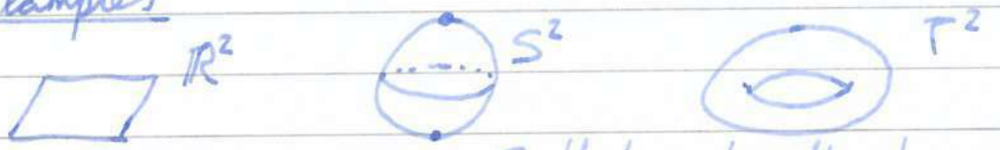
Geometry ①

<u>Examples</u>



A <u>surface</u> is a space S that is locally homeomorphic to, or locally "looks like" an open subset of $\mathbb{R}^2$.

<u>Distance on S</u> : $d(x,y)$ is the length of the shortest path <u>on S</u> joining x to y. Shortest paths, or "straight lines" in our space, are called <u>geodesics</u>

The geodesics on S^2 are the arcs of "great circles", where a great circle is the intersection of S^2 with a plane through the origin. The lines of longitude and the equator are examples.

The cylinder and sphere are both globally different from $\mathbb{R}^2$ as they are compact. However, the cylinder is locally the same as (or <u>isometric</u> to) $\mathbb{R}^2$ whereas the sphere is not.

Let T be a triangle on a surface. Define :
$d(T) = \sum(\text{angles of } T) - \pi$
(Note that if $T = T_1 \cup T_2$, $d(T) = d(T_1) + d(T_2)$)

There is a local metric quantity called <u>curvature</u> $K(p)$ that measures $d(T)$ for small triangles near a point p.

Geometry	Euclidean	Spherical	Hyperbolic
Space	$\mathbb{R}^2$	S^2	$\mathbb{H}^2$
$d(T)$	>0	$\equiv 0$	<0
K	$K\equiv 1$	$K\equiv 0$	$K\equiv -1$

Isometries of Euclidean Space

Definition

If X, X' are metric spaces, $f : X \to X'$ is an isometry if

i) f is bijective   ii) $d'(f(x), f(y)) = d(x, y)$

Note that ii) $\Rightarrow$ f is injective, and continuous.

Lemma

i) Id_X is an isometry   ii) The composition of isometries is an isometry

iii) The inverse of an isometry is an isometry

Proof

i) Is obvious. For ii), if f, g are isometries then:
$d(f(g(x)), f(g(y))) = d(g(x), g(y)) = d(x, y)$. For iii):
$d(x, y) = d(ff^{-1}(x), ff^{-1}(y)) = d(f^{-1}(x), f^{-1}(y))$

Definition

$Isom(X) = \{ f : X \to X \mid f \text{ is an isometry} \}$

Corollary: $Isom(X)$ is the group of isometries of X.

For Euclidean space, $X = \mathbb{R}^n$. $d(\underline{x}, \underline{y}) = \| \underline{x} - \underline{y} \|$ where $\|\underline{u}\| = \sqrt{\underline{u} \cdot \underline{u}}$

$\angle XYZ = \theta$, $\cos\theta = \dfrac{\underline{v} \cdot \underline{w}}{\|\underline{v}\| \|\underline{w}\|}$

Geometry ②

Isometries of $\mathbb{R}^n$

Translation : $\underline{v} \in \mathbb{R}^n$, $T_{\underline{v}}(\underline{x}) = \underline{v} + \underline{x}$

Orthogonal Transformations : $O \in O(n)$ i.e. $O^T O = I$

$\langle O\underline{v}, O\underline{v} \rangle = \langle \underline{v}, O^T O \underline{v} \rangle = \langle \underline{v}, \underline{v} \rangle$

$T_O(\underline{x}) = O\underline{x}$.

$d(T_O(\underline{x}), T_O(\underline{y})) = \| T_O(\underline{x}) - T_O(\underline{y}) \| = \| O(\underline{x} - \underline{y}) \| = \| \underline{x} - \underline{y} \| = d(\underline{x}, \underline{y})$

Theorem

If $f: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then $f = T_{O,\underline{v}}$ for some O, $\underline{v}$, where
$T_{O,\underline{v}}(\underline{x}) = O\underline{x} + \underline{v}$

Lemma 1

If $f: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry and $f(\underline{0}) = \underline{0}$, $f(\underline{e_i}) = \underline{e_i}$ $\forall i$, then f is the identity.

Proof

$f(\underline{x}) = \underline{x}' = (x_1', \dots, x_n')$.

$d(\underline{x}, \underline{0})^2 = d(f(\underline{x}), f(\underline{0}))^2 = d(\underline{x}', \underline{0})^2$

$\sum x_j^2 = \sum x_j'^2$ ①

$d(\underline{x}, \underline{e_i})^2 = d(f(\underline{x}), f(\underline{e_i}))^2 = d(\underline{x}', \underline{e_i})^2$

$\sum_{j \neq i} x_j^2 + (x_i - 1)^2 = \sum_{j \neq i} x_j'^2 + (x_i' - 1)^2$ ②

①, ② $\Rightarrow x_i = x_i' \Rightarrow f(\underline{x}) = \underline{x}$ □

Lemma 2

Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry and $f(\underline{0}) = \underline{0}$. Then $f = T_O$ where O is the matrix with column vectors $f(\underline{e_i})$.

Proof.

We must check that O is orthogonal $\Leftrightarrow \langle f(\underline{e}_i), f(\underline{e}_j)\rangle = \delta_{ij}$

$\langle f(\underline{e}_i), f(\underline{e}_i)\rangle = \|f(\underline{e}_i)\|^2 = \|f(\underline{e}_i) - \underline{0}\|^2 = \|f(\underline{e}_i) - f(\underline{0})\|^2$
$= \|\underline{e}_i - \underline{0}\|^2 = 1$

For $i \neq j$: $\|f(\underline{e}_i) - f(\underline{e}_j)\|^2 = \|\underline{e}_i - \underline{e}_j\|^2 = 2$

$\langle f(\underline{e}_i), f(\underline{e}_i)\rangle - 2\langle f(\underline{e}_i), f(\underline{e}_j)\rangle + \langle f(\underline{e}_j), f(\underline{e}_j)\rangle = 2$
$\Rightarrow \langle f(\underline{e}_i), f(\underline{e}_j)\rangle = 0$

Observe that $T_O(\underline{e}_i) = O\underline{e}_i = f(\underline{e}_i) \Rightarrow T_{O^{-1}} \circ f(\underline{e}_i) = \underline{e}_i \ \forall i$ and $T_{O^{-1}} \circ f(\underline{0}) = \underline{0}$ $\therefore T_{O^{-1}} \circ f = id$ by Lemma 1.

Proof of Theorem

Given an isometry f, let $f(\underline{0}) = \underline{v}$, and let $g = T_{-\underline{v}} \circ f$.
Then $g(\underline{0}) = T_{-\underline{v}} \circ f(\underline{0}) = T_{-\underline{v}}(\underline{v}) = \underline{0}$
$\Rightarrow g = T_O$, for some $O \in O(n)$, by lemma 2.
$f = T_{\underline{v}} \circ T_O = T_{O,\underline{v}}$ $\square$

Applications

Proposition

Isometries preserve angles. If $f : \mathbb{R}^n$ is an isometry, then $\angle xyz = \theta$
$= \angle f(x) f(y) f(z)$ $\quad \underline{v} = \underline{x} - \underline{y}, \ \underline{w} = \underline{z} - \underline{y}$

Proof

This is true for $T_{\underline{u}}$ since $T_{\underline{u}}(\underline{x}) - T_{\underline{u}}(\underline{y}) = \underline{x} - \underline{y}$, and the same for $\underline{z} - \underline{y}$. This is also true for T_O, since $T_O(\underline{x}) - T_O(\underline{y}) = O\underline{v}$,
$T_O(\underline{z}) - T_O(\underline{y}) = O\underline{w}$, and $\langle O\underline{v}, O\underline{w}\rangle = \langle \underline{v}, \underline{w}\rangle$

Then this is true for $T_{\underline{u},O}$, and every isometry is of this form.

Geometry ②

Proposition

If $f \in \mathrm{Isom}(\mathbb{R}^n)$ and fixes $n+1$ points that don't lie in a hyperplane, then $f = id$.

Proof

$f(\underline{v}_i) = \underline{v}_i$, $i = 0, 1, \dots, n$. Define $g = T_{-\underline{v}_0} \circ f \circ T_{\underline{v}_0}$

$g(\underline{v}_i - \underline{v}_0) = -\underline{v}_0 + f(\underline{v}_i - \underline{v}_0 + \underline{v}_0) = -\underline{v}_0 + f(\underline{v}_i) = \underline{v}_i - \underline{v}_0$

$\therefore$ g fixes $\underline{0}$, and $\underline{v}_i - \underline{v}_0$ for $i = 1, \dots, n$.

Then Lemma 2 $\Rightarrow g = T_O$ for some $O \in O(n)$

$\underline{v}_i - \underline{v}_0$ are n linearly independent eigenvectors of O with eigenvalue 1

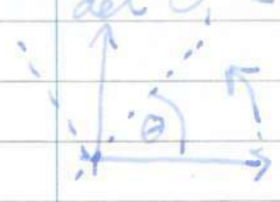
$\Rightarrow O = id$, $f = id$ $\square$

Isometries of $\mathbb{R}^2$

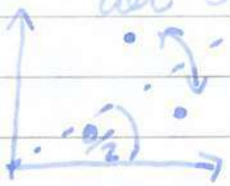
If $O \in O(2)$, either $O = O_1$ or O_2, where

$$O_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad O_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$\det O_1 = 1$ $\qquad$ $\det O_2 = -1$



rotation



reflection

Proposition

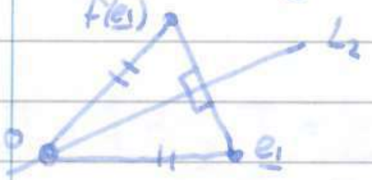
Every $f \in \mathrm{Isom}(\mathbb{R}^2)$ is a composition of at most 3 reflections in lines in $\mathbb{R}^2$.

Proof

Step 1: Consider $f_1 = R_L \circ f$, where R_L is a reflection in L, the perpendicular bisector of $\overrightarrow{\underline{0}f(\underline{0})}$ $\Rightarrow f_1(\underline{0}) = \underline{0}$

Step 2: Reflect in the perpendicular bisector of $\underline{e_1}$ and $f(\underline{e_1})$.
$f_2 = R_{L_2} \circ f_1$ fixes $\underline{0}$ and $\underline{e_1}$.

Either $f_2(e_2) = e_2$, then $f_2 = id$, otherwise, $f_2(e_2) = -e_2$,
then $f_2 = R_{L_3}$ where $L_3 = \overrightarrow{\underline{0}\,\underline{e_1}}$ $\square$



Length of Curves

Suppose X is a metric space. A curve in X is a continuous map $\gamma : [a,b] \to X$.

Definition

If $\{t_i\} \subset [a,b]$ is a finite subset, then let
$$L(\gamma, \{t_i\}) = \sum_{i=1}^{n-1} d(\gamma(t_i), \gamma(t_{i+1}))$$

Notice that if $\{t_i\} \subset \{t_j\}$ then $L(\gamma, \{t_i\}) \leq L(\gamma, \{t_j\})$ by the triangle inequality.

Define $L(\gamma) = \sup\limits_{\text{finite subsets } \{t_i\}} L(\gamma, \{t_i\})$

N.B This could be infinity.

Unproved Proposition:

If $\gamma : [a,b] \to \mathbb{R}^n$ is C^1, then $L(\gamma) = \int_a^b \|\gamma'(t)\| \, dt < \infty$

Geometry ③

Geodesics and Isometries of S^2

$S^2 = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \} = \{ \underline{v} \in \mathbb{R}^3 \mid \|\underline{v}\| = 1 \}$

If $P, Q \in S^2$, let $\pi(P,Q) = \{ \gamma : [0,1] \to S^2 \mid \gamma(0) = P, \gamma(1) = Q, \gamma \in C^1$
We want $d(P,Q) = \inf_{\pi(P,Q)} L(\gamma)$

Using spherical coordinates : $(\theta, \varphi) \mapsto (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$
Let $\gamma(t) = (\theta(t), \varphi(t))$, $L(\gamma) = \int_0^1 \|\gamma'(t)\| \, dt$

$\gamma'(t) = (-\sin\theta \sin\varphi, \cos\theta \sin\varphi, 0)\dot{\theta} + (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi)\dot{\varphi}$
$= \underline{V_\theta}\, \dot{\theta} + \underline{V_\varphi}\, \dot{\varphi}$

$\underline{V_\theta} \cdot \underline{V_\theta} = \sin^2\varphi$, $\underline{V_\theta} \cdot \underline{V_\varphi} = 0$, $\underline{V_\varphi} \cdot \underline{V_\varphi} = 1$
$\|\gamma'(t)\|^2 = \underline{V_\theta} \cdot \underline{V_\theta}\, \dot{\theta}^2 + 2\underline{V_\theta} \cdot \underline{V_\varphi}\, \dot{\varphi}\dot{\theta} + \underline{V_\varphi} \cdot \underline{V_\varphi}\, \dot{\varphi}^2 = \sin^2\varphi\, \dot{\theta}^2 + \dot{\varphi}^2$

Suppose $P = N$. We can get from N to any $Q \in S^2$ by travelling along a path with $\dot{\theta} = 0$

$\Rightarrow L(\varphi) = \int_0^1 \sqrt{\sin^2\varphi\, \dot{\theta}^2 + \dot{\varphi}^2} \, dt \geq \int_0^1 |\dot{\varphi}| \, dt \geq \int_0^1 \dot{\varphi} \, dt = \varphi(Q)$
with equality $\Leftrightarrow$ $\dot{\theta} = 0$, $\dot{\varphi} > 0$.

$\Rightarrow$ A line of longitude is the shortest path from the North pole to any point Q, $d(N, Q) = \varphi(Q) =$ latitude of Q

Proposition

The shortest path from N to $Q \in S^2$ is along a line of longitude.

