leq |\Gamma(A_1)| + |\Gamma(A_2)| - |\Gamma(A_1 \cap A_2)|$$

$$= |A_1| + |A_2| - |\Gamma(A_1 \cap A_2)|$$

$$\leqslant |A_1| + |A_2| - |A_1 \cap A_2| = |A_1 \cup A_2|$$

$$< |A_1 \cup A_2 \cup \{a\}|$$, a contradiction □

Corollary 2·2 (Defect form)

Let G be a bipartite graph with bipartition X, Y. Let $d \in \mathbb{N}$. Then G has $|X| - d$ independent edges

$\Leftrightarrow |\Gamma(A)| \geqslant |A| - d$ for all $A \subset X$.

Proof

Introduce d gregarious men known to all women. In the new graph Hall's Condition holds. Marry the women to all the men, then take away the d gregarious men. □

Corollary 2·3 (polyandrous version)

Let G be a bipartite graph and $d \in \mathbb{N}$. Then, each woman can have d husbands

$\Leftrightarrow |\Gamma(A)| \geqslant d\,|A| \quad \forall A \subset X$

Proof

Add to the set of women $d-1$ clones of each woman, each knowing the same men as their genetic mothers.

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Hall's Condition holds, then we remove the clones and reassign the husbands to the genetic mothers of their respective wives. □

A <u>transversal</u>, or <u>set of distinct representatives</u> for a family of sets $\mathcal{Y} = \{F_1, \ldots, F_n\} \subseteq \mathcal{P}Y$, is a set $\{y_1, \ldots, y_n\}$, $y_i \in F_i$, y_i are distinct.

<u>Corollary 2.4</u>

$\mathcal{Y}$ has a set of distinct representatives $\Leftrightarrow |\bigcup_{i \in I} F_i| \geq |I|$ $\forall I \subset [n]$.

<u>Proof</u>

edge F − y

Construct a bipartite graph with $X = \mathcal{Y}$, $E = \{F_y : F \in \mathcal{Y}, y \in F\}$ □

Y as given

A graph G is <u>k-connected</u> if $|G| > k$ and $G - S$ is connected $\forall S \subseteq V(G)$, $|S| < k$.

<u>Vertex connectivity of G</u> $\kappa(G) = \max\{k : G \text{ is } k \text{ connected}\}$

If G is not complete, $\kappa(G) = \min\{|S| : S \subset V(G), G - S \text{ disconnected}\}$

<u>Local Connectivity</u>

Let $a, b \in V(G)$, $ab \notin E(G)$ then

$\kappa(a,b;G) = \min\{|S| : S \subseteq V(G) \setminus \{a,b\}, \text{no } ab \text{ path in } G \setminus S\}$

If G is not complete, $\kappa(G) = \min_{a,b} \kappa(a,b;G)$

There are corresponding (local) <u>edge-connectivities</u>.

$\lambda(G) = \min\{|F| : F \subset E(G), G \setminus F \text{ is disconnected}\}$

$\lambda(a,b;G) = \min\{|F| : F \subset E(G), \text{no } ab \text{ path in } G \setminus F\}$

Given $a, b \in V(G)$, a set $\{P_1, \dots, P_t\}$ of ab paths is <u>vertex-disjoint</u> if P_i, P_j have no vertices in common except a, b, $1 \leq i < j \leq t$.

How large a set of vertex-disjoint ab-paths can we find?

Clearly, no more than $\kappa(a, b; G)$, because if $S \subset V(G)$ and $G \setminus S$ has no ab path, then each of $P_1, \dots, P_t$ must contain a member of S.

<u>Theorem 2.5</u> (Menger's Theorem)

Let $a, b \in V(G)$, $ab \notin E(G)$. Then there exists a set $\kappa(a, b; G)$ vertex disjoint ab paths.

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Theorem 2.5 (Menger)

If $ab \notin E(G)$ then $\exists$ a set $K(a,b;G)$ vertex disjoint ab path

Proof 1

Apply the Max-Flow Min-Cut Theorem.

Proof 2

Suppose not: let G be ~~the~~ minimal such that it has vertices a, b for which G, a, b is a counterexample. Let $k = K(a,b;G)$.

Suppose that every edge of G lies inside $\Gamma(a)$ or $\Gamma(b)$.

contraction

Then $\Gamma(a) \cap \Gamma(b)$ separates a from b (we say that $S \subset V(G)$ separates a from b if there is no ab path in $G \setminus S$). Thus $|\Gamma(a) \cap \Gamma(b)| \geq K(a,b;G) = k$, and so there exists a set of k ab paths, contradiction ※.

So there exists some edge $e = uv$ not wholly inside $\Gamma(a)$ or $\Gamma(b)$.

Let G/e be the graph obtained by contracting e; that is, take $G - u - v$, add new vertex w, and join w to each vertex in $\Gamma(u) \cup \Gamma(v)$.

Suppose there are k vertex disjoint ab paths $P_1, \dots, P_k$ in G/e. These paths cannot all lie in G, so one of them, P_1 say, contains w. But then, there exists a path P_1' containing one or both of u, v lying in G, disjoint from $P_2, \dots, P_k$ which are also in G, giving k ab-paths in G, a contradiction. ※

So G/e has no such k paths, and since G/e is not

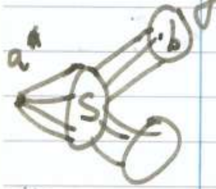
a counter-example, $K(a,b;\, G/e) < k$. So there exists a set S', $|S'| < k$, separating a from b in G/e.
If $w \notin S'$, then S' separates a from b in G itself, contradicting $K(a,b;\,G) = k$ ✗

G

So $w \in S'$, and $S = (S' - \{w\}) \cup \{u, v\}$ separates a, b in G. Now $|S| = |S'| + 1 \le k$, so $|S| = k$, S separates a from b, and (by choice of e) $S \not\subseteq \Gamma(a)$ and $S \not\subseteq \Gamma(b)$.

Form G_a by "contracting" the component C of $G - S$ containing a to a single vertex: take $G - V(C)$, add a new vertex a^* joined to S.

G_a



Since $S \not\subseteq \Gamma(a)$, $G_a \neq G$, so by the theorem applied to G_a, a^*, b, there exist $K(a^*, b, G_a)$ vertex disjoint a^*b paths in G_a. Any set separating a^* from b in G_a separates a from b in G, and so, by definition, $K(a^*, b, G_a) \geq k$.
Similarly, define G_b, and get k vertex disjoint ab^* paths in G_b.
The paths in G_a from S to b plus the paths in G_b from a to S give k vertex disjoint ab paths in G, the final contradiction. □

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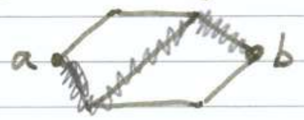
Graph Theory ⑥

Note

If say, $\kappa(a,b;G) = 2$, then Menger says that there are two vertex disjoint ab paths. It does not give you a path disjoint from one you already have.



Corollary 2.6

Let $\kappa(G) \geq k$, and let $X, Y \subset V(G)$, $X \cap Y = \emptyset$, $|X|, |Y| \geq$
Then there exist k vertex disjoint X-Y paths (completely disjoint)

Proof

Form G^* by adding new vertices x joined to all of X and y joined to all of Y. Then $\kappa(x, y; G^*) \geq k$, so there are k vertex disjoint xy paths in G^*, giving the desired paths in G. □

There is an edge form of Menger's Theorem.

Theorem 2.7 (Menger, edge form)

Let G be a graph, $a, b \in V(G)$. Then there is a set of $\lambda(a,b;G)$ edge disjoint ab paths (and clearly no larger,

Proof

Either argue directly, as with the vertex form, or construct the line graph, $L(G)$ whose vertex set is $E(G)$, ~~and~~ with $ef \in E(L(G))$ if e, f are incident in G. Add a new vertex a^* joined to all vertices of ~~the~~ ~~L~~ (G) which are edges of G

incident with a. Add b^* likewise. Then

$\kappa(a^*, b^*, L(G)) \geq \lambda(a, b; G)$, so there exist

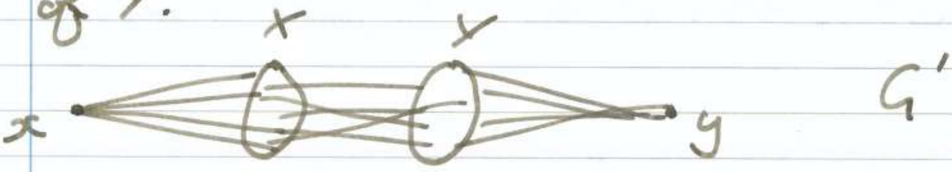
$\lambda(a, b; G)$ vertex disjoint $a^* b^*$ paths in $L(G)$, which give

$\lambda(a, b; G)$ edge disjoint paths in G. $\square$

Menger $\Rightarrow$ Hall

$G = X \cup Y$, a bipartite graph. $\forall A \subset X$, $|\Gamma(A)| \geq |A|$

Add x, vertex connected to all of X, and y connected to all of Y.



Suppose $< |X|$ vertices of $G' \setminus \{x, y\}$ are removed.

Let $C_X = \{$vertices removed from $X\}$, similarly C_Y,

where $|C_X| + |C_Y| < |X|$.

Then by Hall's condition, $|\Gamma(X \setminus C_X)| \geq |X \setminus C_X|$

$$= |X| - |C_X| > |C_Y|$$

So $X \setminus C_X$ still has edges leading to vertices of Y.

$\therefore$ $\kappa(x, y; G) \geq |X|$.

Menger $\Rightarrow \exists$ $|X|$ vertex disjoint paths $X \to Y$.

These give a matching $X - Y$.

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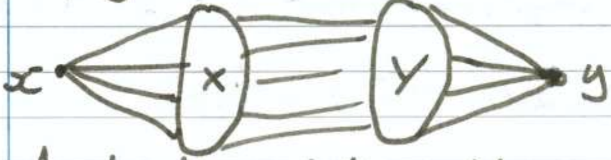
Graph Theory ⑦

Menger $\Rightarrow$ Hall



If $|\Gamma(A)| \geq |A| \ \forall A \subset X$

then $K(x, y; G) \geq |X|$ (check)

Actually, Hall $\Rightarrow$ Menger $\Rightarrow$ Max Flow - Min Cut (all ~~differ~~ equivalent in some sense)

Chapter 3 : Extremal Graph Theory

A Hamiltonian circuit in a graph is a spanning circuit, i.e. meets every vertex exactly once. A graph is Hamiltonian if it has a Hamiltonian circuit. (Related to the travelling salesman problem.

There is no known nice necessary and sufficient condition for HC (Hamiltonian Circuits) "unless" $P = NP$.

Question: How many edges must a graph have before it has a Hamiltonian Cycle?

 X

We need $> \binom{n}{2} - (n-2)$ so this is not very interesting (a nearly complete graph)

Question: How large must $\delta(G)$ be to force a Hamiltonian Cycle? We need $\delta(G) \geq \frac{n}{2}$, e.g.

 X $K_{\lfloor \frac{n+1}{2} \rfloor}$ and $K_{\lceil \frac{n+1}{2} \rceil}$ with a common vertex.

or $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil}$. These examples are very different, so are we far from the correct value?

Theorem 3.1

Let $k \in \mathbb{N}$. Let G be a graph of order $n \geqslant 3$ in which every pair a, b of non-adjacent vertices satisfies $d(a) + d(b) \geqslant k$.

If $k < n$ and G is connected, then G has a path of length k.
If $k = n$, then G is Hamiltonian

Proof

We may assume that G has no HC (otherwise there is also a path of length k and we are done). If $k = n$, then a, b have a common neighbour, so in every case, we may assume that G is connected. Let $P = v_1 v_2 \dots v_L$ be a path of (length $L-1$) maximum length in G. Then G has no L-circuit, for if $L = n$, this is a Hamiltonian Circuit, and if $L < n$, since G is connected, we could get a path of length L. Hence $v_1 v_L \notin E(G)$ so $d(v_1) + d(v_L) \geqslant k$.