Lemma

Suppose $f \in \mathrm{Isom}(\mathbb{R}^3)$. Then $L(f \circ \gamma) = L(\gamma)$, where $\gamma \in \pi(P, Q)$.

Proof

If $\{t_i\} \subset [0,1]$, $L(\gamma, \{t_i\}) = \sum d(\gamma(t_i), \gamma(t_{i+1}))$
$L(\gamma, \{t_i\}) = \sum d(f \circ \gamma\{t_i\}, f \circ \gamma(t_{i+1}))$ since $f \in Isom(\mathbb{R}^3)$
$= L(f \circ \gamma, \{t_i\})$.
$L(\gamma) = \sup L(\gamma, \{t_i\}) = \sup L(f \circ \gamma, \{t_i\}) = L(f \circ \gamma)$

Corollary

Suppose $f \in Isom(\mathbb{R}^3)$, $f(S^2) = S^2$. Then $d(P,Q) = d(f(P), f(Q))$

Proof

There is a bijection $\pi(P,Q) \Leftrightarrow \pi(f(P), f(Q))$, $\gamma \Leftrightarrow f \circ \gamma$ which preserves lengths.
$\therefore d(P,Q) = \inf_{\pi(P,Q)} L(\gamma) = \inf_{\pi(f(P),f(Q))} L(f \circ \gamma) = d(f(P), f(Q))$

Corollary

If $f \in Isom(\mathbb{R}^3)$, $f(S^2) = S^2$, then $f|_{S^2} \in Isom(S^2)$.
Observe that if $O \in O(3)$, then $\langle T_O \underline{v}, T_O \underline{v} \rangle = \langle \underline{v}, \underline{v} \rangle$
so if $\underline{v} \in S^2$, then $T_O \underline{v} \in S^2$.

Definition

A great circle on S^2 is $S^2 \cap H$, where $H \subset \mathbb{R}^3$ is any linear 2D subspace (i.e. plane through the origin)

Theorem

The shortest path from P to Q in S^2 is the arc of a great circle.

Geometry ③

Lemma

$\exists O \in O(3)$ with $T_O(P) = N$

Proof

Start with P and use Gram-Schmidt to extend to an orthonormal basis of $\mathbb{R}^3$, $\{A, B, P\}$. Then A, B, P are the columns of $O' \in O(3)$, where $T_{O'}(N) = O'(\underline{e_3}) = P$, and so $O = (O')^{-1}$.

Proof of Theorem

O is a linear transformation mapping linear subspaces to linear subspaces, so T_O maps great circles to great circles. Pick $O \in O(3)$ with $T_O(P) = N$. Then, the shortest path from $T_O(P)$ to $T_O(Q)$ is a line of longitude, which in particular is the arc of a great circle. Therefore, $T_O^{-1} \circ \gamma$ is also an arc of a great circle. $\square$

Spherical Geometry

If $P, Q \in S^2$, there is a unique line from P to Q if $\underline{P}, \underline{Q}$ are linearly independent vectors in $\mathbb{R}^3$. If $\underline{P} = -\underline{Q}$, one can draw infinitely many lines joining P and Q, and P, Q are called "antipodal" points.

If L_1, L_2 are distinct spherical lines, they correspond to planes H_1 and H_2, which intersect in a 1D linear subspace V, where $V \cap S^2 = 2$ antipodal points.

$\therefore L_1$ and L_2 intersect in a pair of antipodal points.

Isometries

We have already showed that $O(3) \subset \mathrm{Isom}(S^2)$

Theorem

$O(3) = \mathrm{Isom}(S^2)$

Lemma

If $f \in \mathrm{Isom}(S^2)$, $f(\underline{e_i}) = \underline{e_i}$, $i = 1, 2, 3$, then $f = id$.

Proof

If $P = (x, y, z) \in S^2$, then $z = \cos d(P, \underline{e_3})$ since $d(P, \underline{e_3}) = \ell(P)$,
Similarly, $y = \cos d(P, \underline{e_2})$, $x = \cos d(P, \underline{e_1})$

$d(P, \underline{e_i}) = d(f(P), f(\underline{e_i})) = d(f(P), \underline{e_i})$

Then the coordinates of P are all the same.

Geometry ④

Lemma

If $\underline{v}, \underline{w} \in S^2$, then $\underline{v} \cdot \underline{w} = \cos d(\underline{v}, \underline{w})$

Proof

$d(\underline{v}, \underline{w}) =$ the angle between $\underline{v}, \underline{w}$

$\cos d(\underline{v}, \underline{w}) = \frac{\underline{v} \cdot \underline{w}}{\|\underline{v}\|\|\underline{w}\|} = \underline{v} \cdot \underline{w}$

Corollary

If $f \in \mathrm{Isom}(S^2)$, $\underline{v}, \underline{w} \in S^2$, $\underline{v} \cdot \underline{w} = f(\underline{v}) \cdot f(\underline{w})$

$d(\underline{v}, \underline{w}) = d(f(\underline{v}), f(\underline{w})) \Rightarrow \cos d(\underline{v}, \underline{w}) = \cos d(f(\underline{v}), f(\underline{w}))$

Theorem

$\mathrm{Isom}(S^2) = O(3)$

Lemma

If $f \in \mathrm{Isom}(S^2)$ and $f(\underline{e_i}) = \underline{e_i}$, $i = 1, 2, 3$, then $f = id$

Proof

$\underline{v} = (v_1, v_2, v_3)$, $v_i = \underline{v} \cdot \underline{e_i} = f(\underline{v}) \cdot f(\underline{e_i}) = f(\underline{v}) \cdot \underline{e_i}$

$f(\underline{v}) = (v_1', v_2', v_3') \Rightarrow v_i = v_i'$, $\underline{v} = f(\underline{v})$

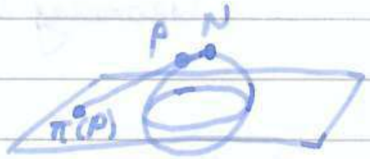
Proof of Theorem

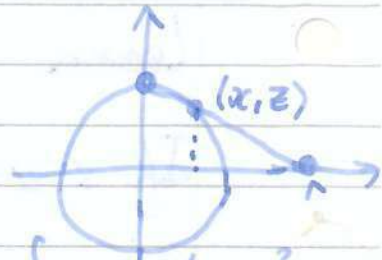
Given $f \in \mathrm{Isom}(S^2)$, $f(\underline{e_i}) \cdot f(\underline{e_j}) = \underline{e_i} \cdot \underline{e_j} = \delta_{ij}$

$\Rightarrow f(\underline{e_i})$ are columns of $O \in O(3)$

$T_o^{-1} \circ f(\underline{e_i}) = \underline{e_i} \Rightarrow T_o^{-1} \circ f = id$, $f = T_o$ $\square$

Proof Möbius Transformations





a) Riemann Sphere

$\pi : S^2 \setminus \{N\} \to \mathbb{R}^2 = \mathbb{C}$, $\pi(P) = \vec{NP} \cap \{x\text{-}y \text{ plane}\}$
π is the stereographic projection map.

Using similar triangles: $\frac{r}{1} = \frac{r-x}{z} \Rightarrow r = \frac{x}{1-z}$, $x = \pm\sqrt{1-z^2}$
$r = \pm\sqrt{\frac{1+z}{1-z}} \Rightarrow (x, z) = \left(\frac{2r}{1+r^2}, \frac{r^2-1}{1+r^2}\right)$

In S^2, radial symmetry $\Rightarrow \pi(x, y, z) = \frac{x+iy}{1-z} \in \mathbb{C}$
$\pi^{-1}(w) = \frac{1}{1+|w|^2}\left(2Re(w), 2Im(w), |w|^2-1\right)$

Identify S^2 with $\mathbb{C} \cup \{\infty\} = \mathbb{C}_\infty$, the Riemann Sphere.
$\underline{v} \mapsto \pi(\underline{v})$, $N \mapsto \infty$

Definition

Complex Projective Space $\mathbb{C}P^1 = \{1D \text{ linear subspaces of } \mathbb{C}^2\}$
$\mathbb{C}P^1 = \{\underline{v} \in \mathbb{C}^2 \mid \underline{v} \neq 0\}$, with $\lambda\underline{v} \sim \underline{v}$ for $\lambda \in \mathbb{C}\setminus\{0\}$
$\mathbb{C}P^1 \Leftrightarrow \mathbb{C}_\infty$, $\underline{v} = (v_1, v_2) \mapsto \frac{v_1}{v_2}$

Möbius Group

$GL_2(\mathbb{C})$ acts on $\mathbb{C}^2$. $A \in GL_2(\mathbb{C})$, $\underline{v} \in \mathbb{C}^2$, $\underline{v} \mapsto A\underline{v}$ (matrix multiplication)
If $\underline{v} \sim \underline{w}$, $A\underline{v} \sim A\underline{w}$ since $A(\lambda\underline{v}) = \lambda A\underline{v}$
$\Rightarrow GL_2(\mathbb{C})$ acts on $\mathbb{C}P^1 = \mathbb{C}_\infty$
If $A = \lambda I$, $A\underline{v} = \lambda\underline{v} \sim \underline{v}$ so $\{\lambda I \mid \lambda \in \mathbb{C}\setminus\{0\}\}$ acts trivially.

Definition

The Möbius group $\mathcal{M} = GL_2(\mathbb{C}) / \{\lambda I \mid \lambda \neq 0\}$
$= PGL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\} = PSL_2(\mathbb{C})$

Geometry (4)

Action of $\mathcal{M}$ on $\mathbb{C}_\infty$:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix} \mapsto \frac{av_1 + bv_2}{cv_1 + dv_2} \in \mathbb{C}_\infty$

$w \in \mathbb{C}_\infty$, $A\,w = \frac{aw+b}{cw+d}$, $w = \frac{v_1}{v_2}$

$\mathcal{M}$ and Isometries

$Isom^+(\mathbb{R}^2) = \{T_{O,\underline{v}} \mid O \in SO(2)\}$ and $Isom^+(S^2)$, $SO(3)$ are the orientation preserving isometries.

Proposition

$$Isom^+(\mathbb{R}^2) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid |a| = 1 \right\} = T$$

Proof

T acts on $\mathbb{C}_\infty$ by $w \mapsto \frac{aw+b}{1}$, $|a| = 1$

$\underline{w} \mapsto T_{O,\underline{v}}\,\underline{w}$, $O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $a = e^{i\theta}$, $b = \underline{v} \in \mathbb{C}^2 = \mathbb{R}^2$

For S^2, $SU(2) \subset GL_2(\mathbb{C})$, $SU(2) = \{u \in GL_2(\mathbb{C}) \mid uu^\dagger = I$

$PSU_2(2) = SU(2) / \{\pm I\} \subset \mathcal{M}$

Proposition

$Isom^+(S^2) = SO(3)$ is realised by elements of $PSU(2) \subset \mathcal{M}$

Proof

R_θ = rotation by θ about the z-axis.

ρ = rotation by $\frac{\pi}{2}$ about the y-axis

Let $G = \langle R_\theta, \rho \rangle \subset SO(3)$

Claim ① : $G = SO(3)$
Claim ② : The actions of R_θ, ρ are realised by elements of $\mathcal{M}$

Lemma

Given $\underline{v} \in S^2$, $\exists g \in G$ with $g(\underline{e_3}) = \underline{v}$

Proof

$$\underline{e_3} = (0,0,1) \xrightarrow{\rho} (1,0,0) \xrightarrow{R_\alpha} (\cos\alpha, \sin\alpha, 0) \xrightarrow{\rho^3} (0, \sin\alpha, \cos\alpha) \xrightarrow{R_{\theta'}} \underline{v}$$

Proof of ① : Any $O \in SO(3)$ is a rotation about some $\underline{v}$, so we can write $O = g R_\theta g^{-1}$, where $g(\underline{e_3}) = \underline{v}$ $\square$

Proof of ②:

R_θ is easy; take $A = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$

We must show that $\pi \rho \pi^{-1} \in \mathcal{M}$

$$\pi\rho\pi^{-1} = \pi\rho\left(\frac{2x}{\alpha}, \frac{2y}{\alpha}, \frac{\alpha-2}{\alpha}\right) \qquad \alpha = x^2+y^2+1$$
$$= \pi\left(\frac{\alpha-2}{\alpha}, \frac{2y}{\alpha}, \frac{2x}{\alpha}\right) = \frac{\alpha-2+2iy}{\alpha-(-2x)}$$
$$= \frac{w\bar{w}+w-\bar{w}-1}{w\bar{w}+w+\bar{w}+1} = \frac{(w-1)(\bar{w}+1)}{(w+1)(\bar{w}+1)} \qquad w = x+iy$$
$$= \frac{w-1}{w+1} \in \mathcal{M}$$
$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in SU(2)$$

30/02/12 Geometry (5)

D) Properties of Möbius Transformations

1. Given z_0, z_1, z_∞ in $\mathbb{C}_\infty$ $\exists! \varphi \in \mathcal{M}$ with $\varphi(z_0) = 0, \varphi(z_1) = 1$
$\varphi(z_\infty) = \infty$, $z \mapsto \frac{z - z_0}{z - z_\infty} \frac{z_1 - z_0}{z_1 - z_\infty}$ is the φ.
2. If $z_1, z_2, z_3, w_1, w_2, w_3 \in \mathbb{C}_\infty$, then $\exists! \varphi \in \mathcal{M}$ with
$\varphi(z_i) = w_i$. Construct ψ_1 with $\psi_1(z_1) = 0, \psi_1(z_2) = 1, \psi_1(z_3) = \infty$
$\psi_2(w_1) = 0, \psi_2(w_2) = 1, \psi_2(w_3) = \infty$, $\varphi = \psi_2^{-1} \circ \psi_1$
3. Cross-ratio: If $w_1, w_2, w_3, w_4 \in \mathbb{C}_\infty$ then the cross-ratio
$[w_1, w_2, w_3, w_4] = \varphi(w_4)$ where $\varphi(w_1) = 0, \varphi(w_2) = 1, \varphi(w_3) = \infty$
$[w_1, w_2, w_3, w_4] = \frac{w_2 - w_3}{w_2 - w_1} \cdot \frac{w_4 - w_1}{w_4 - w_3}$
If $\psi \in \mathcal{M}$, then $[w_1, w_2, w_3, w_4] = [\psi(w_1), \psi(w_2), \psi(w_3), \psi(w_4)]$
4. Generators for $\mathcal{M}$: $\mathcal{M}$ is generated by
$z \mapsto az$, $a \in \mathbb{C} \setminus \{0\}$ dilation
$z \mapsto z + b$, $b \in \mathbb{C}$ translation
$z \mapsto \frac{1}{z}$
$\frac{az+b}{cz+d} = \alpha + \frac{\beta}{cz+d}$
$z \mapsto cz \mapsto cz + d \mapsto \frac{1}{cz+d} \mapsto \frac{\beta}{cz+d} \mapsto \alpha + \frac{\beta}{cz+d}$
5. Möbius Transformations preserve {Euclidean lines and circles} with a line considered a "circle through ∞".