Let $S = \{i : v_1 v_i \in E(G)\}$, $T = \{i+1 \mid v_i v_L \in E(G)\}$ note

Now $S, T \subseteq \{2, \dots, L\}$. By the maximality of P, $|S| = d(v_1)$, $|T| = d(v_L)$, and moreover, $S \cap T = \emptyset$, for if $j \in S \cap T$ then $v_1 v_2 \dots v_{j-1} v_L v_{L-1} \dots v_j$ is an L-circuit. Therefore,

$$L - 1 \geqslant |S \cup T| = |S| + |T| = d(v_1) + d(v_L) \geqslant k$$

If $k = n$ then this is impossible, so G is Hamiltonian.
If $k < n$, then P has length $\geqslant k$. $\square$

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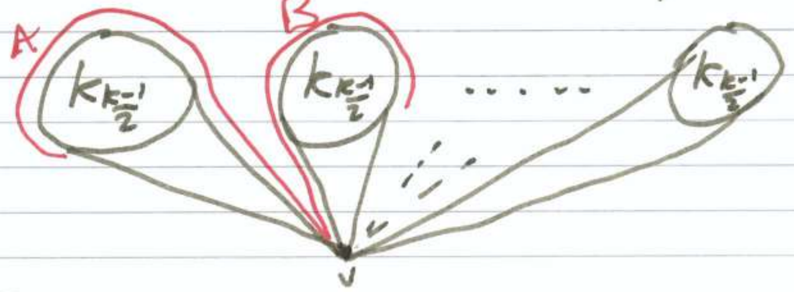
Graph Theory ⑦

Corollary 3.2 (Dirac)

If $\delta(G) \geq |G|/2$ then G is Hamiltonian.

Note : Theorem 3.1 is the best possible if $\frac{k-1}{2} \mid n-1$



Longest path visits all vertices of A, then v, then all vertices of B, WLOG

Theorem 3.3

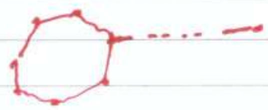
Let G be a graph of order n, with no path of length k.
Then $e(G) \leq (k-1)\frac{n}{2}$ with equality $\Leftrightarrow k \mid n$ and G is a disjoint union of many K_k.

$k<n$, no path of length k $\Rightarrow \exists\, a, b$, non-adjacent $d(a)+d(b)<k$

Proof

By induction on n, $n \leq k$ trivial. In general, if G is disconnected, the result holds by applying induction to each component. If G is connected, by Theorem 3.1, $\boxed{\delta(G) \leq \frac{k-1}{2}}$.
Let x be a vertex, $d(x) \leq \frac{k-1}{2}$. Since G is connected, it cannot contain K_k, so $e(G-x) < (k-1)\frac{n-1}{2}$ by our induction hypothesis. So $e(G) < \frac{k-1}{2}n$ $\square$

Otherwise, since $n > k$, we could have a path of length k



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Graph Theory (8)

Remark

Theorem 3·3 is an example of an <u>extremal</u> theorem. The disjoint union of K_k s was the <u>extremal graph</u>.

How many edges can a triangle free graph have?

Bipartite graphs have no triangles. Can we do better?

Theorem 3·4

Let G be a graph of order n, containing no K_3. Then, there exists a bipartite graph H with $V(H) = V(G)$ and $e(G) \leq e(H)$.

In particular $e(G) \leq \lfloor n^2/4 \rfloor$ → Maximum size of the complete bipartite graph on n vertices.

Proof (Erdős)

Let u be a vertex of maximum degree in G. Let H be the complete bipartite graph with ~~vertex~~ bipartition $\Gamma(u)$, $V(G) - \Gamma(u)$.

If $v \in \Gamma(u)$ then since $\Gamma(u)$ has no edges (because G is triangle free), $d_G(v) \leq |V(G) - \Gamma(u)| = d_H(v)$

If $v \in V(G) - \Gamma(u)$, then $d_G(v) \leq d_G(u) = |\Gamma(u)| = d_H(v)$

Then $e(G) = \frac{1}{2}\sum_v d_G(v) \leq \frac{1}{2}\sum_v d_H(v) = e(H)$ $\square$

Corollary (Mantel's Theorem, 1907)

If $|G| = n$, G triangle free, then $e(G) \leq \lfloor n^2/4 \rfloor$

What about larger complete subgraphs?

Note that no r-partite graph contains K_{r+1}. Which r-partite graphs on n-vertices have most edges? Obviously they should be complete r-partite. Moreover, if two classes satisfy $|X| \geq |Y| + 2$

then reducing X by one, increasing Y by one gains $-|Y| + (|X|-1) > 0$ edges.

The r-partite Turán Graph is the complete r-partite graph of order n with class sizes differing by at most one, i.e. class sizes are $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. It is denoted by $T_r(n)$. Its size is denoted by $t_r(n)$. $\delta(T_{r}) = n - \lceil n/r \rceil$, $\Delta(T_r(n)) = n - \lfloor n/r \rfloor$

It is the unique r-partite graph of maximum size on n-vertices

Note $T_2(n)$ is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ and $t_2(n) = \lfloor n^2/4 \rfloor$

Two Remarks about $T_r(n)$

A vertex of minimum degree in $T_r(n)$ lies in a largest class, and its removal yields $T_r(n-1)$, so

(*) $t_r(n) - \delta(T_r(n)) = t_r(n-1)$

Note that $\Delta(T_r(n)) \leq \delta(T_r(n)) + 1$. So if G is a graph of order n, then

(**) $\delta(G) > \delta(T_r(n)) \Rightarrow e(G) > t_r(n)$

(Compare degree sequences of G and $T_r(n)$)

(In fact, $e(G) \geq t_r(n) + \frac{1}{2}M$, where $M = \#$ vertices in $T_r(n)$ of min degree)

Theorem 3.6 (Turán, 1941)

Let G be a graph of order n with no K_{r+1}.

$e(G) \leq t_r(n)$ with equality $\Leftrightarrow G = T_r(n)$.

Remarks

There are many proofs along the same lines of Theorem 3.4.

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Graph Theory (8)

Giving a Stronger Theorem (including cases of equality) makes the proof easier because our induction hypothesis is stronger.

<u>Proof</u> (induction on n)

The case $n \leq r$ is trivial because then $\underline{T_r(n) \text{ is complete}}$. single vertex ~~class~~ in each class

We prove the theorem by showing that if $\underbrace{e(G) \geq t_r(n)}_{\text{strengthening is here, includes } e(G) = t_r(n)}$, then $G = T_r(n)$.

First, remove edges from G if necessary to get G' with size $e(G') = t_r(n)$.

By (**) applied to G', $\delta(G') \leq \delta(T_r(n))$. Let x be a vertex of degree $\leq \delta(T_r(n))$ in G'. By (*), $e(G'-x) \geq t_r(n) - \delta(T_r(n)) = t_r(n-1)$

But $G'-x$ is K_{r+1} free, so by the induction hypothesis $G'-x = T_r(n-1)$.

Since G' is K_{r+1} free, x cannot have a neighbour in every class of $G'-x$, hence, G' is r-partite. But $e(G') = t_r(n)$ so $G' = T_r(n)$; the unique, r-partite graph of maximum size.

Finally, we cannot add an edge to G' without creating K_{r+1}, so $G = G'$.

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Graph Theory ⑨

Another Proof (Sketch): by induction $\exists K_r$, remove some K_r, no remaining vertex is joined to more than $r-1$ of K_r, and so $e(G) \leq \underbrace{\binom{r}{2}}_{e(K_r)} + \underbrace{(r-1)(n-r)}_{\text{cross edges}} + \underbrace{t_r(n-r)}_{\text{remainder has no } K_{r+1}} \leq t_r(n)$

Think about $T_r(n)$

We might now ask, for any fixed F, what is the extremal function $ex(n,F) = \max\{e(G) : |G| = n,\ F \not\subset G\}$

Mantel gives $ex(n, K_3) = t_2(n)$, Turan gives $ex(n, K_{r+1}) = t_r(n)$

Not surprisingly, there is no exact answer in general.

For small n it can be messy, but as n grows, (F fixed), clarity can emerge.

P  Petersen

$ex(n, C_5) = t_2(n)$, $n \geq 6$.

$ex(n, P) = t_2(n) + n - 2$, $n \geq n_0$ (some n_0)

Usually, even finding $ex(n,F)$ for large n is hopeless.

Can we find it asymptotically? In other words, can we find $\lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{2}}$? N.B. this limit exists (exercise)

Observe $t_r(n) = (1-\frac{1}{r})\binom{n}{2} + O(n)$, so

$ex(n; K_{r+1}) / \binom{n}{2} \to 1 - \frac{1}{r}$

The basis for further results is the remarkable theorem of Erdös and Stone, saying that if $e(G)$ is slightly larger than $t_r(n)$ we get not just K_{r+1} but $K_{r+1}(t) = T_{r+1}((r+1)t)$, the complete $(r+1)$ partite graph with t vertices in each class

[So $K_{r+1}(1) = K_{r+1}$, $K_2(t) = K_{t,t}$]. The Theorem holds "for n sufficiently large".

1

We prove first a weaker version where the minimum degree is large. This is in fact the heart of the proof: the main result follows easily.

Lemma 3·7 (examinable?)

Let $r, t \geq 1$ be integers and $\varepsilon > 0$ real. If n is sufficiently large (to be precise, $\exists\, n_1(r,t;\varepsilon)$ such that if $n \geq n_1$, the statement is true) then every graph G with $|G| = n$ and $\delta(G) \geq (1 - 1/r + \varepsilon)\, n$ contains $K_{r+1}(t)$.

Proof (by induction on r)

Base case : $r = 1$, General case : $r \geq 2$, simultaneously

Let $T = \lceil 2t/\varepsilon r \rceil$. We show 3 simple steps :

a) G contains $K_r(T)$: call it K (by induction)

b) $G - K$ has a large set U of vertices, each joined to at least t in each class of K.

c) Many vertices in U (certainly at least t) are joined to the same t in each class of K. This gives $K_{r+1}(t)$.

It will be evident that each step works providing n is large enough. But for definiteness, the following suffices :

a) $n(1, t, \varepsilon) \geq T$, $n(r, t, \varepsilon) \geq n_1(r-1, T, \frac{1}{r(r-1)})$ for $r \geq 2$

b) $n_1(r, t, \varepsilon) \geq 6rT/\varepsilon$

c) $n_1(r, t, \varepsilon) \geq \frac{2t}{\varepsilon r} 2^{rT}$

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Graph Theory ⑨

a) Certainly, if $r=1$, then G contains $K_1(T)$ if n is large, and if $r \geq 2$ then $\delta(G) \geq \left(1 - \frac{1}{r-1} + \frac{1}{r(r-1)}\right) n$, so by the induction hypothesis, $G \supset K_r(T)$ if n is large.

b) Let U be the set of vertices in $G-K$ having at least $(1-\frac{1}{r} + \varepsilon/2)|K|$ neighbours in K. Writing $e(K, G-K)$ for the number of edges between K and $G-K$, and recalling that every vertex in K has degree $\geq (1-\frac{1}{r}+\varepsilon)n$, we have:

$$|K|\left((1-\tfrac{1}{r}+\varepsilon)n - |K|\right) \leq e(K, G-K) \leq |U||K| + (n-|U|-|K|)(1-\tfrac{1}{r}+\tfrac{\varepsilon}{2})|K|$$

$$\frac{\varepsilon n}{2} - |K| \leq |U|\left(\frac{1}{r} - \varepsilon/2\right)$$

If n is large then $|K| \leq \varepsilon n/6$ so $\varepsilon n/3 \leq |U|/r$ so $|U| \geq \varepsilon r n/3$

c) There are at most $2^{|K|} = 2^{rT}$ ways that a vertex $u \in U$ can be joined to K. So there is some subset $W \subseteq U$, $|W| \geq |U|/2^{rT}$ of vertices joined to exactly the same subset of K. This subset has size $\geq (1-\frac{1}{r}+\frac{\varepsilon}{2})|K| = (1-\frac{1}{r}+\varepsilon/2)rT$
$= (r-1)T + \frac{\varepsilon r T}{2} \geq (r-1)T + t$

Therefore, the subset includes at least t in each class of K.
So if $|W| \geq t$, we have $K_{r+1}(t)$ as desired. This certainly happens because $|W| \geq |U|/2^{rT} \geq \frac{1}{2^{rT}} \varepsilon r/3 \; n \geq t$ if n is large. □

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Graph Theory (10)

Theorem 3·8 (Erdős-Stone 1946)

Let $r, t \geq 1$ be integers, and $\varepsilon > 0$ real. If n is sufficiently large (i.e. there exists $n_0(r,t,\varepsilon)$ so if $n \geq n_0$) then every graph G with $|G| = n$ and $e(G) \geq (1 - \frac{1}{r} + \varepsilon)\binom{n}{2}$ contains $K_{r+1}(t)$.

Proof

It is enough to show that G contains a large subgraph H with $\delta(H) \geq (1 - \frac{1}{r} + \frac{\varepsilon}{2})|H|$. → note! To be precise, we find such an H with $|H| \geq h = \lfloor \sqrt{\varepsilon}\, n/2 \rfloor$. Then taking $n_0 \geq \frac{2}{\sqrt{\varepsilon}} n_1(r, t, \varepsilon/2)$, we have $|G| \geq n_0 \Rightarrow |H| \geq n_1$, so by lemma 3·7 we have ~~$K_{r+1}(t)$ we have~~ $K_{r+1}(t) \subseteq H \subseteq G$.