Why? It is enough to check for the generators of $\mathcal{M}$. This is obvious for dilations and translations. What about $z \mapsto \frac{1}{z}$

Equation of circle: $|z - b|^2 = r^2$, $z\bar{z} - b\bar{z} - \bar{b}z + b\bar{b} = r^2$
$z\bar{z} - b\bar{z} - \bar{b}z = c$

Every equation of the form $az\bar{z} - b\bar{z} - \bar{b}z + c = 0$, $a, c \in \mathbb{R}, b \in \mathbb{C}$
defines a line $(a=0)$ or a circle $(a \neq 0)$

Send $z \mapsto \frac{1}{z}$, $\frac{a}{z\bar{z}} - \frac{b}{\bar{z}} - \frac{\bar{b}}{z} + c = 0 \Rightarrow a - b\bar{z} - \bar{b}z + cz\bar{z} = 0$
$\Rightarrow$ another line or circle.

<u>Spherical Triangles</u>

A) Angles. Suppose $A, B \in S^2$ are not antipodal.

<u>Definition</u>: Line segment $\overline{AB}$ = shorter arc of great circle joining A, B.

<u>Definition</u>: Given points A, B, C

$\angle ABC$ = Angle between the tangent vectors to $\overrightarrow{BA}$, $\overrightarrow{BC}$
(as vectors in $\mathbb{R}^3$)

<u>Lemma</u>

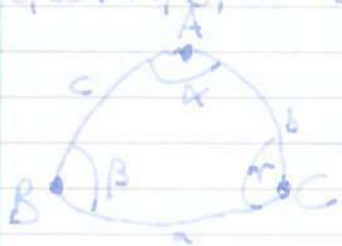
Let $\underline{n_1}$ be the unit normal (in $\mathbb{R}^3$) to the plane containing A, B, O, $\underline{n_2}$ the same but for B, C, O, chosen so $\underline{n_1}$ points towards $\overrightarrow{BC}$ and $\underline{n_2}$ towards ~~AB~~ $\overrightarrow{BA}$. Then the angle between $\underline{n_1}, \underline{n_2}$ is $\pi - \angle ABC$

Proof Let $\underline{v_1}$ = tangent vector to $\overrightarrow{BA}$, $\underline{v_2}$ = tangent vector to $\overrightarrow{BC}$.
Then $\underline{v_1}, \underline{v_2}, \underline{n_1}, \underline{n_2}$ are all in $\underline{B}^\perp$ since {tangent vectors to S^2 at $\underline{B}$} $= \underline{B}^\perp$

$\underline{B}^\perp$:

$\underline{n_1} \perp \underline{v_1}$, $\underline{n_2} \perp \underline{v_2} \Rightarrow$ angle between $\underline{n_1}, \underline{n_2}$
$= \pi - \angle$ between $\underline{v_1}, \underline{v_2}$

Given $A, B, C \in S^2$, $\triangle ABC$ has sides $\overline{AB}, \overline{AC}, \overline{BC}$



06/02/12 Geometry (5).

B) Area of triangles

<u>Theorem</u> Area $(\triangle ABC) = \alpha + \beta + \gamma - \pi$

(The right form of this theorem: $\alpha + \beta + \gamma - \pi = \int_{\triangle ABC} K$ ← curvature

$K \equiv 0$ on $\mathbb{R}^2$, $K \equiv 1$ on S^2)

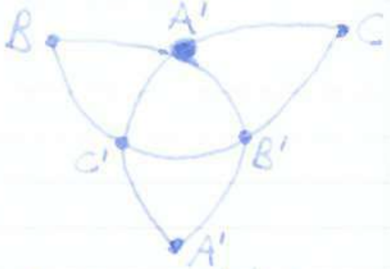
<u>Proof</u>

Let $S_\Theta = \{ (\varphi, \theta) \mid \varphi \in [0, \pi],\ \theta \in [0, \Theta] \}$ = sector with $\angle \Theta$

$$\left.\begin{aligned} \text{Area}(S_\Theta) &= \int_{S_\Theta} \sin\theta \, d\theta \, d\varphi \\ \text{Area}(S_{\Theta+\Theta'}) &= A(S_\Theta) + A(S_{\Theta'}) \end{aligned}\right\} = 2\Theta$$

since $A(S_{2\pi}) = A(S^2) = 4\pi$

~~Write~~ Let A', B', C' be antipodal points to A, B, C. Write $S^2 = R \cup R'$

$R =$  $R' =$ 

If these were Euclidean triangles this would give an octahedron.

Now $R \cong R'$, $A(R) = A(R') = 2\pi$

$R = S_\alpha \cup S_\beta \cup S_\gamma$, with $\triangle ABC$ counted 3 times

$\Rightarrow A(R) = A(S_\alpha) + A(S_\beta) + A(S_\gamma) - 2A(\triangle ABC) = 2\pi$

$\Rightarrow 2\alpha + 2\beta + 2\gamma - A(\triangle ABC$

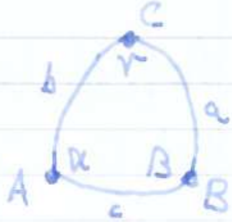
$\Rightarrow \alpha + \beta + \gamma - A(\triangle ABC) = \pi$

$\alpha + \beta + \gamma - \pi = A(\triangle ABC)$

07/02/12

Geometry (6)

c) Spherical Trigonometry



Euclidean | **Spherical**

$c^2=a^2+b^2-2ab\cos\gamma$ $\qquad$ $\sin a\sin b\cos\gamma = \cos c - \cos a\cos b$

$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$ $\qquad$ $\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c}$

Proof

Let $\underline{n_a}, \underline{n_b}, \underline{n_c}$ be the inward pointing normals to OBC, OAC, OAB

Then $\underline{A}\cdot\underline{B} = \cos c\,\|\underline{A}\|\,\|\underline{B}\| = \cos c$ $\qquad$ $\underline{n_a}\cdot\underline{n_b} = \cos(\pi-\gamma) = -\cos\gamma$

$\underline{A}\times\underline{B} = \sin c\ \underline{n_c}$ $\qquad$ $\underline{n_a}\times\underline{n_b} = \sin\gamma\ \underline{C}$

$\underline{B}\times\underline{C} = \sin a\ \underline{n_a}$ $\qquad$ $\underline{n_b}\times\underline{n_c} = \sin\alpha\ \underline{A}$

$\underline{C}\times\underline{A} = \sin b\ \underline{n_b}$ $\qquad$ $\underline{n_c}\times\underline{n_a} = \sin\beta\ \underline{B}$

Lemma

1) $(\underline{A}\times\underline{C})\cdot(\underline{B}\times\underline{C}) = (\underline{C}\cdot\underline{C})(\underline{A}\cdot\underline{B}) - (\underline{A}\cdot\underline{C})(\underline{B}\cdot\underline{C})$

2) $(\underline{A}\times\underline{C})\times(\underline{B}\times\underline{C}) = ((\underline{A}\times\underline{B})\cdot\underline{C})\,\underline{C}$

3) $(\underline{A}\times\underline{B})\cdot\underline{C} = (\underline{B}\times\underline{C})\cdot\underline{A} = (\underline{C}\times\underline{A})\cdot\underline{B}$

1) Becomes $-\sin b\,\underline{n_b}\cdot\sin a\,\underline{n_a} = 1\cdot\cos c - \cos b\cos a$

$\sin b\sin a\cos\gamma = \cos c - \cos b\cos a$

2) Becomes $(-\sin b\ \underline{n_b})\times(\sin a\ \underline{n_a}) = ((\underline{A}\times\underline{B})\cdot\underline{C})\underline{C}$

$\sin a\sin b\sin\gamma\ \underline{C} = ((\underline{A}\times\underline{B})\cdot\underline{C})\underline{C}$

$\sin a\sin b\sin\gamma = (\underline{A}\times\underline{B})\cdot\underline{C} = (\underline{B}\times\underline{C})\cdot\underline{A} = \sin b\sin c\sin\alpha$

$\frac{\sin\gamma}{\sin c} = \frac{\sin\alpha}{\sin a}$

Proof of Lemma

Let $O \in SO(3)$. $T_O \underline{A} \cdot T_O \underline{B} = \underline{A} \cdot \underline{B}$ $\qquad \underline{A} = x\underline{i} + y\underline{j} + z$

$T_O \underline{A} \times T_O \underline{B} = T_O(\underline{A} \times \underline{B}) \Rightarrow$ WLOG $\underline{C} = c\underline{k}$, $\underline{B} = a\underline{j} + b\underline{k}$

5) Euler Characteristic

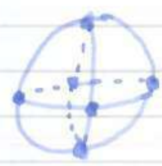
A) Polyhedra

Definition A spherical polyhedron is

1) A set $V \subset S^2$ of vertices $\qquad$ 2) A set E of edges

Edges are such that Each edge is a geodesic arc, length $< \pi$. Endpoints of edges are vertices. Edges are disjoint except at endpoints. Angles at consecutive edges at a vertex are $< \pi$.

3) The set of faces = components of $S^2 \setminus E$



e.g. $S^2 \cap$ coordinate hyperplanes in $\mathbb{R}^3$ is a "spherical octahedron"

6 vertices, 12 edges, 8 faces

A convex Euclidean polyhedron is $P = \bigcap_{i=1}^{n} \{\underline{x} \in \mathbb{R}^3 \mid \underline{x} \cdot \underline{v_i} \le c_i, \underline{v_i} \in \mathbb{R}^3, c_i \in \mathbb{R}\}$

where P is bounded.

Faces = $\{\underline{x} \mid \underline{x} \cdot \underline{v_i} = c_i\}$ for one i.

Edge = " " for two is

Vertex = " " for >2 is

P is regular if all faces are congruent regular polygons and for any 2 vertices of P, there is an isometry of $\mathbb{R}^3$ that leaves P invariant and moves 1 vertex to another.

07/02/12 Geometry (6)

Observe

If $O \in \text{Interior}(P) = \text{Int}(P)$, P a convex Euclidean polyhedron, as $S = S^2(r)$ contains P and is centred at O. Then, the radial projection of P defines a spherical polyhedron on S.

vertices $\Rightarrow$ vertices, edges $\Rightarrow$ edges, faces $\Rightarrow$ faces

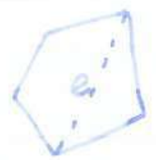
B) Euler's Formula

Theorem If P is a spherical polyhedron, then $\underbrace{|V| - |E| + |F|}_{\chi \text{ - Euler Characteristic}} = 2$ where $|V| = \#$ vertices, $|E| = \#$ edges, $|F| = \#$ faces

Corollary If P is a convex Euclidean polyhedron, $|V| - |E| + |F| = 2$

Proof of Corollary Project P to a spherical polyhedron.

Proof of Theorem



Suppose P has a face with > 3 edges.

We can add a new edge e to get 2 faces with $< n$ edges

$|E|' = |E| + 1$, $|F|' = |F| + 1 \Rightarrow |V| - |E| + |F| = |V| - |E|' + |F|'$

So WLOG, assume all faces are triangles.

In this case, we say P is a <u>Triangulation</u> of S^2.

Each edge is adjacent to 2 faces. Each face has 3 edges $\Rightarrow 2E = 3F$

$\chi = |V| - \frac{1}{2}|F|$

Consider $\sum$ all angles in all faces of P

$$= \sum_{v \in V} \text{angles in } v = \sum_{v \in V} 2\pi = 2\pi|V|$$

$$= \sum_{f \in F} \sum \text{angles in } F = \sum_{f \in F} (\text{Area}(f) + \pi) = \sum_{f \in F} \text{area}(F) + \pi|F| = 4\pi + \pi|F|$$

$2\pi|V| = 4\pi + \pi|F| \Rightarrow |V| - \frac{1}{2}|F| = 2$ $\square$

13/02/12

Geometry (7)

$(0,y)\sim(1,y)$
$(x,0)\sim(x,1)$

C) Euler Characteristic of Other Surfaces

Recall: If P is a polyhedral decomposition of S^2

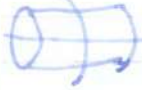
$\chi = |V| - |E| + |F| = 2$. This 2 is an invariant of S^2

Example: The Torus T^2 $= \dfrac{[0,1]\times[0,1]}{\sim}$

$T^2 = \mathbb{R}^2/\mathbb{Z}^2 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) = S^1 \times S^1$

<u>Proposition</u> For a polygonal decomposition of T^2

$\chi = |V| - |E| + |F| = 0$

 - Proof

Exactly the same as for S^2 up until:

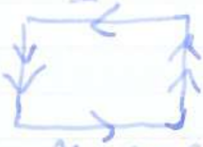
For a triangulation $2\pi |V| = \sum_f \sum \text{angles in } f = \sum_F \pi = \pi |F|$

$\Rightarrow |V| - \frac{1}{2}|F| = 0$ (Not 2)

Example 2: $\mathbb{R}P^2$, real projective space

$= \{$1-d linear subspaces of $\mathbb{R}^3\} = \{\underline{v} \in \mathbb{R}\setminus\{0\}\}/\sim$ $\quad \underline{v} \sim \lambda \underline{v}$, $\lambda \neq 0$, $\lambda \in \mathbb{R}$

$= S^2/\sim$ $\quad \underline{v} \sim -\underline{v}$ $\quad M$

$=$  $=$  $=$  Glue ∂M to a 

$\mathbb{R}P^2$ is not orientable. $\chi(\mathbb{R}P^2) = 1$

Given a polyhedral decomposition P of $\mathbb{R}P^2$, $\pi : S^2 \to \mathbb{R}P^2$

$\pi^{-1}(P) = P'$ is a polyhedral decomposition of S^2

$|V'| = 2|V|$, $|E'| = 2|E|$, $|F'| = 2|F|$

$|V'| + |E'| - |F'| = 2 \Rightarrow |V| + |E| - |F| = 1$

6) Riemannian Metrics

A) Recall that a parametrised surface in $\mathbb{R}^3$ is an open set $U \subset \mathbb{R}^2$ and a map $\underset{(u,v)}{S: U} \to \underset{(x,y,z)}{\mathbb{R}^3}$ so that S is injective and $DS|_p$ is injective at every $p \in U$.

e.g. $S: (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$ Spherical coordinate map

$(\theta, \varphi) \to (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$

Suppose $\gamma: [0,1] \to U$ is a path, $\gamma(t) = (u(t), v(t))$

Then $S \circ \gamma : [0,1] \to \mathbb{R}^3$ is a path on S.

$$L(\gamma) = \int_0^1 \| \Gamma'(t) \| \, dt, \quad \Gamma'(t) = \underline{S}_u \, u'(t) + \underline{S}_v \, v'(t)$$

$$= [\underline{S}_u, \underline{S}_v] \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = DS_{\gamma(t)}(\gamma'(t))$$

$$\Gamma'(t) \cdot \Gamma'(t) = \underline{S}_u \cdot \underline{S}_u \, u'(t)^2 + 2 \underline{S}_u \cdot \underline{S}_v \, u'(t) v'(t) + \underline{S}_v \cdot \underline{S}_v \, v'(t)^2$$

$$= [u'(t) \;\; v'(t)] \begin{bmatrix} \underline{S}_u \cdot \underline{S}_u & \underline{S}_u \cdot \underline{S}_v \\ \underline{S}_u \cdot \underline{S}_v & \underline{S}_v \cdot \underline{S}_v \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix}$$

a quadratic form applied to $\gamma'(t)$.

$\underline{S}_\theta = (-\sin\theta \sin\varphi, \cos\theta \sin\varphi, 0)$, $\underline{S}_\varphi = (\cos\theta \cos\varphi, \sin\theta \cos\varphi, -\sin\varphi)$

$$\begin{bmatrix} \underline{S}_\theta \cdot \underline{S}_\theta & \underline{S}_\theta \cdot \underline{S}_\varphi \\ \underline{S}_\theta \cdot \underline{S}_\varphi & \underline{S}_\varphi \cdot \underline{S}_\varphi \end{bmatrix} = \begin{bmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{bmatrix}$$

B) Riemannian Metrics

Define $\underset{\text{inner products}}{IP(\mathbb{R}^2)} = \{$symmetric +ve definite bilinear forms on $\mathbb{R}^2\}$

$$= \{ B = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, E > 0, EG - F^2 > 0 \} \subset M_2(\mathbb{R})$$

Definition Let $U \subset \mathbb{R}^2$ be an open set.

A Riemannian metric on U is a C^∞ map $g: U \to IP(\mathbb{R}^2)$

$$g(u,v) = \begin{bmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{bmatrix}$$

13/02/12 Geometry (7)

Example:

Suppose $S: U \to \mathbb{R}^3$ is a parametrised surface.

Then there is an induced Riemannian metric defined by

$\langle \underline{w}_1, \underline{w}_2 \rangle_g = \langle D_p \underline{w}_1, D_p \underline{w}_2 \rangle_{\mathbb{R}^3}$

vectors at $p \in U$

Concretely, $\langle \underline{w}_1, \underline{w}_2 \rangle_{g_S} = \underline{w}_1^T \begin{bmatrix} \underline{S}_u \cdot \underline{S}_u & \underline{S}_u \cdot \underline{S}_v \\ \underline{S}_u \cdot \underline{S}_v & \underline{S}_v \cdot \underline{S}_v \end{bmatrix} \underline{w}_2$

Note that $D_p S$ injective $\Rightarrow \underset{\underline{w} \neq \underline{0}}{\langle \underline{w}, \underline{w} \rangle_g} = \langle D_p \underline{w}, D_p \underline{w} \rangle_{\mathbb{R}^3} > 0$

Example:

$S: \mathbb{R}^2 \to \mathbb{R}^3, \quad (u, v) \to \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2} \right)$ Inverse of the stereographic projection map

Find the induced Riemannian metric on $\mathbb{R}^2$

$\underline{S}_u = \left(\frac{2}{1+u^2+v^2} - \frac{4u^2}{(1+u^2+v^2)^2}, -\frac{4uv}{(1+u^2+v^2)^2}, -\frac{4u}{(1+u^2+v^2)^2} \right) \qquad \alpha = 1+u^2+v^2$

$= \frac{1}{\alpha^2} \left(2+2u^2+2v^2-4u^2, -4uv, -4u \right) = \frac{2}{\alpha^2} \left(1-u^2+v^2, -2uv, -2u \right)$

$\underline{S}_v = \frac{2}{\alpha^2} \left(-2uv, 1-v^2+u^2, -2v \right)$

$\underline{S}_u \cdot \underline{S}_u = \frac{4}{\alpha^4} \left(1+u^4+v^4-2u^2-2u^2v^2+2v^2+4u^2v^2+4u^2 \right)$

$= \frac{4\alpha^2}{\alpha^4} = \frac{4}{\alpha^2}$

$\underline{S}_v \cdot \underline{S}_v = \frac{4}{\alpha^2}$

$\underline{S}_u \cdot \underline{S}_v = -2uv \left(1-u^2+v^2+1-v^2+u^2 \right) + 4uv = -4uv + 4uv = 0$

Induced metric $= \begin{bmatrix} \frac{4}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{4}{(1+u^2+v^2)^2} \end{bmatrix}$

Geometry ⑧

Distance and Angles

Suppose $g = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ is a Riemannian metric on $U \subset \mathbb{R}^2$, and $\gamma : [0,1] \to U$ is a path $\gamma(t) = (u(t), v(t))$. Then:
$\langle \gamma'(t), \gamma'(t) \rangle_g = E u'(t)^2 + 2F u'(t) v'(t) + G v'(t)^2$

Write $g = E\,du^2 + 2F\,du\,dv + G\,dv^2$

Length

If $\gamma : [0,1] \to U$ is a C^1 path:
$L_g(\gamma) = \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_g}\, dt = \int_0^1 \| \gamma'(t) \|_g \, dt$

Distance

If $x, y \in U$, define $d_g(x,y) = \inf_{\gamma \in P(x,y)} L_g(\gamma)$

Here $P(x,y) = \{ \gamma : [0,1] \to U \mid \gamma \in C^1, \gamma(0) = x, \gamma(1) = y \}$

N.B. The inf need not be attained by some γ 

$g = du^2 + d$

Angles

Define the angle between $\underline{v}, \underline{w}$ at $p \in U$ to be:
$\cos \angle_p \underline{v}, \underline{w} = \langle \underline{v}, \underline{w} \rangle_{g(p)} / \|\underline{v}\|_{g(p)} \|\underline{w}\|_{g(p)}$

Warning: Like $\|\underline{v}\|_{g(p)}$, this depends on p.

Isometries

If $f \in C^\infty$, f is bijective, and $D_p f$ is bijective $\forall p$, then we sa
that f is a diffeomorphism.

Suppose $f_1 : U_1 \to U_2$ is a diffeomorphism. If g_2 is a Riemannian metric on U_2, then we can define ("pull back") a Riemannian metric $g_1 = f_1^* g_2$ by $\langle \underline{v}, \underline{w} \rangle_{g_1(p)} = \langle D_p f_1 \underline{v}, D_p f_1 \underline{w} \rangle_{g_2(f_1(p))}$, on U_1.

This is bilinear, since $D_p f$ is a linear map, and positive definite since $D_p f$ is injective, and hence a Riemannian Metric.

$$L_{g_2}(f_1 \circ \gamma) = \int_0^1 \sqrt{\langle (f_1 \circ \gamma)', (f_1 \circ \gamma)' \rangle_{g_2}}\, dt$$

$$= \int_0^1 \sqrt{\langle Df_1(\gamma'(t)), Df_1(\gamma'(t)) \rangle_{g_2}}\, dt = \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{g_1}}\, dt$$

$$\Rightarrow L_{g_1}(\gamma) = L_{g_2}(f_1 \circ \gamma)$$

Similarly, if $f_2 : U_2 \to U_3$ is a diffeomorphism, then we can define a Riemannian metric on U_1, using a metric g_3 on U_3:

$$g_1 = f_1^*(f_2^*(g_3)) = (f_2 \circ f_1)^*(g_3)$$

Definition

$\varphi : U_1 \to U_2$ is a Riemannian Isometry if, for metrics g_1 and g_2, $\varphi^*(g_2) = g_1$, and φ is a diffeomorphism.

Note that $\gamma \in P(x, y)$, $x, y \in U$,

$L_{g_1}(\gamma) = L_{g_2}(\varphi \circ \gamma) \Rightarrow d_{g_1}(x, y) = d_{g_2}(\varphi(x), \varphi(y))$

Note that φ is bijective, so if $\Gamma \in P(\varphi(x), \varphi(y))$, then $\varphi^{-1} \circ \Gamma \in P(x, y)$

The Hyperbolic Plane

Unit Disc Model: $U = D = \{ \underline{v} \in \mathbb{R}^2 \mid \|\underline{v}\| < 1 \} \subset \mathbb{R}^2$

$$g = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$$

Geometry (8)

Given any $P, Q \in S^2$, $\underline{v_P}$ tangent to S^2 at P, $\underline{v_Q}$ tangent to S^2 at Q
$\exists\, \varphi \in \mathrm{Isom}^+(S^2)$ with $\varphi(P) = Q$, $d\varphi_P(\underline{v_P}) = \underline{v_Q}$

Proof

We have already seen that $\exists\, \varphi_1$ with $\varphi_1(P) = Q$. If $d\varphi_1(\underline{v_P}) \neq \underline{v_Q}$ then compose φ_1 with a rotation R perpendicular to Q, and let $\varphi = R \circ \varphi_1$ $\square$

$\mathrm{Isom}^+(\mathbb{R}^2)$, $\mathrm{Isom}^+(S^2) \subset \mathcal{M}$. We will find another subgroup and the space it acts on.

Definition

$$G_1 = \left\{ A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \subset M_{2\times 2}(\mathbb{C}) \;\middle|\; \det A > 0 \right\}$$

Lemma

G_1 is a group.

Proof

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} \alpha a + \beta \bar{b} & \alpha b + \beta \bar{a} \\ \bar{\beta} a + \bar{\alpha}\bar{b} & \bar{\beta} b + \bar{\alpha}\bar{a} \end{pmatrix} = \begin{pmatrix} c & d \\ \bar{d} & \bar{c} \end{pmatrix}$$

$$\det A \det B = 0 \Rightarrow \det(AB) > 0$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} \bar{\alpha} & -\beta \\ -\bar{\beta} & \alpha \end{pmatrix}, \quad \det A^{-1} = \frac{1}{\det A} > 0 \qquad \square$$

Lemma

If $G_1 \subset \mathcal{M}$ is the group of Möbius Transformations associated with G_1, then any $\varphi \in G_1$ can be written as $r_\theta \circ \varphi_a$ where
$r_\theta(z) = e^{i\theta} z$, $\varphi_a(z) = \dfrac{z - a}{1 - \bar{a} z}$

3

<u>Proof</u> $\varphi(z) = \frac{az+\beta}{\bar{\beta}z+\bar{a}} = \frac{a}{\bar{a}} \frac{z+\beta/a}{(\overline{\beta/a})z+1}$

<u>Proposition</u>

If $\varphi \in G_1$, $\varphi(D) = D$

<u>Proof</u>

Obvious if $\varphi = r_\theta$. If $\varphi = \varphi_a$, $|z|=1$, then
$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$, $|z-a| = |\bar{z}||z-a| = |1-\bar{z}a| = |1-\bar{a}z|$

$\Rightarrow \cancel{\varphi_a(D) = D}$. $\varphi(S^1) = S^1$ $\blacksquare$

Therefore either $\varphi(D) = D$, or $\varphi(D) = \mathbb{C}^2 \setminus D$
But $\varphi(0) = \frac{\alpha}{1} = \frac{\beta}{\alpha} \Rightarrow |\varphi(0)| = |\frac{\beta}{\alpha}| < 1$ since $|\alpha|^2-|\beta|^2 > 0$

$\Rightarrow \varphi(0) \in D$, $\varphi(D) = D$

Geometry 9

We would like a Riemannian metric g on D so that $G \subset \mathrm{Isom}(D, g)$.
If so, g invariant under $r_\theta \iff g = f(r)(dx^2 + dy^2)$
$\varphi_a(a) = 0$, so if g is invariant under φ_a, then
$\langle \underline{v}, \underline{w} \rangle_{g(a)} = \langle D\varphi_a|_a \underline{v}, D\varphi_a|_a \underline{w} \rangle_{g(0)}$

$\varphi_a(z)$ is complex differentiable : $\varphi_a'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}$
$\varphi_a'(a) = \frac{1}{1-|a|^2}$

So $\langle \underline{v}, \underline{w} \rangle_{g(a)} = \langle \frac{1}{1-|a|^2} \underline{v}, \frac{1}{1-|a|^2} \underline{w} \rangle_{g(0)} = f(0) \frac{1}{(1-|a|^2)^2} \langle \underline{v}, \underline{w} \rangle_{Euc}$

We take $g_\rho = \frac{4(dx^2 + dy^2)}{(1-r^2)^2}$, the Poincaré metric on D.