(We shall also choose n_0 large enough so $\binom{h+1}{2} \geq n$ for a technical reason in the proof; we can do this because $\binom{h+1}{2}$ is of order n^2.)

Suppose that, on the contrary, there is no such H. Then there is a sequence of graphs $G = G_n \supset G_{n-1} \supset \ldots \supset G_h$ where $|G_j| = j$ and the vertex in G_j but not in G_{j-1} has degree $\leq (1 - \frac{1}{r} + \frac{\varepsilon}{2})|G_j|$ in G_j.

→ note!

But then $e(G_h) \geq (1 - \frac{1}{r} + \varepsilon)\binom{n}{2} - \sum_{j=h+1}^{n} (1 - \frac{1}{r} + \frac{\varepsilon}{2}) j$

$$= \left(1 - \frac{1}{r} + \frac{\varepsilon}{2}\right)\left(\binom{n+1}{2} - n\right) + \frac{\varepsilon}{2}\binom{n}{2} > \binom{h}{2}$$

which is impossible since $|G_h| = h$. This completes the proof. $\square$

The Erdős-Stone theorem gives information about $ex(n, F)$ for all F. The <u>chromatic number</u> $\chi(F)$ of F is the smallest number of colours wherewith we can colour the vertices of F so that no two adjacent vertices have the same colour, i.e. the smallest k such that F is k-partite.

<u>Corollary 3.9</u>

$$\lim_{n\to\infty} \frac{ex(n;F)}{\binom{n}{2}} = 1 - \frac{1}{\chi(F)-1} \quad \text{for all } F.$$

<u>Proof</u>

Let $r+1 = \chi(F)$. Then F is not r-partite, so $F \not\subset T_r(n)$, so $ex(n;F) \geq t_r(n) = \left(1-\frac{1}{r} + o(1)\right)\binom{n}{2}$ so $\lim_{n\to\infty} \frac{ex(n;F)}{\binom{n}{2}} \geq 1-\frac{1}{r}$.

On the other hand, if $e(G) \geq \left(1-\frac{1}{r}+\varepsilon\right)\binom{n}{2}$, and $|G|$ is large $\geq n_0(r, |F|, \varepsilon)$ then Erdős–Stone $G \supset K_{r+1}(|F|) \supset F$

so $\lim_{n\to\infty} \frac{ex(n;F)}{\binom{n}{2}} \leq 1-\frac{1}{r}+\varepsilon$. This is true for all $\varepsilon > 0$ so the result follows. $\square$

This means we know $ex(n;F)$ asymptotically, unless $\chi(F) = 2$, i.e. F is bipartite. In this case, we know only that $ex(n;F) = o(n^2)$.

Can we be more precise?

The natural analogue to the Turán problem is to seek $ex(n, K_{t,t})$. But, in fact the answer is not known. Even the order of magnitude is unknown except for $t=2$, $t=3$ (hard).

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Graph Theory ⑩

It is natural in this case to ask for the maximum size of a <u>bipartite</u> graph not containing $K_{t,t}$. This is known as the problem of Zarankiewicz. We define

$Z(n,t) = \max\{e(G) : G \text{ is } n\times n \text{ bipartite}, G \not\supset K_{t,t}\}$

Here "G is $n\times n$ bipartite" means that G is bipartite with two vertex classes (of size n), each of n vertices. So $|G| = 2n$ and $G \subseteq K_{n,n}$. In fact, $Z(n,t)$ is closely related to $ex(n;K_{t,t})$.

The following simple idea is more or less all that is known:

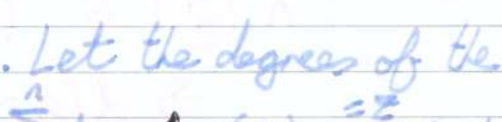
<u>Theorem 3·9</u>

$$Z(n,t) \leq (t-1)^{1/t}(n-t+1)n^{1-1/t} + (t-1)n \underset{t \text{ fixed},\, n \text{ large}}{=} O(n^{2-1/t})$$

<u>Proof</u>

Consider a bipartite $K_{t,t}$ free graph G with $e(G) = Z(n,t)$ with bipartition X, Y, $|X| = |Y| = n$. Let the degrees of the vertices in X be $d_1, d_2, \dots, d_n$ where $\sum_{i=1}^{n} d_i = e(G) = Z(n,t) = nd$.

Notice that $d_i \geq t-1$ or else we could add an edge to G without creating $K_{t,t}$, contradicting $e(G) = Z$.



Count "t-fans" or "t-stars"

There are $\binom{d_i}{t}$ t-fans centred at the i^{th} vertex of X.

On the other hand, no set of t vertices in Y is part of more than $t-1$ t-stars from X (else $G \supset K_{t,t}$). Since there are $\binom{n}{t}$ t-subsets we have $\sum_{i=1}^{n}\binom{d_i}{t} \leq (t-1)\binom{n}{t}$

3

The polynomial $\binom{x}{t}$ is convex for $x \geq t-1$, so

$$n\binom{d}{t} \leq \sum_{i=1}^{n}\binom{d_i}{t} \leq (t-1)\binom{n}{t}$$

d not necessarily integer.

$$\binom{d}{t} = \frac{d(d-1)\dots(d-t+1)}{t!}$$

Hence $\left(\frac{d-t+1}{n-t+1}\right)^t \leq \frac{d(d-1)\dots(d-t+1)}{n(n-1)\dots(n-t+1)} \leq \frac{t-1}{n}$ □

$$\Rightarrow \frac{d-t+1}{n-t+1} \leq \left(\frac{t-1}{n}\right)^{\frac{1}{t}}$$

$$\Rightarrow d \leq \left(\frac{t-1}{n}\right)^{\frac{1}{t}}(n-t+1) + (t-1)$$

$$\Rightarrow z(n,t) = nd \leq (t-1)^{\frac{1}{t}} n^{1-\frac{1}{t}} + (t-1)n$$

$$n\binom{d}{t} \leq (t-1)\binom{n}{t}$$

$$n\binom{d}{2} \leq \binom{n}{2}$$

$$\Rightarrow d(d-1) \leq n-1$$

$$d^2 - d - (n-1) \leq 0$$

$$d \leq \frac{1}{2}\left(1+\sqrt{4n-3}\right)$$

$$z(n,2) \leq \frac{1}{2}n\left(1+\sqrt{4n-3}\right)$$

N.B. $K_{2,2} = C_4$

27/10/12 Graph Theory (11)

Theorem 3·11

$Z(n,2) \leq \frac{1}{2} n (1+\sqrt{4n-3})$ and equality holds infinitely often.

In the proof above, we have $n\binom{d}{t} \leq (t-1)\binom{n}{t}$ or $d(d-1) \leq n-1$

so $d \leq \frac{1}{2}(1+\sqrt{4n-3})$

Note $n\binom{d}{t} \leq \sum \binom{d_i}{t} \leq (t-1)\binom{n}{t}$

The proof shows that equality holds if and only if d is an integer, and there is a d-regular bipartite graph such that any two vertices in X have exactly one common neighbour in Y (and, of course, the same is true with X and Y interchanged).

This is equivalent to the existence of a projective plane of order p, where $p^2+p+1=n$. This is a set of n points together with n subsets called lines: each line contains $p+1$ points, each point lies in $p+1$ lines, every two points lie in exactly one common line, every two lines intersect in one point.

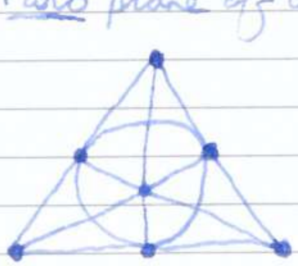
Let X = points, Y = lines, $xy \in E \Rightarrow$ point x is in line y.

$d = p+1$, no $K_{2,2}$.

It is known that these exist when p is prime (or a prime power).

It is known that no plane exists of order 6 (easy) or 10 (hard).

The Fano plane of order 2.



A Sidon set (in $[n]$ or $\mathbb{Z}/n\mathbb{Z}$) is a set S whose pairwise sums are distinct: i.e. if $a, b, c, d \in S$ and $a+b = c+d$ then $a=c, b=d$ or $a=d, b=c$.

Corollary

If $S \subseteq \mathbb{Z}/n\mathbb{Z}$ is a Sidon set then ~~$|S| \leq n^{1/2}+1$~~ $|S| \leq n^{\frac{1}{2}}+1$

Proof

Given S, consider the ~~set~~ bipartite graph with vertex classes X, Y, both copies of $[n]$, with $xy \in E \Leftrightarrow y = x+s$, some $s \in S$.

If G contains $K_{2,2}$, there exist $a, b, c, d \in S$ such that $x_1 + a = x_2 + c$ and $x_1 + d = x_2 + b$, some $x_1, x_2 \in X$.

Thus $a - c = x_2 - x_1 = d - b$ so $a+b = c+d$. Since S is Sidon, $K_{2,2} \not\subseteq G$. So $e(G) \leq \frac{1}{2} n \left(1 + \sqrt{4n-3}\right)$. But $e(G) = n|S|$.

Therefore $|S| \leq \frac{1}{2}\left(\sqrt{4n-3}+1\right)$ $\square$

4 Eigenvalue methods

Some extremal properties of graphs are constrained by algebraic factors lurking beneath the surface.

Let G be a graph with vertex set $[n]$. The adjacency matrix of G, $A(G)$, is the $n \times n$ matrix $A = (a_{ij})$.

$$a_{ij} = \begin{cases} 1 & ij \in E(G) \\ 0 & ij \notin E(G) \end{cases}$$

Since a is real symmetric, it is diagonalisable, and has n real eigenvalues (which in principle, are easily calculated).

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Graph Theory (11)

Observe that $(A^2)_{ij} = \sum_l a_{il} a_{lj}$ is the number of <u>walks</u> of length 2 from i to j, and more generally $(A^k)_{ij}$ is the number of walks, length k, from i to j (that is, sequences of vertices $ipqr\dots uvj$, consecutive pairs being adjacent vertices)

In particular, if G is connected with diameter

$$\mathrm{diam}(G) = \max\{d(u,v) : u, v \in G\}$$

then $\{I, A, A^2, \dots, A^{\mathrm{diam}(G)}\}$ forms a linearly independent set.

So A has at least $1 + \mathrm{diam}(G)$ distinct eigenvalues.

Recall that there is an orthonormal basis of eigenvectors $e_1, \dots, e_n$ for A, with eigenvalues $\lambda_1, \dots, \lambda_n$. So for any vector $x = \xi_1 e_1 + \dots + \xi_n e_n$ with $\|x\|^2 = \sum \xi_i^2 = 1$ we have $x^T A x = \sum_i \lambda_i \xi_i^2$. Therefore, writing λ_{min} and λ_{max} for the min and max eigenvalues, we have

$$\lambda_{min} = \min_{\|x\|=1} x^T A x, \quad \lambda_{max} = \max_{\|x\|=1} x^T A x$$

If $E(G) \neq \emptyset$, say $12 \in E$, then $x = (1, 1, 0, 0, \dots, 0)/\sqrt{2}$ shows $\lambda_{max} > 0$ and $x = (1, -1, 0, 0, \dots, 0)/\sqrt{2}$ shows $\lambda_{min} < 0$.

Let $H = G[W]$ be an induced subgraph of G. If y is a unit vector in $\mathbb{R}^W$ let x be the unit vector in $\mathbb{R}^n$ equal to y on W and 0 on $[n] - W$.

We can choose y, for example, so that

$x^T A x = \lambda_{min}(H)$ so $\lambda_{min}(G) \leq \lambda_{min}(H) \leq \lambda_{max}(H)$
$\leq \lambda_{max}(G)$

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Graph Theory ⑫

If G is bipartite, we can write $A = \begin{array}{c} \overset{k}{\longleftrightarrow} \\ k\updownarrow \left(\begin{array}{c|c} 0 & \\ \hline & 0 \end{array}\right) \updownarrow n-k \end{array}$

Consider $\det(\lambda I - A)$. If you pick t λs from the left hand k columns, then we must have $k-t$ entries from the bottom left. So we must pick $n-k-(k-t)$ λs from the bottom right hand side giving a term const. $\times\, \lambda^{n-2k+2t}$. So if n is even, we get a polynomial in λ^2, and if n is odd, we get λ times such a polynomial. Hence if λ is an eigenvalue, so is $-\lambda$.

<u>Theorem 4.1</u>

Let G be a graph.

a) $\delta(G) \leq \lambda_{max} \leq \Delta(G)$

b) $|\lambda_i| \leq \Delta(G)$ for all i.

Moreover, if G is connected:

c) Δ is an eigenvalue $\Leftrightarrow$ G is regular, and the multiplicity is 1.

d) $-\Delta$ " " $\Leftrightarrow$ G is regular and bipartite.