<u>Proposition</u>

$G \subset \mathrm{Isom}(D, g_\rho)$

<u>Proof</u>

We must show that for $\varphi \in G$, $\varphi(b) = a$:
$\langle \underline{v}, \underline{w} \rangle g_\rho(b) = \langle D\varphi|_b \underline{v}, D\varphi|_b \underline{w} \rangle_{g_\rho(a)}$
i.e. $\varphi^* g_\rho(a) = g_\rho(b)$

Observe: $\varphi_a \circ \varphi(b) = \varphi_a(a) = 0$
$\Rightarrow \varphi_a \circ \varphi = r_\theta \circ \varphi_b$, $\quad \varphi^* \circ \varphi_a^* = \varphi_b^* \circ r_\theta^*$
$\varphi^*(\varphi_a^*(g_\rho(0))) = \varphi_b^*(r_\theta^*(g_\rho(0)))$
$= \varphi^*(g_\rho(a)) = \varphi_b^*(g_\rho(0)) = g_\rho(b)$ $\quad \square$

Hyperbolic Lines and Distances

What is the shortest path between $0, a \in D$? In polar coordinates:
$$g_\rho = \frac{4}{(1-r^2)^2}(dx^2 + dy^2) = \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\theta^2)$$

Proposition

The shortest path from 0 to a is along a line segment.

Proof

Let $\gamma \in P(0,a)$. $L(\gamma) = \int_0^1 \|\gamma'(t)\| dt = \int_0^1 \sqrt{\frac{4(r'^2 + r^2\theta'^2)}{(1-r^2)^2}}$
$$\geq \int_0^1 \frac{2r'(t)}{1-r(t)^2} dt$$
with equality $\Leftrightarrow \theta' \equiv 0$, $r'(t) > 0$
Then, $L(\gamma) = \int_0^{|a|} \frac{2dr}{1-r^2} = 2 \operatorname{artanh} |a|$

Let $\rho(z_1, z_2)$ be the distance from z_1 to z_2 in D, with respect to g_ρ.
$\rho(0, a) = 2 \operatorname{artanh} |a|$

Definition

A hyperbolic line L is $C \cap D$, where C is a Euclidean line or circle which is perpendicular to S^1 where $C \cap S^1$, and $C \cap S^1$ is two points.
For example, Euclidean lines through 0 are hyperbolic lines.

Proposition

Suppose L is a hyperbolic line and $\varphi \in G$. Then $\varphi(L)$ is also a hyperbolic line.

Proof

$\varphi \in M$ so φ preserves {Euclidean lines and circles}

Geometry (9)

$\therefore \varphi(L)$ is a Euclidean line or circle.
Also, since φ is holomorphic with $\varphi'(z) \neq 0$, φ preserves angles so the angle between $\varphi(L)$ and S^1 is $90°$ because the angle between L and S^1 is $90°$.

Corollary.

If $a, b \in D$, then the shortest path between them is a hyperbolic line segment.

Proof:

φ_a is an isometry, so if γ is the shortest path from $\varphi_a(a) = 0$ to $\varphi_a(b)$, then $\varphi_a^{-1} \circ \gamma$ is the shortest path from a to b. The shortest path from 0 to $\varphi_a(b)$ is a hyperbolic line segment, therefore so is $\varphi_a^{-1} \circ \gamma$ $\square$

Proposition

If $P, Q \in S^1$, $P \neq Q$, then there is a unique hyperbolic line between P and Q.

Proof:

We are looking for a circle tangent to OP at P and OQ at Q. Taking Euclidean perpendiculars to OP at P, OQ at Q, we take the centre of our circle to be their intersection point X. If they do not intersect, then PQ itself is a hyperbolic line. $\square$

Then, a hyperbolic line through O is a Euclidean line, since if P, Q, O are not collinear, the common tangent circle to P and Q does not pass through O.

22/02/12 Geometry (10)

C) Lines, Angles, and Isometries

Recall 1) If $a \in D$, $\exists\, \varphi_a \in G \subset \mathrm{Isom}(D, g_D)$ with $\varphi_a(a) = 0$
2) Elements of G preserve hyperbolic lines
3) Hyperbolic lines through 0 are Euclidean lines in D.

1) Two hyperbolic lines intersect in at most one point.
2) There is a unique hyperbolic line through any 2 points.
3) If L is a line, and $p \notin L$, there are infinitely many lines through p disjoint from L. (n.b. the ones tangent to L are parallel. Others are ultraparallel.)
4) If L is a line, $\exists\, \varphi \in G$ with $\varphi(L)$ = real axis.
$\Leftrightarrow$ If L_1, L_2 are lines, $\exists\, \varphi \in G$ with $\varphi(L_1) = L_2$
5) If $p \in D$, $\underline{v}$ is a vector at p, there is a unique hyperbolic line through p tangent to ~~$\underline{v}$~~ $\underline{v}$.

Theorem Suppose φ is a Riemannian isometry of (D, g_D). Then

orientation preserving → either $\varphi \in G$, or $\bar{\varphi} \in G$ (where $\bar{\varphi}(z) = \varphi(\bar{z})$)
$\varphi = \varphi_1 \circ c$, $\varphi_1 \in G$, $c(z) = \bar{z}$ ← orientation reversing

Lemma
Suppose $\varphi \in \mathrm{Isom}(D, g_D)$ with $\varphi(0) = 0$ and $D\varphi|_0 = id$
Then $\varphi = id$

Proof $\varphi \in \mathrm{Isom}(D, g_D) \Rightarrow \varphi$ preserves shortest paths
$\Rightarrow$ If L is a line, $\varphi(L)$ is a line
$\Rightarrow$ If $L_{\underline{v}}$ is a line through 0 tangent to $\underline{v}$, then $\varphi(L_{\underline{v}})$ is a line tangent to $D\varphi(\underline{v}) = \underline{v}$
$\Rightarrow \varphi(L_{\underline{v}}) = L_{\underline{v}}$ $\quad$ $\therefore\ \varphi$ preserves distance on $L_{\underline{v}}$ hyperbolic distance
$\Rightarrow$ If $\underline{x} \in L_{\underline{v}}$, $\rho(x, 0) = \rho(\varphi(x), \varphi(0)) = \rho(\varphi(x), 0)$
$\Rightarrow$ ~~$\varphi(\underline{x}) = \underline{x}$ or $\varphi(x$~~ $\varphi(x) = x$ or $\varphi(x) = -x$
Then, since $D\varphi|_0 = id$, $\varphi(x) = x$

Proof of Theorem Suppose $\varphi \in \mathrm{Isom}(D, g_D)$
$\varphi(0) = a$. Let $\varphi_1 = \varphi_a \circ \varphi \Rightarrow \varphi_1(0) = 0$
$\langle D\varphi_1 \underline{v}, D\varphi_1 \underline{w} \rangle_{g_D(0)} = \langle \underline{v}, \underline{w} \rangle_{g_D(0)}$, $g_D(0) = 4 \times$ Euclidean Metric
$\Rightarrow D\varphi_1 \in O(2)$. So either $D\varphi_1 = r_\theta$ or $D\varphi_1 = r_\theta \circ c$
$\varphi_2 = r_{-\theta} \circ \varphi_1$ $\qquad$ $\varphi_2 = c \circ r_{-\theta} \circ \varphi_1$

$$\left.\begin{aligned} D\varphi_2 &= D r_{-\theta} \circ D\varphi_1 \text{ or } D(c \circ r_{-\theta}) \circ D\varphi_1 \\ &= r_{-\theta} \circ D\varphi_1 \text{ or } c \circ r_{-\theta} \circ D\varphi_1 \end{aligned}\right\} = id$$

$\Rightarrow \varphi_2 = id$ by our lemma

$\Rightarrow \varphi = \varphi_a^{-1} \circ r_\theta$ or $\varphi_a^{-1} \circ r_\theta \circ c$, and $\varphi_a^{-1} \circ r_\theta \in G$ $\square$

Reflections If L is a hyperbolic line, reflection in L, $R_L : D \to D$

If L = real axis, $R_L(z) = \bar{z}$ (Euclidean reflection)

~~If~~ In general, pick $\varphi \in G$ with $\varphi(L)$ = real axis. $R_L = \varphi^{-1} \circ c \circ \varphi$

(We have to check that this is well defined and doesn't depend on the choice of φ

Angles

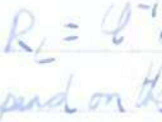
Proposition If $\underline{v}, \underline{w}$ are vectors at a point $p \in D$

$\angle_{g_{P(p)}} \underline{v}, \underline{w} = \angle_{Eucl} \underline{v}, \underline{w}$

Proof $\langle \underline{v}, \underline{w} \rangle_{gP} = \frac{4}{(1-r^2)^2} \langle \underline{v}, \underline{w} \rangle_{Eucl} = k^2 \langle \underline{v}, \underline{w} \rangle_{Eucl}$

$$\cos\theta_{gP} = \frac{\langle \underline{v}, \underline{w} \rangle_{gP}}{\sqrt{\langle \underline{v}, \underline{v} \rangle_{gP} \langle \underline{w}, \underline{w} \rangle_{gP}}} = \frac{k^2 \langle \underline{v}, \underline{w} \rangle_{Eucl}}{\sqrt{k^2 \langle \underline{v}, \underline{v} \rangle_{Eucl} k^2 \langle \underline{w}, \underline{w} \rangle_{Eucl}}} = \frac{\langle \underline{v}, \underline{w} \rangle_{Eucl}}{\sqrt{\langle \underline{v}, \underline{v} \rangle_{Eucl} \langle \underline{w}, \underline{w} \rangle_{Eucl}}} = \cos\theta_{E}$$

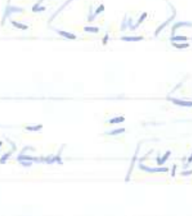
8. Hyperbolic Triangles

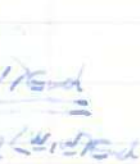


A. Motivation

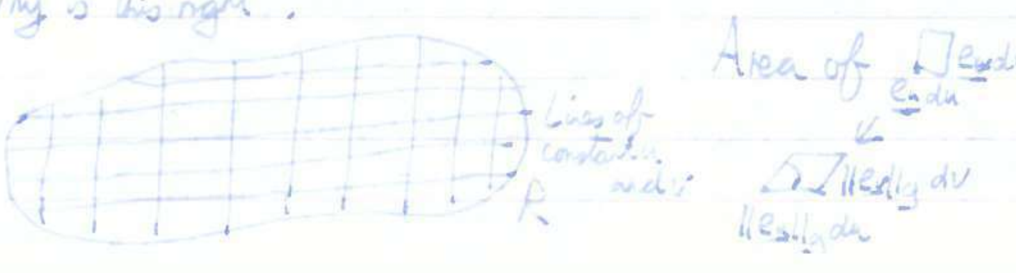
An ideal hyperbolic triangle has vertices on S^1. Edges are hyperbolic lines.
circles are tangent $\Rightarrow$ angle between sides of triangle is 0.
Perturb this to an 'honest' hyperbolic triangle with small angles.

Theorem Suppose $\triangle ABC$ is a hyperbolic triangle. Then $\pi - \alpha - \beta - \gamma = \text{Area}_{\triangle ABC}$
In particular, $\alpha + \beta + \gamma < \pi$.





B. Area Suppose G is a Riemannian Metric on $U \subset \mathbb{R}^2$ and
$R \subset U$. Then we define $\text{Area}(R) = \int_R \sqrt{EG - F^2}\, dA = \int_R \sqrt{\det(g)}\, dA$
Why is this right?



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Geometry (10)

Area of the parallelogram is $\|\underline{e_u}\| \|\underline{e_v}\| \sin\theta \, du \, dv$

$$\text{Area}^2 = \langle \underline{e_u}, \underline{e_u} \rangle_g \langle \underline{e_v}, \underline{e_v} \rangle_g \sin^2\theta$$
$$= \langle \underline{e_u}, \underline{e_u} \rangle_g \langle \underline{e_v}, \underline{e_v} \rangle_g (1-\cos^2\theta)$$
$$= \langle \underline{e_u}, \underline{e_u} \rangle_g \langle \underline{e_v}, \underline{e_v} \rangle_g - \langle \underline{e_u}, \underline{e_v} \rangle_g^2 = EG - F^2$$

$\Rightarrow$ Area element $= \sqrt{EG-F^2} \, du \, dv$

<u>Proposition</u> Suppose $\varphi : U_1 \to U_2$ is a diffeomorphism, and g is a Riemannian metric on U_2. If $R \subset U_1$, then

$$\text{Area}_{\varphi^*(g)}(R) = \text{Area}_g(\varphi(R))$$

<u>Proof</u> $\langle \underline{v}, \underline{w} \rangle_{\varphi^*(g)} = \langle D\varphi \underline{v}, D\varphi \underline{w} \rangle_g = (D\varphi \underline{v})^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} D\varphi \underline{w}$

$\Rightarrow \varphi^*(g)$ has matrix $D\varphi^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} D\varphi$

$\Rightarrow \det(\varphi^*(g)) = \det(D\varphi)^2 \det(g)$

$$\text{Area}_{\varphi^*(g)}(R) = \int_R \sqrt{\det(\varphi_*(g))} = \int_R |\det D\varphi| \sqrt{\det g}$$

$\llcorner$ change of variable

$$= \int_{\varphi(R)} \sqrt{\det g} = \text{Area}_g(\varphi(R))$$

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Geometry (11)

C) Upper half-plane model

$z = x + iy$, $H = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

$\varphi(z) = \frac{z-i}{z+i}$ $\quad \varphi(\mathbb{R}) = S^1$, $\varphi(i) = 0 \Rightarrow \varphi(H) = D$

Define : g_H = Poincaré metric on $H = \varphi^*(g_D)$

$\Rightarrow (H, g_H) \cong (D, g_D)$ (isometric)

What is g_H? $\qquad \theta = \operatorname{Arg} \varphi'(z)$

$$\langle \underline{v}, \underline{w} \rangle_{g_H} = \langle D\varphi\, \underline{v}, D\varphi\, \underline{w} \rangle_{g_D} = \langle |\varphi'| r_\theta \underline{v}, |\varphi'| r_\theta \underline{w} \rangle_{g_D}$$

$$= |\varphi'(z)|^2 \langle r_\theta \underline{v}, r_\theta \underline{w} \rangle_{g_D} = |\varphi'(z)|^2 \frac{4}{(1-r^2)^2} \langle \underline{v}, \underline{w} \rangle_{Eucl}$$

$\varphi'(z) = \frac{2i}{(z+i)^2}$, $\quad r = \left| \frac{z-i}{z+i} \right| \Rightarrow \frac{1}{1-r^2} = \frac{1}{1 - \frac{x^2+(y-1)^2}{x^2+(y+1)^2}}$

$$= \frac{x^2+(y+1)^2}{x^2+(y+1)^2-(x^2+(y-1)^2)} = \frac{|z+i|^2}{4y}$$

$$\langle \underline{v}, \underline{w} \rangle_{g_H} = \left(\frac{2}{|z+i|^2} \right)^2 \cdot 4 \left(\frac{|z+i|^2}{4y} \right)^2 \langle \underline{v}, \underline{w} \rangle_{Eucl}$$

$$g_H = \frac{1}{y^2}(dx^2 + dy^2)$$

Geodesics in H are Euclidean lines and circles perpendicular to $\mathbb{R}$ (as φ preserves {lines + circles} and angles)

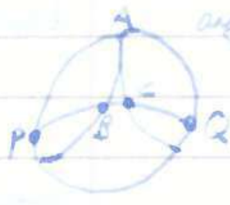
$\operatorname{Isom}^+(H) = PSL_2(\mathbb{R}) \subset \mathcal{M} = PSL_2(\mathbb{C})$

Angles in H are the same as Euclidean angles.