<u>Proof</u>

If $y = (a_1, \dots, a_n)$ then $(Ay)_i = \sum_{j\in\Gamma(i)} a_j$ and $y^T A y = 2\sum_{ij\in E(G)} a_i a_j$. Taking $x = (1, 1, \dots, 1)/\sqrt{n}$, we have $x^T A x = \frac{2e(G)}{n} \geq \delta(G)$, proving the first part of a).

Now pick y with $Ay = \lambda_i y$, find m with $|a_m| \geq |a_p|\ \forall p$ and we may assume that $a_m = 1$. So $|\lambda_i| = |\sum_{j\in\Gamma(m)} a_j| \leq \sum_{j\in\Gamma(m)} |a_j|$
$\sum_{j\in\Gamma(m)} |a_j| \leq d(m) \leq \Delta(G)$, proving b) and the 2^{nd} part of a).

$|\lambda_i| = |\lambda_i x_p| = |(Ax)_p| = |\sum^{n} a_{pl} x_l| \leq \sum^{n} a_{pl}|x_l| \leq |x_p| d(v_p) \leq \Delta$

Moreover, if $\lambda_i = \Delta$ then $d(m) = \Delta$ and furthermore $a_j = 1$ $\forall j \in \Gamma(m)$. Applying the same argument to some $j \in \Gamma(m)$, we find that $d(l) = \Delta$ $\forall l \in \Gamma(j)$, and a_l is 1 likewise. By connectivity, we find that G is Δ-regular, and $y = (1, 1, \dots, 1)$, proving c).

Finally, if $\lambda_i = -\Delta$, then $d(m) = \Delta$ and $a_j = -1$ $\forall j \in \Gamma(m)$. Applying the argument to j, we find that $d(j) = \Delta$ and $a_l = 1$ for all $l \in \Gamma(j)$. In the end, we find that G is δ-regular, with $a_j = \pm 1$ and if jl is an edge, then $a_j a_l = -1$. $\square$

Let $\vec{G}$ be an orientation of G (give a direction to every edge). The <u>incidence matrix</u> B of $\vec{G}$ is the $n \times e(G)$ matrix

$$B = (b_{ve}) \qquad b_{ve} = \begin{cases} 1 & \text{if } e = uv \\ -1 & \text{if } e = vu \\ 0 & \text{otherwise} \end{cases}$$

The <u>(combinatorial) Laplacian</u> is $L = BB^t$. Note that $L = D - A$, where $D = (d(1), d(2), \dots, d(n))$ is the diagonal matrix of degrees. In particular, L does <u>not</u> depend on the orientation.

The fact that $L = BB^t$ means that L is positive semi-definite.

Let the eigenvalues of L be $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ (L is real symmetric) so $\mu_1 \geq 0$. Indeed, if $x = (x_1, \dots, x_n)$, then

$$x^t L x = (B^t x)^t (B^t x) = \sum_{ij \in E(G)} (x_i - x_j)^2$$

In particular, if $x = (1, 1, \dots, 1)$, then $Lx = 0$, so $\mu_1 = 0$.

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Graph Theory (12)

If G is Δ-regular, then the eigenvalues of $L = \Delta I - A$ are just $\Delta - \lambda_i$, but in general the μ_i and λ_is are unrelated.

The value of μ_2 is of special importance.

<u>Theorem 4·2</u>

Let G be a graph, and $U \subseteq V = V(G)$. Then, there are at least $\mu_2 \frac{|U||V-U|}{|V|}$ edges between U and $V-U$.

<u>Proof</u>

We may assume that $0 < |U| = k < n = |G|$.

Let $x = (x_1, \dots, x_n)$, where $x_i = n-k$ for $i \in U$ and $x_i = -k$ for $i \notin U$. L, being real symmetric, has an orthonormal basis of eigenvalues $e_1, \dots, e_n$, where $Le_i = \mu_i e_i$, and we may assume $e_1 = (1, \dots, 1)/\sqrt{n}$. Now $\langle x, e_1 \rangle = 0$ so

$x = \sum_{i=2}^{n} \xi_i e_i$.

So $Lx = \sum_{i=2}^{n} \xi_i \mu_i e_i$, and $x^T L x = \sum_{i=2}^{n} \mu_i \xi_i^2 \geq \mu_2 \sum \xi_i^2$

$= \mu_2 \|x\|^2 = \mu_2 nk(n-k)$. But $x^T L x = \sum_{ij \in E} (x_i - x_j)^2$

$= n^2 |F|$, where F is the number of edges from U to $V-U$ $\square$

Theorem 4·2 is one important way of showing that a graph "expands".

A graph of diameter 2 and maximum degree d has at most $1+d^2$ vertices

 $e(G) \leq 1 + d + d(d-1) = 1 + d^2$.

We attain $n = d^2 + 1$ if and only if G is d-regular and has girth 5 (all vertices in the picture must be distinct). Such a graph is called a <u>Moore</u> graph. Do they exist?

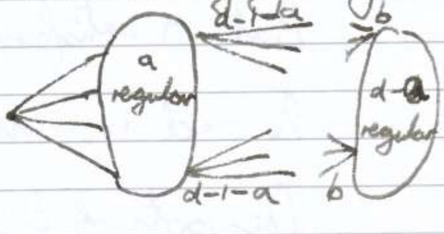
$d = 2$  $d = 3$  Petersen Graph

$d = 4$  Can't fill in the picture. For $d = 4$, no Moore graph.

1/11/12 Graph Theory (13)

A ***strongly-regular*** graph with parameters (d, a, b) is a d-regular graph in which every pair of adjacent vertices has

numbers → a common neighbours, every pair of non-adjacent vertices having b-common neighbours. A Moore graph is our strongly regular graph with parameters $(d, 0, 1)$.



From every vertex, the graph looks like

Note that $|G| = 1 + d + \frac{d(d-1-a)}{b}$

Let $J = $ ($n \times n$ matrix of 1s). Let $B = J - I - A$, the adjacency matrix of the complement $\bar{G}$. $(\bar{G} = (V(G), V(G)^{(2)} - E(G)))$.

Then $A^2 = dI + aA + bB$ (think of walks of the form edge-edge i to j).

$BA = (d-1-a)A + (d-b)B$ (think of walks of the form nonedge-edge i to j).

So $A^3 = dA + aA^2 + bBA = dA + dA^2 + b(d-1-a)A + \cancel{b(d-b)B}$
$+ (d-b)[A^2 - dI - aA]$

that is, A satisfies the cubic $A^3 - (d-b+a)A^2 - [d(b-a)+d-b]A + d(d-b)I = 0$

<u>Theorem 4·3</u>

Let G be strongly regular with parameters (d, a, b), $b > 0$

Then $\frac{1}{2}\left\{n - 1 + \frac{(n-1)(b-a) - 2d}{\sqrt{(a-b)^2 + 4(d-b)}}\right\} \in \mathbb{Z}$, where $n = |G|$.

<u>Remark</u>

We exclude the case where $b = 0$, which happens when

1

when each component of G is K_{d+1}. Since $b > 0$, we know that G is connected, having diameter 2.

Proof

We know that A has eigenvalue d with multiplicity 1 (Theorem 4.1

But A satisfies a cubic, and so there are only two other eigenvalue λ and μ, with multiplicities r and s, where $r+s = n-1$.

The roots of the cubic $t^3 - (d-b+a)t^2 - [d(b-a)+d-b]t + d(d-$

$= (t-d)(t^2 - (a-b)t - (d-b)) = 0$ are d, λ, μ.

$$\lambda, \mu = \frac{1}{2}\left\{a - b \pm \sqrt{(a-b)^2 + 4(d-b)}\right\}$$

Also, $r+s = n-1$ and $Tr(A) = 0 = d + r\lambda + s\mu$

Then $r(\lambda-\mu) + (r+s)\mu = -d$ so $r\sqrt{\cdots\cdots} = -d-(n-1)\mu$

$= \frac{1}{2}\left\{(n-1)(b-a) - 2d + (n-1)\sqrt{\cdots\cdots}\right\}$ $\square$

So for the Moore graph, where $n-1 = d^2$, $\frac{d}{2}\left\{d + \frac{d-2}{\sqrt{4d-3}}\right\} \in \mathbb{Z}$

Either $d = 2$ or $4d-3 = \ell^2$, a perfect square

Corollary 4.4

There exists a Moore graph only if $d = 2, 3, 7$, or (possibly) 57.

Proof

If $d \neq 2$, then $4d-3 = \ell^2$ and $r = \frac{d}{2}(d + \frac{d-2}{\ell})$

so $d^2\ell - 2r\ell + d(d-2) = 0$, then

$16d^2\ell - 32r\ell + (\ell^2+3)(\ell^2-5) = 0$

So $\ell \mid 15$. $\ell = 1, 3, 5, 15$, so $d = 1, 3, 7, 57$

1/11/12

Graph Theory (13)

The case $d=1$ makes no sense. For $d=2$, C_5 is unique, $d=3$, Petersen is unique. For $d=7$, the Hoffman-Singleton graph is unique. $d=57$ is unknown, but cannot be 'symmetrical' □

5 Colouring

A (vertex) k-colouring of a graph G is a map $c: V(G) \to [k]$ such that $c(u) \neq c(v)$ if $uv \in E(G)$.

Chromatic number $\chi(G) = \min\{k : G \text{ is } k\text{-colourable}\}$

Unlike for $k=2$, there is no easy criterion for $\chi(G)=k$, $k \geq 3$.

Likewise, there is no known good algorithm for testing $\chi(G)=k$, $k \geq 3$ (NP-complete).

The Greedy algorithm produces an upper bound on χ by colouring the graph. Order the vertices $v_1, \ldots, v_n$ in some way.

Then colour them in this order, using for v_j the least colour not used in any previously coloured neighbour.

i.e. $c(v_j) = \min\left([k] - \{c(v_i) : i<j, v_iv_j \in E(G)\}\right)$

Note that the number of colours used depends on the ordering.

Theorem 5.1

$\chi(G) \leq \Delta(G)+1$

Proof

Regardless of the ordering, if $\Delta(G)+1$ colours are available, then the algorithm never gets stuck. □

By choosing the ordering we can do a bit better.

<u>Theorem 5.2</u>

$\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$, the maximum being over all (induced) subgraphs H of G.

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<u>Theorem 5·2</u>

$\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$

<u>Proof</u>

Let v_n be a vertex of minimum degree in $H_n = G$. Let v_{n-1} be of minimum degree in $H_{n-1} = G[V(G) - \{v_n\}]$ and in general let v_j be of minimum degree in ~~$H_j = G[V(G) - \{v_n$~~ $H_j = G[V(G) - \{v_n, v_{n-1}, \dots, v_{j+1}\}]$. Let $d = \max_j \delta(H_j)$

Run the greedy algorithm. When colouring v_j, at most $\delta(H_j)$ colours are forbidden, so with $d+1$ colours we never get stuck $\square$

Clearly $d \leq \max_{H \subset G} \delta(H)$, but in fact, equality holds. For if $H \subset G$ is induced, and let $m = \max\{j : v_j \in H\}$ then $\delta(H) \leq \delta(H_m)$

If $K(G) = 0$ then clearly $\chi(G) = \max \chi(C)$ where C runs over the components of G.

If G has a <u>cut vertex</u> x, i.e. $K(G) = 1$ but $K(G - x) = 0$ then $\chi(G) = \max \chi(G[V(C) \cup \{x\}])$ where C runs over the components of $G - x$. So when colouring, we usually assume that ~~$\chi(G)$~~ $K(G) \geq 2$

$\chi(G) \leq \Delta(G) + 1$

Which graphs attain equality in <u>Theorem 5·1</u>? $\chi(G) = 1 + \Delta(G)$ attained?

By Theorem 5·2, G must be regular.

<u>Theorem 5·3</u> (Brookes, 1941)

If G is connected, and $\chi(G) = 1 + \Delta(G)$, then $G \cong K_{\Delta+1}$ or $\Delta = 2$ and G is an odd circuit.