D) Angle Defect Theorem

<u>Theorem</u> Suppose $\triangle ABC$ is a hyperbolic triangle with interior angles α, β, γ. Then $\operatorname{Area}(\triangle ABC) = \pi - \alpha - \beta - \gamma$.

(we can work with either H or D, since they are isometric.



<u>Lemma:</u>

The theorem holds if $A \in S_1$ ($\Leftrightarrow \alpha = 0$)

Work in H. After applying an element of $\operatorname{Isom}^+(H) = PSL_2(\mathbb{R})$, we can assume $A = \infty$, $P = -1$, $Q = +1$.

$$\operatorname{Area} = \int_R |\det g|\, dA = \int_R \frac{dx\, dy}{y^2}$$

$R : \; -\cos\beta \leq x \leq \cos\gamma$

$\quad \sqrt{1-x^2} \leq y \leq \infty$

$$= \int_{-\cos\beta}^{\cos\gamma} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2}\, dx$$

$$= \int_{-\cos\beta}^{\cos\gamma} \frac{dx}{\sqrt{1-x^2}} = \left[\cos^{-1} x \right]_{\cos\gamma}^{-\cos\beta}$$

$$= -\gamma + \pi - \beta = \pi - \beta - \gamma$$

ABC has angles α, β, γ

Proof of Theorem Work in D.

$$\text{Area}(ABC) = \text{Area}(ABP) - \text{Area}(BCP)$$
$$= \pi - (\beta + \beta') - \alpha - (\pi - (\beta' + \pi - \gamma))$$
$$= \pi - \beta - \beta' - \alpha - (-\beta' + \gamma) = \pi - \alpha - \beta - \gamma \quad \square$$

9) Hyperboloid Model + Hyperbolic Trig

Spherical

$V = \mathbb{R}^3 \quad \langle \underline{v}, \underline{w} \rangle_E = \sum_{i=1}^{3} v_i w_i$

$S = \{ x \in \mathbb{R}^3 \mid \|x\|_E^2 = 1 \}$

$\pi : S^1 \setminus N \to \mathbb{C}$ (Stereographic Projection)

$\pi(x, y, z) = \frac{x + iy}{1 - z}$

$\pi^{-1}(w) = \left(2\frac{\text{Re}\, w}{1+|w|^2}, 2\frac{\text{Im}\, w}{1+|w|^2}, \frac{|w|^2 - 1}{1+|w|^2} \right)$

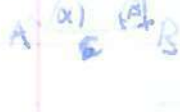
$(\pi^{-1})^*(g_E) = g_{\text{spherical}}$

$g_{\text{sph}} = \frac{4}{(1+u^2+v^2)^2}(du^2 + dv^2)$

$\text{Isom}(S^2, g_{\text{sph}}) \overset{\text{preserves } \langle \cdot,\cdot \rangle_E}{=} O(3)$

H a plane through O

$\text{Geodesics} = \{ S^2 \cap H \}$



$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}$

$\sin a \sin b \cos \gamma = \cos c - \cos a \cos b$

$\underline{A} \cdot \underline{B} = \cos c$

$\underline{A} \times \underline{B} = \sin c \; \underline{n}_c$

Hyperbolic

$V = \mathbb{R}^3, \; \langle \underline{v}, \underline{w} \rangle = v_1 w_1 + v_2 w_2 - v_3 w_3$
$= \underline{v}^T L \underline{w}$ with $L = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$

$S = \{ x \in \mathbb{R}^3 \mid \langle x, x \rangle_L = -1, \; x_3 > 0 \}$

$x^2 + y^2 - z^2 = -1$

$N = (0, 0, -1)$

$\pi : S \to \mathbb{C} \; (D)$

$\pi(x, y, z) = \frac{x + iy}{1 + z} = w, \; |w| < 1 \qquad z^2 - x^2 - y^2 = 1, \; z^2 > x^2 + y^2$

$\pi^{-1}(w) = \left(2\frac{\text{Re}\, w}{1-|w|^2}, 2\frac{\text{Im}\, w}{1-|w|^2}, \frac{1+|w|^2}{1-|w|^2} \right)$

$(\pi^{-1})^*(g_L) = \frac{4}{(1-u^2-v^2)^2}(du^2 + dv^2), \; w = u + iv$
$= g_P$ on D

$\text{Isom}(D, g_L) \overset{\text{preserves } \langle \cdot,\cdot \rangle_L}{=} O(2,1)$
$= \{ O \in M_{3\times3}(\mathbb{R}) \mid O^T L O = L \}$

Why is $O^+(2,1) \cong PSL_2(\mathbb{R})$?

$\text{Geodesics} = \{ S \cap H \mid H \text{ a plane through the origin} \}$

Proof: After an isometry, we can assume $p \in$ Geodesic $= (0,0,1)$, $0 \in D$. Geodesics through 0 are straight lines.

$\frac{\sinh \alpha}{\sinh a} = \frac{\sinh \beta}{\sinh b} = \frac{\sinh \gamma}{\sinh c}$

$\sinh a \sinh b \cos \gamma = \cosh a \cosh b - \cosh c$

$\langle \underline{A}, \underline{B} \rangle_L = \cosh c$

$\underline{A} \times_L \underline{B} = \sinh c \; \underline{n}_c$

How to define $\times_L$?

$\alpha(\underline{v}) = (v_1, v_2, iv_3), \; \langle \underline{v}, \underline{w} \rangle = \langle \alpha(\underline{v}), \alpha(\underline{w}) \rangle_E \qquad \underline{v} \times_L \underline{w} = \alpha^{-1}(\alpha(\underline{v}) \times \alpha(\underline{w}))$

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Geometry (12)

10) Embedded Surfaces in $\mathbb{R}^3$

Recall that a parametrised surface is $S : U \to \mathbb{R}^3$, $U \subseteq \mathbb{R}^2$, S, DS injective

Definition: An embedded surface in $\mathbb{R}^3$ is $\Sigma \subset \mathbb{R}^3$ such that for every $p \in \Sigma$, we can find N_p open in $\mathbb{R}^3$, $p \in N_p$,
and a parametrised surface $S_p : U_{p \subset \mathbb{R}^2} \to \mathbb{R}^3$ so that
$\operatorname{im} S_p = \Sigma \cap N_p = V_p$

Example (1)

$\Sigma = \{ (x, y, z) \mid x^2 + y^2 = 1 \}$. The S_p are called charts for Σ.

$S_1 : \left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right) \times \mathbb{R} \to \mathbb{R}^3$, $(\theta, z) \mapsto (\cos\theta, \sin\theta, z)$

$S_2 : \left(\frac{\pi}{4}, \frac{7\pi}{4}\right) \times \mathbb{R} \to \mathbb{R}^3$, $(\theta, z) \mapsto (\cos\theta, \sin\theta, z)$

Example (2)

$\Sigma = S^2$, $S_1 : \mathbb{R}^2 \to \mathbb{R}^3$, $S_1 = \pi_N^{-1} : (X, Y) \mapsto \left(\frac{2X}{\alpha}, \frac{2Y}{\alpha}, \frac{X^2+Y^2-1}{\alpha}\right)$

$\alpha = X^2 + Y^2 + 1$

$S_2 = \pi_S^{-1} (X, Y) \to \left(\frac{2X}{\alpha}, \frac{2Y}{\alpha}, \frac{1 - X^2 - Y^2}{\alpha}\right)$

$U_i' = S_i^{-1}(V_1 \cap V_2)$

$\varphi : U_1' \to U_2'$, $\varphi = S_2^{-1} \circ S_1$

Lemma

φ is a diffeomorphism.

Proof: φ is bijective by construction. Given $p \in U'$, we must show that φ is differentiable at p. Pick a hyperplane $H \subset \mathbb{R}^3$ with
$H \cap \operatorname{im}(DS_1|_p)^\perp = 0$, $H \cap \operatorname{im}(DS_2|_p)^\perp = 0$

Let $\pi : \mathbb{R}^3 \to H$ be orthogonal projection.

$F_i = \pi \circ S_i$. Then $DF_i|_p = D\pi \circ DS_i|_p = \pi \circ DS_i|_p$

$\Rightarrow DF_i|_p$ is an isomorphism.

$\Rightarrow$ (Inverse function Theorem) F_i is invertible near p and F_i^{-1} is differentiable. So observe that $\varphi = S_2^{-1} \circ S_1 = F_2^{-1} \circ F_1$ is differentiable near p, and $D\varphi|_p = DF_2^{-1} \circ DF_1|_p$ is an isomorphism.

$\Rightarrow \varphi$ is a diffeomorphism.

$S_1 = S_2 \circ \varphi$

Corollary

$\operatorname{Im} DS_1|_p = \operatorname{Im} DS_2|_{\varphi(p)}$

Proof: $DS_1|_p = DS_2|_{\varphi(p)} \circ D\varphi|_p$ $\square$

<u>Definition</u> If φ is an embedded surface, and $p \in \Sigma$, we define
$T_p \Sigma = \operatorname{Im} DS|_{p'}$, where S is any chart with $S(p') = p$
$T_p \Sigma$ = Tangent space to Σ at p.
e.g. $\Sigma = S^2$, $T_{\underline{x}} \Sigma = \underline{x}^{\perp}$

Suppose $f : \Sigma \to \mathbb{R}^n$
$\qquad DF|_p : T_p\Sigma \to \mathbb{R}^n \quad (= T_{F(p)}\mathbb{R}^n)$
We say F is differentiable at p if $F \circ S$ is differentiable at p'
where S is a chart with $S(p') = p$
$DF = D(F \circ S) \circ (DS|_{p'})^{-1}$
Check that this is well defined:
If $S_1 = S_2 \circ \varphi \qquad S_2 = S_1 \circ \varphi^{-1}, \; DS_2 = DS_1 \circ D\varphi^{-1}$
$DF = D(F \circ S_1) \circ (DS_1)^{-1} = D(F \circ S_2 \circ \varphi) \circ (DS_1)^{-1}$
$\quad = D(F \circ S_2) \circ [D\varphi \circ (DS_1)^{-1}] = D(F \circ S_2) \circ (DS_2)^{-1}$

3) <u>First Fundamental Form</u> (FFF)
If $\Sigma \subset \mathbb{R}^3$ is an embedded surface, $p \in \Sigma$, the FFF is a bilinear form on $T_p\Sigma$
$B_1(\underline{v}, \underline{w}) = \langle \underline{v}, \underline{w} \rangle \rightarrow$ Euclidean inner product on $\mathbb{R}^3$
In a chart, it is often convenient to think of $S^*(B_1)$
to $T_{p'} U_{\subset \mathbb{R}^2}$. This is the Riemannian metric on U induced by S.
With respect to the basis $\langle \underline{e}_u, \underline{e}_v \rangle$ of $T_{p'}U$, this is given by the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} S_u \cdot S_u & S_u \cdot S_v \\ S_u \cdot S_v & S_v \cdot S_v \end{bmatrix}$

4) <u>Second Fundamental Form</u>
<u>Definition</u> We will say Σ is orientable if there is a continuous map $\underline{N} : \Sigma \to S^2$ such that $\underline{N}(p)$ is perpendicular to $T_p\Sigma$.

Non-Orientable.

We will assume that Σ is orientable.

<u>Lemma</u> If $\underline{v} \in T_p\Sigma$, $D\underline{N}(\underline{v}) \in T_p\Sigma$.
<u>Proof</u> $\underline{N} \cdot \underline{N} \equiv 1$, $D(\underline{N} \cdot \underline{N})(\underline{v}) = 0$
$\quad D\underline{N}(\underline{v}) \cdot \underline{N} + \underline{N} \cdot D\underline{N}(\underline{v}) = 2\underline{N} \cdot D\underline{N}(\underline{v}) = 0$
$\quad \therefore D\underline{N}(\underline{v}) \in \underline{N}_p^{\perp} = T_p\Sigma$

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Geometry (12)

__Definition__ The second fundamental form is the bilinear form on $T_p\Sigma$ defined by $B_2(\underline{v}, \underline{w}) = \langle DN(\underline{v}), \underline{w} \rangle$ ← Euclidean inner product

In a chart, with respect to $\langle e_u, e_v \rangle$

$$B_2 = \begin{bmatrix} \underline{N}_u \cdot \underline{S}_u & \underline{N}_u \cdot \underline{S}_v \\ \underline{N}_v \cdot \underline{S}_u & \underline{N}_v \cdot \underline{S}_v \end{bmatrix}$$ where $\underline{N}_u$ is the partial derivative of $N \circ S$ wrt u.