Proof

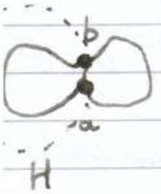
We apply induction on $|G|$. The case $K(G) = 1$ is taken care of by our previous remarks (think). So we assume $K(G) \geq 2$.
Also, if $\Delta = 1$ then $G = K_2$ and if $\Delta = 2$, then G is a path or a circuit, so we may assume that $\Delta \geq 3$.
We shall show that, unless G is complete, there exists a vertex v_n with neighbours v_1, v_2, such that $v_1 v_2 \notin E(G)$, and $G - \{v_1, v_2\}$ is connected.
In that case, we may label the vertices of $G - \{v_1, v_2\}$ as $v_3, \dots, v_n$, so that every vertex v_j, $3 \leq j \leq n-1$, has a neighbour v_i, $i > j$. To see this, we just label the vertices in reverse order: having picked $\{v_{j+1}, v_{j+2}, \dots, v_n\}$ we can find v_j with a neighbour in this set because $G - \{v_1, v_2\}$ is connected.
We run the Greedy Algorithm on this order. Then $c(v_1) = c(v_2) =$ because $v_1 v_2 \notin E(G)$. Moreover, $c(v_j) \leq \Delta$ for $3 \leq j \leq n-1$ because when we colour v_j it has $\leq \Delta - 1$ already coloured neighbours. Also, v_n has two neighbours of the same colour, so $c(v_n) \leq \Delta$.
To find v_n, v_1, v_2 if $K(G) \geq 3$, pick a vertex v_n of degree Δ. Since $G \neq K_{\Delta+1}$, v_n has two neighbours v_1, v_2 with $v_1 v_2 \notin E(G)$

03/11/12 Graph Theory (14)

$\kappa(G) \geq 3$, so $G - \{v_1, v_2\}$ is connected

There remains the case $\kappa(G) = 2$. Let $\{a, b\}$ be a cut-set.

For each component C of $G - \{a, b\}$, Let $H(C) = G[V(C) \cup \{a, b\}]$ with the edge ab added if not already present.



If $\chi(H(C)) \leq \Delta$ for all C, since these colourings can be fitted together (because $c(a) \neq c(b)$), we get $\chi(G) \leq \Delta$.

But if not, $H(C)$ is a complete graph, $H(C) = K_{\Delta+1}$, some C.

But then, $ab \notin E(G)$ and there are single edges aa', bb', joining a, b to the rest of the graph.

Take $v_1 = a$, $v_2 = a'$, $v_1 \in V(C)$ $\square$

The function $p_G(x)$ is defined, for $x \in \mathbb{N}$, to be the number of ways to colour G, with palette $[x]$.

<u>Example</u>

G = complete, K_r. $P_G(x) = x(x-1)\dots(x-r+1)$

$p_G(x) = x^r$ for $G = \overline{K_r}$

Recall from the proof of Mengers theorem the definition of G/e.

<u>Theorem 5.4</u>

$p_G(x) = P_{G-e}(x) - P_{G/e}(x)$ for all edges $e \in E(G)$

Proof

Let $e = uv$. The colourings of $G - e$ in which $c(u) \neq c(v)$ are exactly colourings of G, and the other colourings are colourings of G/e $\square$

It follows by induction on $|G| + e(G)$ that this function, $p_G(x)$ is a polynomial in x (though this is easily seen directly - Exercise)

This is called the chromatic polynomial of G.

Some obvious properties:

- $p_G(x) = \prod p_C(x)$ where C runs over the components of G.

The following is done by induction:

Corollary 5.5

$p_G(x)$ is a polynomial in x. Moreover

$$p_G(x) = x^n - a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + (-1)^k a_k x^k$$

where $n = |G|$, $a_{n-1} = e(G)$, $a_j \geq 0 \ \forall j$ and

$\min\{j : a_j > 0\} = k =$ number of components $\square$

06/11/12 Graph Theory (15)

A **k-edge colouring** of a graph G is a colouring $c : E(G) \to [k]$ such that $c(e) \neq c(f)$ if e, f are incident.

It is precisely a k-vertex colouring of the line graph $L(G)$, but edge colourings have special properties worthy of note.

The **chromatic index** is $\chi'(G) = \min\{k : G \text{ is } k\text{-edge-colourable}\}$

Clearly $\chi'(G) \geq \Delta(G)$. Also, $\chi'(G) \leq 1 + \Delta(L(G)) \leq 2\Delta(G) - 1$ and Brook's Theorem applied to $L(G)$ shows that $\chi'(G) \leq 2\Delta(G) - 2$ unless $G = K_2$ or an odd circuit. But far stronger results hold.

Theorem 5.6

If G is a bipartite multigraph then $\chi'(G) = \Delta(G)$

Remark 1

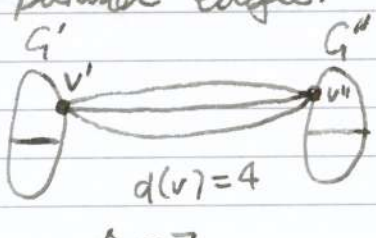
A multigraph is a graph where parallel edges are allowed. The degree of a vertex is the number of incident edges.

Remark 2

The theorem can be proved for graphs via Hall's Theorem (exercise)

Proof (Rizzi, 1998)

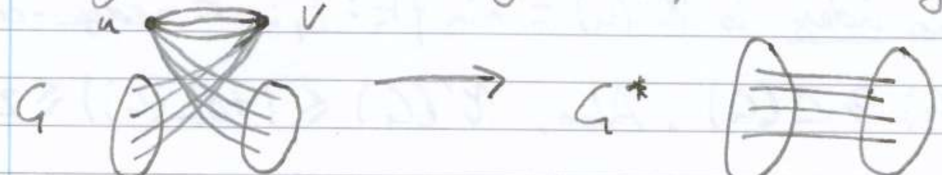
First, embed G in a Δ-regular bipartite multigraph, e.g. by taking two copies G', G'' and joining v' to v'' by $\Delta - d(v)$ parallel edges.



$\Delta = 7$

We prove the theorem for Δ-regular bipartite multigraphs, by induction on $|G| + \Delta(G)$.

Pick an edge uv of multiplicity $m \geq 1$. Add a set of $\Delta(G) - m$ parallel edges to $G - \{u, v\}$ between $\Gamma(u)$ and $\Gamma(v)$ to form a new Δ-regular bipartite multigraph G^* (possible).



Colour G^* with Δ colours, by induction. Some colour, red say is not used on the 'new' edges. The red edges together with one edge uv form a 1-factor of G. Remove this 1-factor: recolour the remaining $(\Delta-1)$-regular bipartite multigraph with $\Delta - 1$ colours (not including red), and now, colour the 1-factor red to get a Δ-edge colouring of G. $\square$

The following theorem is more remarkable:

<u>Theorem 5.7</u> (Vizing 1965)

$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for every graph G.

<u>Proof</u>

The lower bound is trivial. For the upper bound it suffices to show that if all but one edge of G is coloured using $\Delta + 1$ colours then we can recolour G so all edges are coloured (repeating as necessary).

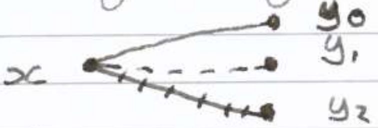
06/11/12

Graph Theory (15)

Let xy_0 be the uncoloured edge. At every vertex, there is (at least ~~one~~) one unused colour. Produce a maximal sequence of distinct edges $xy_0, xy_1, \ldots, xy_k$. Thus, given xy_i, let c_i be a colour missing at y_i, and let xy_{i+1} have colour c_i.

This sequence must end, and it does so for one of two reasons:

a) C_k is unused at x, or



b) the edge coloured C_k at x already appeared as xy_{j+1}, some $j \geq 0$

i.e. $C_k = C_j$, for some $j < k$.

In case a), recolour xy_i with c_i, $0 \leq i < k$, and colour xy_k with C_k.

In case b), we first recolour xy_i with c_i for $0 \leq i < j$, and leave xy_j uncoloured. Let c be a colour missing at x. Let H be the subgraph of G consisting of the edges coloured using C_k and c. If x and y_j are in different components of H, then swap the colours in the component meeting y_j and colour xy_j with c.

So we may assume that x, y_j, lie in the same component of H.

But $\Delta(H) \leq 2$ so its components are paths and circuits.

Moreover $d_H(x), d_H(y_j), d_H(y_k) \leq 1$, so x, y_k, y_j lie at the ends of paths of H. Since x, y_j lie in the same component, y_k lies in another component. Swap the colours in this component, recolour xy_i with c_i, $j \leq i < k$, and colour xy_k with c □

2

08/11/12

Graph Theory (16)

Let G be planar. Then $e(G) \leq 3|G| - 6$ and so $\delta(G) \leq 5$. Moreover, if $H \subseteq G$ then H is planar so $\delta(H) \leq 5$. By Theorem 5·2, $\chi(G) \leq 6$.   Chromatic number not index

We can do better.

Theorem 5·8 (Heawood 1890, 'Five-colour Theorem')

$\chi(G) \leq 5$ if G is planar.

Proof

Suppose not, and let G be a counterexample with $|G|$ minimal. Then $\delta(G) = 5$, because if for some v, $d(v) \leq 4$, colour $G - v$ with 5 colours, and there is still a colour that can be used on v.

Let v be a vertex with neighbours $u_1, u_2, \dots, u_5$. There exist i, j with $u_i u_j \notin E(G)$ (since $K_6 \not\subseteq G$), say $u_1 u_3 \notin E(G)$.

Remove v, and identify u_1 and u_3 (this is equivalent to removing vu_2, vu_4, vu_5 and then contracting vu_1, vu_3.

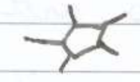
The resultant graph is planar, so colour it with 5 colours.

Now colour G as follows:

- All vertices except u_1, u_3 keep their colours.
- Give u_1 and u_3 the colour of the identified vertex.
- There is a colour free for v; since u_1, u_3 are the same colour, the neighbours of v have at most 4 colours

$\square$

The Four-Colour Problem (Guthrie 1852) is to show that every planar graph is 4-colourable. By considering dual-maps, this is equivalent to saying that the faces of any bridgeless plane map (plane graph) can be coloured with 4-colours so that contiguous faces get different colours.

Kempe (1879) "proved" the 4-colour theorem using methods akin to the proof of Vizing ('Kempe chains'). Heawood found a flaw in the proof. Tait (1880) "found" a beautiful equivalent form of the 4-colour Theorem, which he then "proved". First observe that it is enough to face colour cubic maps (3-regular plane graphs): to see this, either triangulate the dual, or replace  by .

Theorem 5·10 ('Tait 1880')

The 4-colour Theorem holds if and only if $\chi'(G) = 3$ for every cubic bridgeless plane graphs G.

Proof

We must show that a cubic bridgeless planar graph G is 4-face colourable iff it is 3-edge colourable. We use face colours $00, 01, 10, 11 \in \mathbb{Z}_2 \times \mathbb{Z}_2$, that is, we can add component-wise (mod 2). We use $01, 10, 11$ on edges.

Given a face colouring, colour an edge by the sum of the bordering face colours. Since G is bridgeless, the edge colour is not 00.

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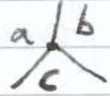
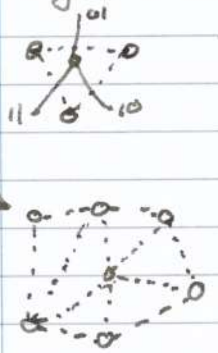
Graph Theory (16)

Since $a+c \neq b+c$ for 

, the edge colouring is proper, so $\chi'(G) = 3$.

Given an edge colouring, pick a face F_0 and colour it 00. Given a face F, travel a route from F_0 to F and colour F the sum of the edge colours crossed. This guarantees that contiguous faces get distinct colours, but is the colouring well defined? Do two different routes give the same answer?

This is equivalent to saying that if we travel from F_0 to F and back via another route, the sum of the traversed edge colours is 00. The dual is a triangulated map and we can label its edges with the label of the corresponding original edges. Then, if we go round a cycle in the dual, from the vertex corresponding to F_0, do the edges sum to ~~[illegible]~~ 00? The edges around a face total 00.



dotted lines are for the dual graph.

The edges around a circuit sum (mod 2) to the sum (mod 2) of the sums (mod 2) around the interior triangles. $\square$

Tait's conjecture (which he thought that he had proved) was that every cubic, bridgeless planar map has a Hamiltonian circuit. Since cubic graphs have even order, colour the circuit's edges red and blue, and the others green. Thus $\chi'(G) = 3$.

But Tutte (1946) found a counterexample. The smallest counterexample has 38 vertices.

10/11/12

Graph Theory 17

Call H a 'minor' of G if H can be obtained by a sequence of vertex and edge deletions and edge contractions, written $G > H$. Hadwiger (1943) conjectured that if $\chi(G) = r$, then $G > K_r$. For $r = 5$ this implies the 4-colour-theorem, and Wagner (1935) proved that the case $r=5$ is equivalent to the 4-colour theorem. The conjecture is easy for $r \leq 4$. For $r=6$ it is now known to be equivalent to the four-colour-theorem. For $r \geq 7$ no more is known.