__Remark__ $\underline{N} = \dfrac{\underline{S}_u \times \underline{S}_v}{\|\underline{S}_u \times \underline{S}_v\|}$

To compute, observe that $S^*(B_2) = -\begin{bmatrix} \underline{N} \cdot \underline{S}_{uu} & \underline{N} \cdot \underline{S}_{vu} \\ \underline{N} \cdot \underline{S}_{uv} & \underline{N} \cdot \underline{S}_{vv} \end{bmatrix}$

__Proof__

$\underline{N} \cdot \underline{S}_u = 0$, so $(\underline{N} \cdot \underline{S}_u)_u = \underline{N}_u \cdot \underline{S}_u + \underline{N} \cdot \underline{S}_{uu} = 0$

$\Rightarrow \underline{N}_u \cdot \underline{S}_u = -\underline{N} \cdot \underline{S}_{uu}$

and similarly for other entries

__Corollary__

B_2 is a symmetric form, $B_2(\underline{v}, \underline{w}) = B_2(\underline{w}, \underline{v})$

__Proof__

$\underline{N} \cdot \underline{S}_{uv} = \underline{N} \cdot \underline{S}_{vu}$ $\square$

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Geometry ⑬

Proposition

$B_2(\underline{w}, \underline{w}) = -f_{\underline{w}}''(0)$

Proof

Parametrise S as $S(s,t) = \underline{p} + t\underline{w} + s\underline{w_1} + f(t,s)\underline{N}$

where $\underline{w_1} \in T_p\Sigma$, $\underline{w_1} \perp \underline{w}$, $\|\underline{w_1}\| = 1$. $\Rightarrow f_w(t) = f(t,0)$

Then $S^*B_2(e_t, e_t) = -N \cdot f_{tt}(0) = -\underline{N} \cdot f_{\underline{w}}''(0)$

$S^*(B_2(e_t, e_t)) = B_2(DS(e_t), DS(e_t)) = B_2(\underline{S_t}, \underline{S_t})$

$\underline{S_t} = \underline{w} + f_t(0)\underline{N} = \underline{w}$ since Σ is tangent to span$(\underline{w}, \underline{w_1})$ at p

So $S^*(B_2(e_t, e_t)) = B_2(\underline{w}, \underline{w})$ $\square$

Definition If $\underline{w} \in T_p\Sigma$, $\|\underline{w}\| = 1$, let

$K_{\underline{w}}(p) = -f_{\underline{w}}''(0)$ = curvature of $\Sigma \cap H_{\underline{w}}$.

Remark

$K_{\underline{w}}(p) = \frac{1}{r}$ where r is the radius of the circle in H_w tangent to 2^{nd} order to γ_w at p.

Proposition Either $K_w(p)$ is constant $\forall \underline{w} \in T_p\Sigma$ with $\|\underline{w}\| = 1$ or there are orthogonal vectors $\underline{w}_{max}$, $\underline{w}_{min} \in T_p\Sigma$, $\|\underline{w}_{max}\| = \|\underline{w}_{min}\| = 1$ so that $K_p(\underline{w}_{min})$ is minimal, $K_p(\underline{w}_{max})$ is maximal.

$K_{\underline{w}_{min}}(p)$ $K_{\underline{w}_{max}}(p)$

Proof

$K_{\underline{w}} = B_2(\underline{w}, \underline{w}) = \langle DN(\underline{w}), \underline{w} \rangle$

and DN is a self adjoint linear map.

~~Let~~ Either $dN = K\,Id$ (i.e. 2 eigenvalues the same),

or there are eigenvectors $\underline{w}_{max}$, $\underline{w}_{min}$ so that

$DN = \begin{bmatrix} K_{max} & 0 \\ 0 & K_{min} \end{bmatrix}$ perpendicular

with respect to this basis, $K_{max} > K_{min}$

$\Rightarrow B_2$ is maximised $\underline{w}_{max}$, minimised on $\underline{w}_{min}$ $\square$

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Geometry (14)

II) Curvature

$\Sigma \subset \mathbb{R}^3$ an embedded surface, $p \in \Sigma$, $DN : T_p\Sigma \to T_p\Sigma$

Definition

$K(p) = \det DN$ = Gaussian Curvature of Σ at p

$K : \Sigma \to \mathbb{R}$

E.g. $\Sigma = \mathbb{R}^2$, $DN \equiv 0 \Rightarrow K \equiv 0$

$\Sigma = S^2(r)$, $DN = \pm Id \Rightarrow K = \det DN = \frac{1}{r^2}$

$\Sigma = S^2$, $K = 1$

3 Ways of Thinking about K:

1) K measures the "local area distribution" of N near p.

If R is very small, $A(N(R)) \approx K(p)\,\text{Area}(R)$

Why? Let $\pi : \Sigma \to T_p\Sigma$ be orthogonal projection

Then $A(R) \approx A(\pi(R))$, $A(N(R)) \approx A(\pi(N(R)))$

By definition, $dN : T_p\Sigma \to T_p\Sigma$ distorts area by K

2) $K(p) = K_{max}(p) K_{min}(p)$, where K_{max}, K_{min} are eigenvalues of DN aka principle curvatures at p.

Why? $DN = \begin{bmatrix} K_{max} & 0 \\ 0 & K_{min} \end{bmatrix}$ with respect to the basis v_{max}, v_{min} of eigenvectors.

What is the sign of K? $K > 0$ if K_{max}, K_{min} have the same sign $\Leftrightarrow$ Σ is locally on one side of $T_p\Sigma$

$K < 0$ if K_{max}, K_{min} have opposite signs and Σ is on both sides of $T_p\Sigma$

e.g. $z = xy$ in $\mathbb{R}^3$

≈ saddle

3) In a chart $S: \mathcal{U} \to \Sigma$, $S(p') = p$

$K = \det(M_2) / \det(M_1)$

where $M_2 = \begin{bmatrix} k & l \\ l & m \end{bmatrix}$ represents $S^* B_2$

and $M_1 = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ represents $S^* B_1$ $\qquad K = \frac{km - l^2}{EG - F^2}$

Lemma Let V be a vector space, β_1, β_2 two bilinear forms on V. Suppose β_i is represented by matrices M_i and M_i', with respect to two different bases $\{\underline{e_i}\}$ and $\{\underline{e_i'}\}$.

Then $\det(M_2') / \det(M_1') = \det(M_2) / \det(M_1)$

Proof

$M_i' = A^T M_i A$ where A is the change of basis matrix

$\det M_i' = (\det A)^2 \det(M_i)$

$\det(M_2') / \det(M_1') = \dfrac{\det(A)^2 \det(M_2)}{\det(A)^2 \det(M_1)} = \det(M_2) / \det(M_1)$

We normalise $\underline{w}_{max}, \underline{w}_{min}$ so that $B(\underline{w}_{max}, \underline{w}_{max}) = B(\underline{w}_{min}, \underline{w}_{min}) = 1$

Then with respect to this basis $M_2' = \begin{bmatrix} K_{max} & 0 \\ 0 & K_{min} \end{bmatrix}$, $M_1' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

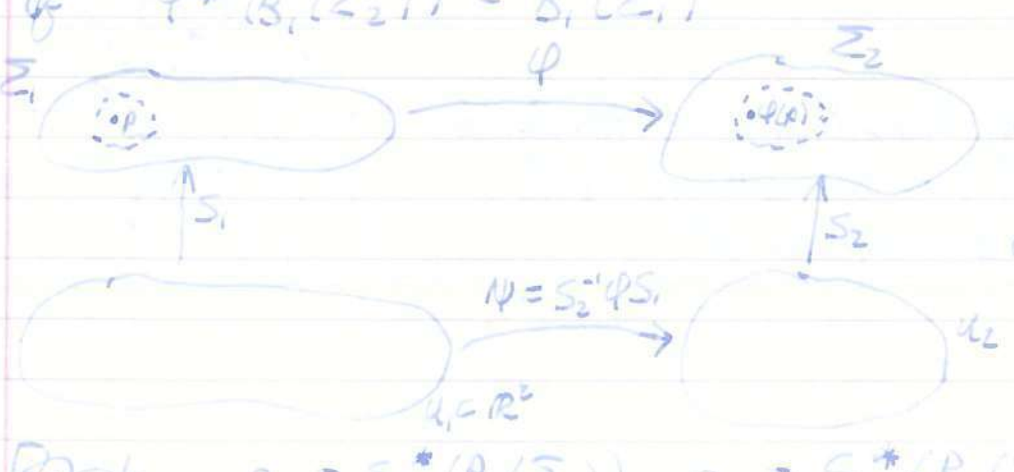
With respect to the basis $\{DS(\underline{e_1}), DS(\underline{e_2})\}$

$M_2 = \begin{bmatrix} k & l \\ l & m \end{bmatrix}$, $M_1 = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$

Then $\det(M_2) / \det(M_1) = \det(M_2') / \det(M_1') = \frac{K_{min} K_{max}}{1} = K$

B) Isometries and Curvature

Definition Suppose $\varphi : \Sigma_1 \to \Sigma_2$ (Σ_i an embedded surface) is a diffeomorphism. We say that φ is an isometry if $\varphi^*(B_1(\Sigma_2)) = B_1(\Sigma_1)$



φ is an isometry $\iff \psi : \mathcal{U}_1 \to \mathcal{U}_2$ is an isometry of Riemannian Manifolds

Example $g_1 = S_1^*(B_1(\Sigma_1))$, $g_2 = S_2^*(B_1(\Sigma_2))$

$\psi^*(S_2^*(B_1(\Sigma_2))) = S_1^*(\varphi^*((S_2^*)^{-1} S_2^* B_1(\Sigma_2))) = S_1^*(B_1(\Sigma_1))$

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E.g. $\Sigma_1 = (0, 2\pi) \times \mathbb{R} \rightarrow \Sigma_2 = \{(x,y,z) \mid x^2+y^2=1\}$
$\hat{\mathbb{R}}^2$

$\varphi(\theta, z) = (\cos\theta, \sin\theta, z)$. Then $\varphi^* B_1(\Sigma) = \begin{bmatrix} \varphi_\theta \cdot \varphi_\theta & \varphi_\theta \cdot \varphi_z \\ \varphi_z \cdot \varphi_\theta & \varphi_z \cdot \varphi_z \end{bmatrix} = I$
= Euclidean B_1 on Σ_1, φ is an isometry

<u>Theorem</u> (Theorema Egregia (Gauss))
If $\varphi : \Sigma_1 \rightarrow \Sigma_2$ is an isometry, then $K_{\Sigma_1} = K_{\Sigma_2} \circ \varphi$
$\Leftrightarrow$ K only depends on B_1, not on B_2!
<u>Idea of Proof</u>
1) Find a chart $S(r, \theta)$ near $p \in \Sigma$ so that $S^* B_1 = dr^2 + G(r,\theta)\, d\theta^2$
2) Prove that $K(p) = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}$
e.g. $\Sigma = \mathbb{R}^2$, $B_1 = dr^2 + r^2 d\theta^2$, $\sqrt{G} = r$, $K = -\frac{(r)_{rr}}{r} = 0$

12) Geodesics

A) Equations for Geodesics

<u>Setup</u>: $U \subseteq \mathbb{R}^2$ is a Riemannian manifold (i.e. U is the domain for a chart $S : U \rightarrow$
g = Riemannian metric on U ($g = S^* B_1$)
$x, y \in U$, $P_{(x,y)} = \{\gamma : [0,1] \rightarrow U \mid \gamma(0) = x, \gamma(1) = y, \gamma \text{ smooth}\}$
$L(\gamma) = \int_0^1 \sqrt{\langle \gamma', \gamma' \rangle_{g(\gamma(t))}}\, dt$ Length of γ
$E(\gamma) = \int_0^1 \langle \gamma', \gamma' \rangle_{g(\gamma(t))}\, dt$, Energy of γ

<u>Proposition</u> γ minimises E $\Leftrightarrow$ γ minimises L and $\|\gamma'\|_{g(\gamma(t))}$ is constant
<u>Proof</u> Recall that $(\int fg)^2 \leq \int f^2 \int g^2$ (Cauchy-Schwartz, $\langle f, g \rangle = \int fg$)
so
$L(\gamma)^2 = \left(\int \sqrt{\langle \gamma', \gamma' \rangle_g}\, dt\right)^2 \leq \int_0^1 \langle \gamma', \gamma' \rangle_g\, dt \int_0^1 1\, dt = E(\gamma)$
Equality $\Leftrightarrow$ $\langle \gamma', \gamma' \rangle \equiv c$

Suppose γ minimises L, $\|\gamma'\| = c$. Then $L(\gamma)^2 = E(\gamma)$
and $E(\gamma) = L(\gamma)^2 \leq L(\gamma_1)^2 \leq E(\gamma_1)$
So $E(\gamma)$ is minimal as well $\square$

If $\gamma \in P_{(x,y)}$, $T_\gamma P = \{\underline{v} : [0,1] \rightarrow \mathbb{R}^2 \mid \underline{v}(0) = \underline{v}(1) = 0, \underline{v} \text{ smooth}\}$

$\tilde{r}_\varepsilon = \tilde{r} + \varepsilon v$

Directional derivative: $D_{\underline{v}} \underline{E} = \lim_{\varepsilon \to 0} \frac{\underline{E}(\tilde{r} + \varepsilon \underline{v}) - \underline{E}(\tilde{r})}{\varepsilon}$

If $\tilde{r}$ minimises $\underline{E}$, we must have $D_{\underline{v}} \underline{E} = 0 \quad \forall \underline{v} \in T_{\tilde{r}} P$.