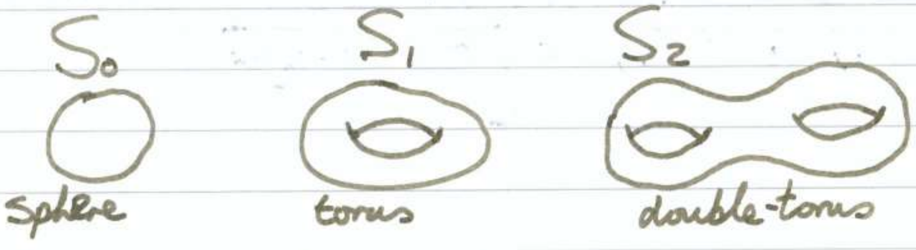
In 1976, Appel and Haken proved the 4-colour-theorem by computer using ideas of Heesch. Nobody has verified their proof. In 1997, Robertson, Sanders, Seymour and Thomas developed a simpler proof, again based on Heesch and computing.

R, S, S, T "The Four-Colour-Theorem"
Journal of Combinatorial Theory (Series B) 70 (1997) 2-44

Note that G is planar $\Leftrightarrow$ G is embeddable on the sphere. What about other (closed, bounded - compact!) surfaces that "look locally planar"?

The orientable surface S_g of genus $g \geq 0$ is a sphere with g handles added (a handle means to cut two discs out of the sphere joined by a tube).

e.g.



The non-orientable surface N_g of genus $g \geq 1$ is a sphere formed by ~~a sphere~~ cutting out g discs and identifying antipodal points with a disc.

e.g. N_1  projective plane N_2  klein bottle

It can be shown that every closed bounded surface is one of S_g or N_g.

Every surface has an <u>Euler-characteristic</u> E. For any map drawn on the surface with n vertices, m edges, f faces such that the faces are simply connected $n - m + f = E$.

For the sphere, $E = 2$. In general, $E = 2 - 2g$ for orientable surfaces and $E = 2 - g$ for non-orientable surfaces.

Given a graph G with n vertices and $e(G)$ edges drawn on a surface, add edges until every face is a triangle. (multiple edges are ok). You now have $2m = 3f = 3(E-n+m$ so $m = 3(n-E)$. Hence $\underline{e(G) \leq 3(n-E)}$

For the projective plane $E = 1$, $e(G) \leq 3n - 3$, so $\delta(G) \leq 5$ for any graph on N_1. Likewise $\delta(H) \leq 5$ for all subgraphs H of G, so $\chi(G) \leq 6$ (Theorem 5·2).

<u>Theorem 5·11</u> (Heawood 1890)

If G can be drawn on a surface of Euler characteristic $E \leq 1$ then $\chi(G) \leq H(E) = \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor$

<u>Proof</u>

$E = 1$ is already done, so consider $E \leq 0$. Let G be

10/11/12 Graph Theory (17)

Let G be a minimal graph embeddable on the surface with $\chi(G) = k$. Then $\delta(G) \geq k-1$
$(\chi(G-v) \leq k-1$ but $\chi(G) \geq k$ so $d(v) \geq k-1)$

Thus $k-1 \leq \delta(G) \leq \frac{2e(G)}{|G|} \leq \frac{6}{|G|}(|G|-E) = 6 - 6\frac{E}{|G|}$
$\leq 6 - \frac{6E}{k}$
since $E \leq 0$ and $|G| \geq k$.
So $k^2 - 7k + 6E \leq 0$, so $k \leq H(E)$ □

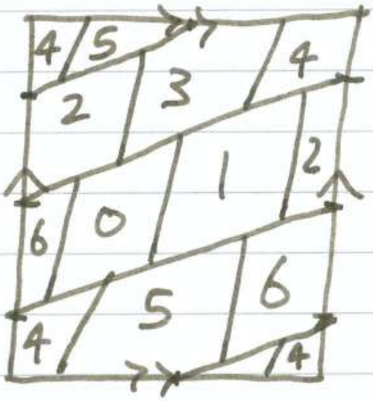
Note that $H(2) = 4$, interesting.
Can equality hold? Consider $E = 0$, $H(0) = 7$.
Then $e(G) \leq 3n$, and a minimal 7-chromatic graph on the surface has $\delta \geq 6$, so G must be 6-regular. But then $\chi(G) \leq 6$ unless $G = K_7$, by Brooke's Theorem.
Thus $\chi(G) = 7$ is attainable for some G iff K_7 is embeddable on the surface.

$\chi(G) = 1 + \Delta(G)$

$\Rightarrow G = K_{\Delta+1}$
or $\Delta = 2$, G an odd circuit

For the torus we can do it.

The dual of the required graph:



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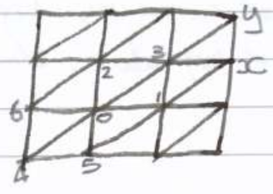
Graph Theory (18)

Can we draw K_7 on the Klein Bottle? Considerations of equality show that we must embed K_7 on the Klein Bottle so that every face is a triangle. So locally we have a triangular tiling. How are the vertices identified with one another?



WLOG, fix 0 and colour 1-6 around it.

x cannot be 3 or 1, or 2 (3 has neighbour 2) or 5 or 0, so $x = 4$ or 6. WLOG (or by symmetry) $x = 4$. Likewise $y = 4$ or 5, but $x = 4$ so $y = 5$. Hence the labelling is uniquely defined (it is the same as the labelling for the torus). But colours always appear in the <u>same</u> cyclic order around any vertex, so the surface is orientable $\times$

So K_7 cannot be embedded in the Klein bottle.

Amazingly, the Klein-Bottle is the only surface for which

$\max\{\chi(G) : G \text{ embedded in } S, S \text{ has Euler Characteristic } E\} = H(E)$

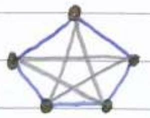
fails. Heawood "proved" this but there was a problem, so the full proof was given in 1965.

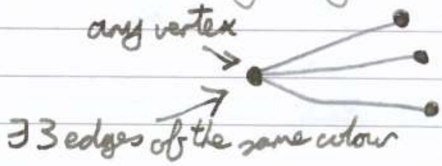
Note that for $E = 2$, it is the upper bounding which is hard rather than the embedding of graphs.

6. Ramsey's Theorem

Ramsey Theory – 'finding order in chaos'

We can colour the edges of K_5 without getting a monochromatic K_3. Here, a colouring is just any partition into two classes.





If any two of these three are joined by red, we get a red K_3. If not, they are all joined by blue, blue K_3

But we cannot so colour K_6.

Is there a number similar to "6" for monochromatic K_4?

Let $R(s)$ be the smallest n (if it exists) such that if K_n is 2-coloured then there is a monochromatic K_s. We proved $R(3) = 6$.

It helps to define $R(s,t)$ to be the smallest n (if it exists) such that every red/blue colouring of K_n has a red K_s or a blue K_t (or both).

Clearly $R(s) = R(s,s)$. $R(s,t) = R(t,s)$. $R(s,2) = s$.

<u>Theorem 6.1</u> (Ramsey 1930, Erdős-Szekeres 1935)

$R(s,t)$ exists. Moreover, $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

<u>Proof</u>

It suffices to demonstrate the inequality. Let $a = R(s-1,t)$, and $b = R(s,t-1)$. Colour K_{a+b} red and blue. Pick a vertex v. Now $d(v) = a+b-1$, so either there are (at least) a red edges at v or there are b blue edges.

By symmetry, we assume the latter.

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Graph Theory ⑱

Hence v is joined to a $K_{R(s,t-1)}$ by blue edges. Either this $K_{R(s,t-1)}$ contains a red K_s, or it contains a blue K_{t-1}, which together with v, gives a blue K_t. Thus $R(s,t) \leq a+b$ □

Remarks

1. By induction, the inequality gives $R(s,t) \leq \binom{s+t-2}{s-1} \leq 2^{s+t-3}$
2. The inequality is not always the best possible. For example, equality can hold only if K_{a+b-1} has an $(a-1)$-regular subgraph (red) which cannot happen if a, b are both even.
3. Only $R(3) = 6$, $R(4) = 18$ are known.

What if we use more than 2 colours? Let $R_k(s_1, \ldots, s_k)$ be the smallest n so that if K_n is coloured with colours in $[k]$ then there exists K_{s_i} in colour i, for some i.

Theorem 6·2

Let $k \geq 2$, $s_1, \ldots, s_k \geq 2$. Then $R_k(s_1, \ldots, s_k)$ exists.

$$a_1 = R(s_1 - 1, s_2, \ldots, s_k)$$
$$a_2 = R(s_1, s_2 - 1, \ldots, s_k)$$
$$\cdots$$
$$a_k = R(s_1, s_2, \ldots, s_k - 1)$$

$d(v) = \sum a_i - 1$

Consider $K_{a_1 + \cdots + a_k}$, $[k]$ coloured. Pick $v \in G$. WLOG v has a_1 edges of colour 1.

Look at the K_{a_1} joined to v by 1-coloured edges. This contains a $1 - K_{s_1 - 1}$ or an $i - K_{s_i}$ □

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Theorem 6·2

$R_k(s_1, \dots, s_k)$ exists.

Proof (By induction on $k \geq 2$)

We know $R_2(s_1, s_2) = R(s_1, s_2)$ exists. If $k > 2$, we 'go colourblind' on colours 1 and 2, so we percieve only $k-1$ colours. If $n \geq R_{k-1}(R_2(s_1, s_2), s_3, \dots, s_k)$ and we colour K_n with pallette $[k]$, there is either a $K_{R_2(s_1,s_2)}$ in colours 1 and 2, or there exists K_{s_i} in colour i, for some $i \geq 3$. In the first case, regain your colour vision and observe one of K_{s_1} or K_{s_2} in colour 1 or 2 respectively. □

This shows $R_k(s_1, \dots, s_k) \leq R_{k-1}(R_2(s_1, s_2), s_3, \dots, s_k)$

Alternatively, we could have gone colourblind on colours 1 to 4, say to show $R_k(s_1, \dots, s_k) \leq R_{k-3}(R_4(s_1, s_2, s_3, s_4), s_5, \dots, s_k)$

Or we could mimic the proof of Theorem 6·1 to show

$$R_k(s_1, \dots, s_k) \leq R_k(s_1 - 1, s_2, \dots, s_k) + R_k(s_1, s_2 - 1, s_3, \dots, s_k) + \dots + R_k(s_1, \dots, s_{k-1}, s_k - 1) + 2 - k$$

Recall that an edge of K_n is just a 2-subset of $V(K_n)$

An <u>r-uniform</u> <u>hypergraph</u> is a pair $H = (V, E)$ where $E \subset V^{(r)} = \{Y \subset V : |Y| = r\}$. So a graph is a 2-uniform hypergraph.

The <u>complete</u> r-uniform hypergraph is $K_n^{(r)} = ([n], [n]^{(r)})$

Define $R^{(r)}(s,t)$ to be the smallest n, if it exists, such that if the edges of $K_n^{(r)}$ are coloured red/blue then there is a red $K_s^{(r)}$ or a blue $K_t^{(r)}$.

Clearly $R^{(r)}(s,t) = R^{(r)}(t,s)$ (Why?)

$R^{(r)}(s,r) = s$

Note that $R^{(1)}(s,t) = s+t-1$; this is just the Pigeonhole Principle.

We used this to prove Theorem 6·1.

<u>Theorem 6·3</u> (Ramsey for r-sets)

$R^{(r)}(s,t)$ exists.

<u>Proof</u>

Let $a = R^{(r)}(s-1,t)$, $b = R^{(r)}(s,t-1)$, $n = 1 + R^{(r-1)}(a,b)$

Colour $K_n^{(r)}$ red and blue. Select $v \in V(K_n^{(r)})$. Consider now the complete $(r-1)$ uniform hypergraph $K_{n-1}^{(r-1)}$ with vertex set $V(K_n^{(r)}) - \{v\}$. Colour each of its edges Z the colour that $Z \cup \{v\}$ had in the original colouring. Since $n-1 = R^{(r-1)}(a,b)$ this $K_{n-1}^{(r-1)}$ contains either a red $K_a^{(r-1)}$ or a blue $K_b^{(r-1)}$. By symmetry, we can assume the latter. This means that there is a set of $b = R^{(r)}(s,t-1)$ vertices, all of whose $r-1$ subsets, together with v are blue r-subsets in the original.

Hence in the original, either we find $K_s^{(r)}$ red, or we find a blue $K_{t-1}^{(r)}$, all of whose $r-1$-subsets with v are blue, i.e. we have a blue $K_t^{(r)}$. Hence $R^{(r)}(s,t) \le n$ $\square$

'Colour blindness' shows that $R_k^{(r)}(s_1, \dots, s_k)$ exists with the obvious definition.

What if we allow infinite vertex sets -, say $V = \mathbb{N}$?

Suppose $\mathbb{N}^{(2)}$ is coloured red/blue. Will there be an <u>infinite</u> subset $M \subseteq \mathbb{N}$ such that $M^{(2)}$ is monochromatic?

e.g. colour ij $\begin{cases}\text{red}\\ \text{blue}\end{cases}$ if $i+j$ is $\begin{matrix}\text{even}\\ \text{odd}\end{matrix}$: $M = \{$even numbers$\}$

colour ij $\begin{cases}\text{red}\\ \text{blue}\end{cases}$ if $\max\{k : 2^k | i+j\}$ is $\begin{matrix}\text{even}\\ \text{odd}\end{matrix}$: $M = \{4^k : k \in \mathbb{N}\}$

ij $\begin{cases}\text{red}\\ \text{blue}\end{cases}$ if $i+j$ has an $\begin{matrix}\text{even}\\ \text{odd}\end{matrix}$ number of prime factors.

Here, no explicit M is known.

The treatment of the infinite case is in some ways easier than the finite because no counting is needed.

<u>Theorem 6.4</u> (Infinite Ramsey)

Let $\mathbb{N}^{(r)}$ be coloured with k colours. Then there exists an infinite subset $M \subseteq \mathbb{N}$ with $M^{(r)}$ monochromatic.

<u>Proof</u>

By induction on r, the case $r=1$ being the Pigeonhole Principle.

Let $\mathbb{N}^{(r)}$ be coloured with palette $[k]$. Let v_1 be the first element of $\mathbb{N}$. Colour $(\mathbb{N} - \{v_1\})^{(r-1)}$ by giving Z the colour of $Z \cup \{v_1\}$. By the induction hypothesis, there exists an

infinite subset $M_1 \in \mathbb{N} - \{v_1\}$ with $M_1^{(r-1)}$ monochromatic.
i.e. in the original, every r-subset inside $\{v_1\} \cup M_1$, with first element v_1, has some colour, $c(v_1)$, say. Now let v_2 be the first element of M_1. In the same way, there is an infinite subset $M_2 \subset M_1$, so every r-set in $v_2 \cup M_2$ with first element v_2 has colour $c(v_2)$, say. Now find v_3, M_3, etc.
So we construct an infinite sequence $\mathbb{N} \supset M_1 \supset M_2 \supset \dots$
v_i is the first element of M_{i-1} ($M_0 = \mathbb{N}$), all r-sets with first element v_i have colour $c(v_i)$.
Let $M^* = \{v_1, v_2, \dots\}$. In M^*, an r-set Y has colour $c(v_i)$ where $v_i = \min Y$.
But we have k-colours, so there is an infinite subset $M = \{v_{i_1}, v_{i_2}, \dots\}$ on which $c(\cdot)$ is constant. Then $M^{(r)}$ is monochromatic. $\square$

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Theorem 6.4 (Infinite Ramsey)

$\mathbb{N}^{(r)}$ k-coloured $\Rightarrow \exists$ infinite monochromatic $M \subseteq \mathbb{N}$.

Corollary 6.5

Every real sequence $(a_n)_{n=1}^{\infty}$ has a monotonic subsequence.

Proof

Colour i, j red if $a_i \leq a_j$ and blue otherwise. This gives a 2-colouring on $\mathbb{N}^{(2)}$. So there is an infinite monochromatic $M = \{i_1, i_2, i_3, \ldots\}$. Then $a_{i_1}, a_{i_2}, \ldots$ is monotonic. $\square$

Other structures support Ramsey-like theorems. For example, ~~Hard~~ Van der Waerden's theorem says that if $n \geq \omega(k, l)$ an $[n]$ are k-coloured then there is a monochromatic Arithmetic Progression of length l.

The Hales-Jewett theorem says that if $[n]^d$ is k-coloured, then there is a monochromatic line, provided d is large.

Define $Y \subseteq \mathbb{N}$ to be 'big' if $|Y| \geq \min Y$

e.g. $\{3,4,5\}$ is big, $\{12, 918, 1473, 96201\}$ is not.

Theorem 6.6

Given s, k, r, $\exists B(s,k,r)$ such that, if $n \geq B(s,k,r)$ and $\{s, s+1, \ldots, n\}^{(r)}$ is coloured with k-colours, then there exists $Y \subset \{s, s+1, \ldots, n\}$, Y big, Y monochromatic.

Remark

This is very similar to Ramsey's finite r-set k-colour theorem, because $|Y| \geq S$. This says a little more.

Proof

If not, for every $n \geq S$, there is a colouring $c_n : \{S, S+1, \dots, n\}^{(r)} \to [k]$ with no big monochromatic set.

Enumerate $\{S, S+1, \dots\}^{(r)}$ as $Z_1, Z_2, Z_3, \dots$

There is an infinite subsequence of $c_S, c_{S+1}, c_{S+2}, \dots$ called $c_1^{(1)}, c_1^{(2)}, c_1^{(3)}, \dots$ on which the colour of Z_1 is constant. Call the colour $c(Z_1)$

This has, in turn, an infinite subsequence $c_2^{(1)}, c_2^{(2)}, \dots$ on which the colour of Z_2 is also constant. Call this colour $c(Z_2)$.

Now we take a subsequence $c_3^{(1)}, c_3^{(2)}, c_3^{(3)}, \dots$ on which the colour of Z_3 is $c(Z_3)$ etc. We obtain a colouring $c : \{S, S+1, \dots\}^{(r)} \to [k]$.

By Ramsey's Theorem 6.4, there is an infinite set $M \subset \{S, S+1, \dots\}$ with c constant on $M^{(r)}$. Let $L = \min M$, and let Y be the first L elements of M. So Y is big, and c is constant on $Y^{(r)}$.

But let $j = \max\{i : Z_i \in Y^{(r)}\}$. But c agrees with $c_j^{(1)}$ on $Y^{(r)}$ $\#$ ($c_j^{(1)}$ had no big monochromatic set) $\square$

The argument here is quite general (compactness).

Theorem 6.6 is a finite theorem about finite sets, so is true in Peano Arithmetic. But amazingly (Paris-Harrington) there is no proof in Peano Arithmetic.

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We have to use infinite sets to prove it. This is (the first natural) example of Gödel Incompleteness.

Known Ramsey Numbers

$R(3,3)=6$, $R(3,4)=9$, $R(3,5)=14$, $R(3,6)=18$, $R(3,7)=23$

$R(3,8)=28$, $R(3,9)=36$ (1982)

example sheet ← $\underbrace{R(4,4)}=18$, $R(4,5)=25$ (1992, "All the computers in Australia for 8 months")

$R_3(3,3,3)=17$ $R^{(3)}(4,4)=13$ (1991: $\exists\, 2\times 10^5$ colourings)

Upper bounds $R(s) \leq \binom{2s-2}{s-1} \sim \frac{4^s}{4\sqrt{2\pi s}}$

The only improvements have been an extra factor $\frac{1}{\sqrt{s}}$ (1989)

and $\frac{c_k}{s^k}$ $\forall k$ (Conlon 2009)

But, for example, $R(s) < (3.99\ldots9)^s$ unknown.

Lower bounds $R(s) > (s-1)^2$ is easy, $> s^3$ is tricky.

$e^{\log^2(s)/4\log\log s}$ is the best construction known.

20/11/12 Graph Theory (21)

7. Probabilistic Methods

We know that $R(s) < 4^s$ but our best constructions (lower bounds) are sub-exponential.

Theorem (Erdős, 1947)

$R(s) > \sqrt{2}^s$ for $s \geq 3$

Proof

Colour the edges of K_n independently, at random red or blue with probability $\frac{1}{2}$. Let X be the random variable which is the number of monochromatic K_s. There are $\binom{n}{s}$ s-subsets of vertices, each spanning a monochromatic K_s with probability $2 \times 2^{-\binom{s}{2}}$. So

$$\mathbb{E}(X) = \binom{n}{s} 2^{1-\binom{s}{2}} < \frac{n^s}{s!} 2^{1-\binom{s}{2}} = \frac{2^{1+\frac{s}{2}}}{s!} \left[\frac{n}{\sqrt{2}^s}\right]^s < \left[\frac{n}{\sqrt{2}^s}\right]^s$$

since $2^{1+\frac{s}{2}} < s!$ if $s \geq 3$.

Therefore, if $n \leq \sqrt{2}^s$, then $\mathbb{E}(X) < 1$, so there must be at least one colouring where $X = 0$. Hence $R(s) > n$ □

Remarks

1. We used the fact that if $X \geq 0$ is an integer valued random variable and $\mathbb{E}(X) < 1$, then $X = 0$ (else $\mathbb{E}(X) \geq 1$)
2. We could 'de-randomise' the proof by saying, equivalently, the average number of monochromatic K_s over all colourings is < 1. But in doing so, we rob ourselves of valuable insight; that approaches using randomness can give very powerful results with very little effort.

3. The theorem has had essentially no improvement.

Elementary Probability

Ω - a finite probability space, for example, the set of red/blue colourings of K_n, $V(K_n) = [n]$, each with probability $2^{-\binom{n}{2}}$

An event A is a subset $A \subset \Omega$

A random variable is a function $X : \Omega \to \mathbb{R}$

The expected value of X is $\mathbb{E}(X) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) X(\omega)$ $\left(= \int_\Omega X \, d\mathbb{P}\right)$

Profoundly useful fact: expectation is linear.

$\mathbb{E}(\alpha X + \beta Y) = \sum_{\omega \in \Omega} \mathbb{P}(\omega)(\alpha X(\omega) + \beta Y(\omega)) = \alpha \mathbb{E}(X) + \beta \mathbb{E}(Y)$

The indicator function 1_A of the event A is $1_A(\omega) = \begin{cases} 0 & \omega \notin A \\ 1 & \omega \in A \end{cases}$

Then $\mathbb{E}(1_A) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) 1_A(\omega) = \sum_{\omega \in A} \mathbb{P}(\omega) = \mathbb{P}(A)$

A variable that counts generally can be expressed as a sum of indicators. e.g. $X(\omega) = \#$ monochromatic K_s in the colouring ω

For each $\alpha \in [n]^{(s)}$ let 1_A be the indicator that $K_n[\alpha]$ is monochromatic. Then $X = \sum_\alpha 1_A$,

$\mathbb{E}(X) = \sum_\alpha \mathbb{E}(1_\alpha) = \sum_\alpha \mathbb{P}(1_\alpha = 1) = \binom{n}{s} 2^{1-\binom{s}{2}}$

Theorem 7.2

The graph G has an independent (edgeless) set of vertices of cardinality at least $\sum_{v \in G} \frac{1}{d(v)+1} \geq \frac{|G|}{d+1}$ where d is the average degree.

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Remark

1. This is Turan's Theorem, viewed from the complement.

2. Recall that the Arithmetic mean $\geq$ Geometric mean $\geq$ Harmonic mean

Proof

Take a random ordering of the vertices, and let X be the number of vertices coloured 1 by the greedy algorithm.

Then $X = \sum_{v \in G} \mathbb{1}_v$ where $\mathbb{1}_v$ indicates whether v gets colour 1.

Now, $\mathbb{P}(\mathbb{1}_v = 1) \geq \mathbb{P}(\{v \text{ precedes all of its neighbours}\}) = \frac{1}{d(v)+1}$

Thus $\mathbb{E}(X) = \sum_v \mathbb{E}(\mathbb{1}_v) \geq \sum_v \frac{1}{d(v)+1}$, so there is some ordering in which $X \geq \sum_v \frac{1}{d(v)+1}$ $\square$

The space $\mathcal{G}(n,p)$ comprises all $2^{\binom{n}{2}}$ graphs on vertex set $[n]$ whose edges are chosen independently at random with probability p.

This means that the probability that the outcome is some particular graph G is $p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}$.

A related model comprises bipartite graphs with two disjoint sets of n vertices, edges being chosen independently with probability p from the n^2 possible edges.

Recall: we showed that $z(n,t) = O(n^{2-\frac{1}{t}})$ as $n \to \infty$

To get a lower bound, note that the expected number of $K_{t,t}$'s in this model is $\binom{n}{t}^2 p^{t^2}$ which is <1 if $p \simeq n^{-2/t}$ (large n) giving (after more work) about $pn^2 \simeq n^{2-2/t}$ edges

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$Z(n,t) = O(n^{2-1/6})$, t fixed, n large.

We can do better, rigourously, and more easily, as follows.

Theorem 7·3

$Z(n,t) > \frac{1}{2} n^{2-\frac{2}{t+1}}$

Proof

Let J be the number of edges in our random $n \times n$ bipartite graph.
Let k be the number of $K_{t,t}$'s. By removing an edge from each $K_{t,t}$, we get a $K_{t,t}$ free graph with $\geq J-k$ edges.

Now $\mathbb{E}(J) = pn^2$, and $\mathbb{E}(k) = p^{t^2}\binom{n}{t}^2 < \frac{1}{2}n^{2t}p^{t^2}$

By linearity of expectation, $\mathbb{E}(J-k) = \mathbb{E}(J) - \mathbb{E}(k) > pn^2 - \frac{1}{2}p^{t^2}n^{2t}$

By taking $p = n^{-2/t+1}$ we have $\mathbb{E}(J-k) > \frac{1}{2}n^{2-2/t+1}$ and some graph has $J-k > \frac{1}{2}n^{2-2/t+1}$ $\square$

Theorem 7·1 shows that there exist graphs with no complete subgraph bigger than $2\log_2 n$ but also no independent set larger than $2\log_2 n$, so $\chi(G) \geq \frac{n}{2\log_2 n}$, far larger than the largest complete subgraph.

Can we find still more dramatic examples?

Recall that the girth of a graph is the length of the shortest circuit.

We recall the simple inequality of Markov:

since for real $t>0$, $1_{\{|X|>t\}} < \frac{|X|}{t}$

By taking expectations, $\mathbb{P}(|X| > t) < \frac{\mathbb{E}(|X|)}{t}$

<u>Theorem 7.4</u> (Erdős 1959)

Let $k, g \in \mathbb{N}$. Then there exists a graph of girth at least g and chromatic number at least k.

<u>Proof</u>

Consider a random graph $G \in \mathcal{G}(n,p)$, where $p = n^{-1+1/g}$

Here n will be chosen large enough (depending on k, g) so that certain estimates hold.

Let X_l be the number of circuits of length l in G, and let $X = X_3 + \dots + X_{g-1}$. Then

$$\mathbb{E}(X) = \sum_{l=3}^{g-1} \mathbb{E}(X_l) \leq \sum_{l=3}^{g-1} n^l p^l = \sum_{l=3}^{g-1} n^{l/g} < g n^{\frac{g-1}{g}} < \frac{n}{4}$$

if n is large. Let A be the event $\{X > \frac{1}{2}\}$. $\mathbb{P}(A) < \frac{1}{2}$ by Markov. Let Y be the number of independent sets of size $t = \lceil \frac{n}{2k} \rceil$.

$1-p \leq e^{-p}$

$$\text{Then } \mathbb{E}(Y) = \binom{n}{t}(1-p)^{\binom{t}{2}} \leq n^t e^{-p\binom{t}{2}}$$

$$= [n e^{p/2}]^t e^{-pt^2/2} \leq \exp\left\{(\log n + \tfrac{1}{2})(\tfrac{n}{2k}+1) - \frac{n^{1+1/g}}{8k^2}\right\} < \frac{1}{2}$$

if n is large. Let B be the event $\{Y \geq 1\}$.

Then $\mathbb{P}(B) < \frac{1}{2}$.

Since $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B) < 1$, there exists a graph G where neither A nor B happens, i.e. $X \leq \frac{n}{2}$ and $Y = 0$.

Remove a vertex from each circuit of short length to leave a graph G' of girth $\geq g$, and $|G'| \geq \frac{n}{2}$

Since G' has no independent set size t, we have $\chi(G') \geq \frac{|G'|}{t} \geq k$

$\square$

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end of proof : $\chi(G') \geqslant \frac{|G'|}{t-1} \geqslant k$

Let A be an event in $\mathcal{G}(n,p)$, where p may be a function of n. We say that A holds <u>with high probability</u> whp if $\mathbb{P}(A) \to 1$ as $n \to \infty$.

If X counts something, we can never infer that $X \geqslant 1$ just by knowing $\mathbb{E}(X)$ is large.

e.g. $\mathbb{P}(X=0) = 0.999$, $\mathbb{P}(X=10^6) = 0.001$

In such cases, Chebyshev's Inequality can be useful :

$\mathbb{P}(|X| \geqslant t) = \frac{\mathbb{E}|X|}{t}$

The variance of X is $\mathbb{E}((X-\mathbb{E}(X))^2) = Var(X)$. So

$$\mathbb{P}(|X-\mathbb{E}(X)| > t) = \mathbb{P}((X-\mathbb{E}X)^2 > t^2) < \frac{Var X}{t^2} \text{ by Markov}$$

How do we compute $Var X$? Note that

$$Var X = \mathbb{E}((X-\mathbb{E}(X))^2) = \mathbb{E}(X^2 - 2X \cdot \mathbb{E}(X) + (\mathbb{E}(X))^2)$$
$$= \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

Suppose X is a sum of indicators, $X = \sum_\alpha 1_\alpha$

$$1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Then $\mathbb{E}(X) = \sum_\alpha 1_\alpha = \sum_\alpha \mathbb{P}(\alpha)$, so

$(\mathbb{E}(X))^2 = (\sum_\alpha \mathbb{P}(\alpha))^2 = \sum_{\alpha,\beta} \mathbb{P}(\alpha)\mathbb{P}(\beta)$ (α, β run over possibilit

Now $X^2 = (\sum_\alpha 1_\alpha)^2 = \sum_{\alpha,\beta} 1_\alpha 1_\beta = \sum_{\alpha,\beta} 1_{\alpha\cap\beta}$

So $\mathbb{E}(X^2) = \sum_{\alpha,\beta} \mathbb{E}(1_{\alpha\cap\beta}) = \sum_{\alpha,\beta} \mathbb{P}(\alpha\cap\beta)$

$$= \sum_{\alpha,\beta} \mathbb{P}(\alpha)\mathbb{P}(\beta|\alpha)$$

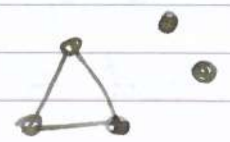
Thus $Var(X) = \sum_\alpha \mathbb{P}(\alpha) \sum_\beta [\mathbb{P}(\beta|\alpha) - \mathbb{P}(\beta)]$

$1_\alpha \times 1_\beta = 1_{\alpha\cap\beta}$

Note that if β, α are independent then $[\dots] = 0$

Observe that $\mathbb{P}(X=0) < \frac{Var(X)}{t^2}$ for all $t < \mathbb{E}(X)$

so $\mathbb{P}(X=0) \leqslant \frac{Var(X)}{(\mathbb{E}(X))^2}$ $(*)$



An <u>isolated vertex</u> is a vertex of degree 0.

<u>Theorem 7.5</u>

Let $\omega(n) \to \infty$. Let $G \in \mathcal{G}(n,p)$. If $p = \frac{1}{n}(\log(n) - \omega(n))$

then with high probability, G has isolated vertices.

If $p = \frac{\log(n) + \omega(n)}{n}$, then with high probability G has no isolated vertices.

<u>Remark</u>

This kind of "threshold phenomenon" is typical.

<u>Proof</u>

Let 1_v be the indicator of v being isolated, $v \in \underset{\{1,2,\dots,n\}}{[n]}$

Let X be the number of isolated vertices, so $X = \sum_v 1_v$

$\mathbb{E}(X) = \sum_v \mathbb{P}(v \text{ is isolated}) = n(1-p)^{n-1}$

Also, $Var(X) = \sum_v \mathbb{P}(v \text{ is isolated}) \sum_w \mathbb{P}(w \text{ isolated} \mid v \text{ isolated}) - \mathbb{P}(w \text{ isolated})$

$$\cancel{= \sum_v} = n(1-p)^{n-1}\Big\{\underbrace{1-(1-p)^{n-1}}_{w=v} + \underbrace{(n-1)\left[(1-p)^{n-2} - (1-p)^{n-1}\right]}_{w \neq v}\Big\}$$

$$\leqslant \mathbb{E}(X) + n^2(1-p)^{n-1}p(1-p)^{n-2} = \mathbb{E}(X) + \frac{p}{1-p}(\mathbb{E}(X))^2$$

If $p = \frac{1}{n}(\log(n) + \omega(n))$, then $\mathbb{E}(X) = \frac{1}{1-p}n(1-p)^n \leqslant \frac{1}{1-p}ne^{-pn} \to 0$

$1-x \leqslant e^{-x}$

so $\mathbb{P}(X \geqslant 1) \to 0$ By Markov's inequality.

i.e. $X = 0$ with high probability.

If $p = \frac{1}{n}(\log(n) - \omega(n))$, then $\mathbb{E}(X) \to \infty$ (e.g. use $1-p > e^{-p-\frac{p^2}{4}}$ for $p < \frac{1}{4}$), and so $\frac{Var(X)}{(\mathbb{E}(X))^2} = \frac{1}{\mathbb{E}(X)} + \frac{p}{1-p}$.

So by Chebyshev, $\mathbb{P}(X=0) \to 0$ i.e. $X \geq 1$ with high probability. $\square$

As a final example, we consider complete subgraphs of $\mathcal{G}(n,p)$

<u>Theorem 7.6</u>

Let $0 < \varepsilon, p < 1$ be fixed. Let $G \in \mathcal{G}(n,p)$

If $d = \lfloor (2-\varepsilon)\log_{1/p} n \rfloor$ then $K_d \subset G$ with high probability.

If $d = \lceil (2+\varepsilon)\log_{1/p} n \rceil$ " $K_d \not\subset G$ " " "

$= K_4$

<u>Proof</u>

Let X be the number of K_d's in G. If $d = \lceil (2+\varepsilon)\log_{1/p} n \rceil$ then $\mathbb{E}(X) = \binom{n}{d} p^{\binom{d}{2}} < \left[n p^{\frac{d-1}{2}} \right]^d < \left[n p^{(1+\frac{\varepsilon}{4})\log_{1/p} n} \right]^d$

$\to 0$ as $n \to \infty$.

By Markov's Inequality, $\mathbb{P}(X \geq 1) \to 0$, so $X = 0$ with high probability.

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Theorem 7.6

$d = \lfloor (2-\varepsilon)\log_{1/p} n \rfloor \Rightarrow K_d \subset G$ with high probability

$X =$ the number of K_ds $\qquad \mathbb{E}(X) = \binom{n}{d} p^{\binom{d}{2}}$

Now $X = \sum_\alpha \mathbb{1}_\alpha$ where α runs over the $\binom{n}{d}$ events

$G[A] = K_d$ where $A \in [n]^{(d)}$.

If β is the event $G[B] = K_d$ then α, β are independent if $|A \cap B| \leq 1$, and in general $\mathbb{P}(\beta|\alpha)$ depends only on the value of $l = |A \cap B|$. Hence

$$Var(X) = \sum_\alpha \mathbb{P}(\alpha) \sum_\beta \left[\mathbb{P}(\beta|\alpha) - \mathbb{P}(\beta)\right]$$
$$\binom{n}{d} p^{\binom{d}{2}} \sum_{l=2}^{d} \binom{d}{l}\binom{n-d}{d-l}\left[p^{\binom{d}{2}-\binom{l}{2}} - p^{\binom{d}{2}}\right]$$

So
$$Var(X)/(\mathbb{E}(X))^2 = \sum_{l=2}^{d} \frac{\binom{d}{l}\binom{n-d}{d-l}}{\binom{n}{d}}\left[p^{-\binom{l}{2}} - 1\right]$$

$$\leq \sum_{l=2}^{d} \frac{\binom{d}{l}\binom{n-l}{d-l}}{\binom{n}{d}} p^{-\binom{l}{2}} = \sum_{l=2}^{d} \binom{d}{l} \frac{d}{n}\cdot\frac{d-1}{n-1}\cdot\ldots\cdot\frac{d-l+1}{n-l+1} p^{-\binom{l}{2}}$$
$$\leq \sum_{l=2}^{d} \left[\frac{d^2}{n} p^{-\frac{l-1}{2}}\right]^l$$

Put $d = \lfloor (2-\varepsilon)\log_{1/p} n \rfloor$. So for large n,
$$\frac{Var(X)}{(\mathbb{E}(X))^2} \leq \sum_{l=2}^{d}\left[\frac{1}{n^{\varepsilon/4}}\right]^l \leq \frac{2}{n^{\varepsilon/2}} \to 0 \text{ as } n \to \infty.$$
Hence, by Chebyshev's Inequality, $\mathbb{P}(X=0) \to 0$, so $X \geq 1$ with high probability. $\square$

Remarks

With more care, the "correct" value of d can be estimated sharply.

The theorem shows that with high probability $\chi(G) \geq (1+o(1))\frac{n}{2\log_{1/q} n}$ $\quad q = 1-$

It is easily shown that the greedy algorithm gives $\chi(G) \leq (1+o(1))\frac{n}{\log_{1/q} n}$

To show that the first bound is the correct value, we need to replace "$\mathbb{P}(X=0) \to 0$" by "$\mathbb{P}(X=0)$ is exponentially small" in our proof. This was done separately by Bollobás, Janson, and Talagrand

Recall Hadwiger's Conjecture : $\chi(G) \geq k \Rightarrow G > K_k$

Hajós' Conjecture : $\chi(G) \geq k \Rightarrow G$ contains a subdivision of K_k.

Hajós $\Rightarrow$ Hadwiger.

In fact, almost every graph is a counterexample to Hajós.

A random graph doesn't contain a subdivision of $K_{4\sqrt{n}}$