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A) Equations for Geodesics

$U \subset \mathbb{R}^2$ open, $g = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ a Riemannian metric

$r : [a,b] \to U$ a path in U, "Energy of r" $= E(r) = \int_a^b \langle r'(t), r'(t) \rangle_{g(r(t))}$

$V : [0,1] \to \mathbb{R}^2$ $\qquad \underline{v}$ a vector field on r

$D_{\underline{v}} E|_r = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(E(r + \varepsilon V) - E(r) \right) = \frac{d}{d\varepsilon} \left(E(r + \varepsilon V) \right) \Big|_{\varepsilon = 0}$

Definition: r is a geodesic if $D_V E|_r = 0$ for all V with $V(a) = V(b) = 0$ i.e. r is a critical point for E on $P_{(x,y)}$ $\quad x = r(a)$, $y = r(b)$

If r minimises $E|_{P(x,y)}$, it is a geodesic.

Theorem $r = (r_1(t), r_2(t))$ is a geodesic $\iff 2\frac{d}{dt}(E\dot{r}_1 + F\dot{r}_2) = E_u \dot{r}_1^2 + 2F_u \dot{r}_1 \dot{r}_2 + G_u \dot{r}_2^2$

$2\frac{d}{dt}\left(E(r(t))\dot{r}_1(t) + F(r(t))\dot{r}_2(t) \right)$

Similarly $2\frac{d}{dt}(F\dot{r}_1 + G\dot{r}_2) = E_v \dot{r}_1^2 + 2F_v \dot{r}_1 \dot{r}_2 + G_v \dot{r}_2^2$

Proof:

$D_V E|_r = \frac{d}{d\varepsilon} \left(\int_a^b \langle r' + \varepsilon V', r' + \varepsilon V' \rangle_{g(r(t))} dt \right) \Big|_{\varepsilon = 0}$

$= \frac{d}{d\varepsilon} \Big(\int_a^b E(r(t) + \varepsilon V)(\dot{r}_1 + \varepsilon \dot{V}_1)(\dot{r}_1 + \varepsilon \dot{V}_1)$ ①

$+ 2F(r(t) + \varepsilon V)(\dot{r}_1 + \varepsilon \dot{V}_1)(\dot{r}_2 + \varepsilon \dot{V}_2)$ ②

$+ G(r(t) + \varepsilon V)(\dot{r}_2 + \varepsilon \dot{V}_2)(\dot{r}_2 + \varepsilon \dot{V}_2) \; dt$ ③

① $= \int_a^b 2E(r(t))\dot{r}_1 \dot{v}_1 + E_u v_1 \dot{r}_1^2 + E_v v_2 \dot{r}_1^2 \, dt$

(Integrate by parts) $= 2v_1 E\dot{r}_1 \Big|_a^b - 2\int_a^b \frac{d}{dt}(E\dot{r}_1)v_1 \, dt + \int_a^b E_u v_1 \dot{r}_1^2 + E_v v_2 \dot{r}_1^2$

Doing a similar procedure on ②, ③, we obtain

$2(E\dot{r}_1 + F\dot{r}_2)v_1 + 2(F\dot{r}_1 + G\dot{r}_2)v_2 \Big|_a^b$ (a)

$-2\int_a^b \frac{d}{dt}(E\dot{r}_1 + F\dot{r}_2)v_1 + \frac{d}{dt}(F\dot{r}_1 + G\dot{r}_2)v_2 \, dt$ (b)

$+ \int_a^b (E_u \dot{r}_1^2 + 2F_u \dot{r}_1 \dot{r}_2 + G_u \dot{r}_2^2)v_1 + (E_v \dot{r}_1^2 + 2F_v \dot{r}_1 \dot{r}_2 + G_v \dot{r}_2^2)v_2$ (c)

The terms in a) are 0, since $v(a) = v(b) = 0$

In order for (b), (c) to vanish for all choices of v_1, v_2 we need the coefficients of v_1, v_2 in the integral to be 0.

These equations are the ones in our theorem statement.

Proposition Suppose that $S : U \to \mathbb{R}^3$ is a chart for Σ, $g = S^*(B_1)$, $\Gamma(t) = S(r(t))$. Then r is a geodesic $\iff \underset{\in \mathbb{R}^3 \text{ at } \Gamma(t)}{\Gamma''(t)} \perp T_{\Gamma(t)}\Sigma$

Proof. $\Gamma'(t) = dS(\gamma') = S_u \gamma_1' + S_v \gamma_2'$

$\Gamma''(t) \perp T_{\Gamma(t)}\Sigma \Leftrightarrow S_u \cdot \Gamma''(t) = 0,\ S_v \cdot \Gamma''(t) = 0$

Write $S_u \cdot \Gamma''(t) = (S_u \cdot \Gamma'(t))' - (S_u)' \cdot \Gamma'(t)$

$= (S_u \cdot (S_u \gamma_1' + S_v \gamma_2'))' - (S_{uu}\gamma_1' + S_{uv}\gamma_2') \cdot (S_u\gamma_1' + S_v\gamma_2')$

$E = S_u \cdot S_u$
$F = S_v \cdot S_u$
$G = S_v \cdot S_v$

$= (E\gamma_1' + F\gamma_2')' - (S_{uu} \cdot S_u \gamma_1'^2 + (S_{uv} \cdot S_u + S_{uu} \cdot S_v)\gamma_1'\gamma_2' + S_{uv} \cdot S_v \gamma_2'^2)$

$E_u = 2S_u \cdot S_{uu}$ etc

$= (E\gamma_1' + F\gamma_2')' - \frac{1}{2}(E_u \gamma_1'^2 + 2F_u \gamma_1'\gamma_2' + G_u \gamma_2'^2)$

So $S_u \cdot \Gamma''(t) = 0 \Leftrightarrow 1^{st}$ geodesic equation is satisfied

$S_v \cdot \Gamma''(t) = 0 \Leftrightarrow 2^{nd}$ equation is satisfied

B) Geodesic Polar Coordinates

$\subseteq \mathbb{R}^2$

Proposition Given $p \in U$, $\underline{v} \in T_pU$, there is a unique geodesic $\gamma_{\underline{v}}$ with $\gamma_{\underline{v}}(0) = p$, $\gamma_{\underline{v}}'(0) = \underline{v}$

Proof

If we expand out the Geodesic equations, we get $E\gamma_1'' + F\gamma_2'' = \alpha(u, v, \dot{\gamma}_1, \dot{\gamma}_2)$

$F\gamma_1'' + G\gamma_2'' = \beta(u, v, \dot{\gamma}_1, \dot{\gamma}_2)$

$\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ is invertible, $\gamma_1'' = a(u, v, \dot{\gamma}_1, \dot{\gamma}_2)$, $\gamma_2'' = b(u, v, \dot{\gamma}_1, \dot{\gamma}_2)$

i.e. the Geodesic equations form a 2^{nd} order ODE, so to determine a solution, we must specify $\gamma(0)$, $\gamma'(0)$.

(r, θ), polar coordinates $\qquad V_\theta = (\cos\theta, \sin\theta)$

Define a map $S : B(\varepsilon) \to U$ by $S(r, \theta) = \gamma_{rV_\theta} = \gamma_{V_\theta}(r)$

geodesic polar coordinates

Proposition

In these coordinates, $S^*(g) = dr^2 + \tilde{G}(r, \theta)\, d\theta^2$

Proof

(postponed for now)

C) Theorem Egregium

Suppose that $S : U \to \mathbb{R}^3$ is a chart for Σ.

$S^*(B_1) = dr^2 + G(r, \theta)\, d\theta^2$

$$K = \frac{-(\sqrt{G})_{rr}}{\sqrt{G}}$$

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Proof

$\underline{e} = \underline{S}_r$, $\underline{f} = \underline{S}_\theta / \sqrt{G}$, $\underline{n}$ = unit normal to Σ

These are functions of r, θ, and for each, this is an orthonormal basis for $\mathbb{R}^3$

Write $\begin{pmatrix} \underline{e}_r \\ \underline{f}_r \\ \underline{n}_r \end{pmatrix} = A \begin{pmatrix} \underline{e} \\ \underline{f} \\ \underline{n} \end{pmatrix}$

Claim 1 $A = -A^T$, i.e. $\begin{pmatrix} \underline{e}_r \\ \underline{f}_r \\ \underline{g}_r \end{pmatrix} = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix} \begin{pmatrix} \underline{e} \\ \underline{f} \\ \underline{g} \end{pmatrix}$ differentiate

Proof

The coefficient of $\underline{e}$ in $\underline{e}_r$ is $\underline{e} \cdot \underline{e}_r$, but $\underline{e} \cdot \underline{e} = 1$, $\underline{e} \cdot \underline{e}_r = 0$

Similarly, $\underline{e} \cdot \underline{f} = 0 \Rightarrow \underline{e}_r \cdot \underline{f} + \underline{e} \cdot \underline{f}_r = 0$

and similarly for the other elements.

Also $\begin{pmatrix} \underline{e}_\theta \\ \underline{f}_\theta \\ \underline{n}_\theta \end{pmatrix} = \begin{pmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{pmatrix} \begin{pmatrix} \underline{e} \\ \underline{f} \\ \underline{n} \end{pmatrix}$

$\Rightarrow \underline{n}_r = DN(\underline{S}_r) = -a_2 \underline{e} - a_3 \underline{f} = -a_2 \underline{S}_r - a_3 \frac{\underline{S}_\theta}{\sqrt{G}}$

$\underline{n}_\theta = DN(\underline{S}_\theta) = -b_2 \underline{e} - b_3 \underline{f} = -b_2 \underline{S}_r - b_3 \frac{\underline{S}_\theta}{\sqrt{G}}$

$K = \det DN = \det \begin{pmatrix} -a_2 & -b_2 \\ -\frac{a_3}{\sqrt{G}} & -\frac{b_3}{\sqrt{G}} \end{pmatrix} = \frac{1}{\sqrt{G}} (a_2 b_3 - a_3 b_2)$

Claim 2 $a_1 = 0$, $b_1 = (\sqrt{G})_r$

Proof

$a_1 = \underline{e}_r \cdot \underline{f} = \underline{S}_{rr} \cdot \underline{S}_\theta / \sqrt{G}$ $\qquad \underline{S}_r \cdot \underline{S}_r = 1 = E$

$\underline{S}_r \cdot \underline{S}_\theta = 0$, $\underline{S}_\theta \cdot \underline{S}_\theta = G$

$\Rightarrow \underline{S}_{r\theta} \cdot \underline{S}_r = 0$, $\underline{S}_{rr} \cdot \underline{S}_\theta + \underline{S}_r \cdot \underline{S}_{r\theta} = 0$

$a_1 = -\underline{S}_r \cdot \underline{S}_{r\theta} / \sqrt{G} = 0$

$b_1 = \underline{e}_\theta \cdot \underline{f} = \underline{S}_{r\theta} \cdot \underline{S}_\theta / \sqrt{G} = \frac{1}{2} \frac{G_r}{\sqrt{G}} = (\sqrt{G})_r$

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Geometry (16)

<u>Recall</u> Choose geodesic polar coordinates

$S^* g = dr^2 + G\,d\theta^2$

$\underline{e} = \underline{S}_r$, $\underline{f} = \underline{S}_\theta / \sqrt{G}$, $\underline{n}$ normal to $\Sigma \subset \mathbb{R}^3$ } our orthonormal basis

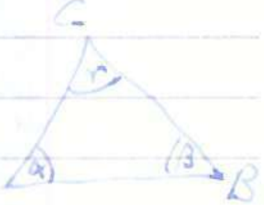
$$\begin{pmatrix} e_r \\ f_r \\ n_r \end{pmatrix} = A \begin{pmatrix} \underline{e} \\ \underline{f} \\ \underline{n} \end{pmatrix} \qquad \begin{pmatrix} e_\theta \\ f_\theta \\ n_\theta \end{pmatrix} = B \begin{pmatrix} \underline{e} \\ \underline{f} \\ \underline{n} \end{pmatrix} \quad \text{where } A = -A^T,\ B = -B^T \quad \Big| \; A = \{a_{ij}\},\ B = \{b_{ij}\}$$

$a_1 = 0$, $b_1 = (\sqrt{G})_r$, $K = \frac{1}{\sqrt{G}} (a_2 b_3 - a_3 b_2)$

<u>Step</u>: $a_2 b_3 - a_3 b_2 = e_r \cdot f_\theta - e_\theta \cdot f_r = (e_r \cdot f)_\theta - (e_\theta \cdot f)_r$
$= (a_1)_\theta - (b_1)_r = 0 - (\sqrt{G}_r)_r = -(\sqrt{G})_{rr}$

So $K = -\dfrac{(\sqrt{G})_{rr}}{\sqrt{G}}$ $\square$

15) Gauss-Bonnet Theorem and Abstract Surfaces



A) Gauss Bonnet for △s.

Setup: $U \subset \mathbb{R}^2$, g a Riemannian metric

$\triangle ABC \subset U$. A, B, C vertices, and the sides are geodesics.

Angle defect $\delta(\triangle ABC) = \alpha + \beta + \gamma - \pi$

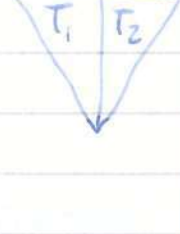
<u>Theorem</u> $\delta(\triangle ABC) = \int_{\triangle ABC} K\,dA$ ← with respect to g.

<u>Example</u> g the spherical metric, $K \equiv 1$, this becomes $\delta(\triangle ABC) = $ area

<u>Proof</u>

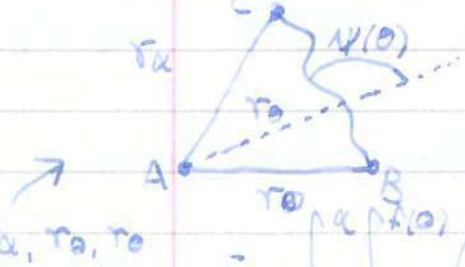
$T = T_1 \cup T_2$, $\delta(T) = \delta(T_1) + \delta(T_2)$

$\int_T K\,dA = \int_{T_1} K\,dA + \int_{T_2} K\,dA$



So to prove this for any T, chop T into small △s, each of which is contained in a geodesic polar chart. Now assume that we have geodesic polar coordinates centred at A. $g = dr^2 + G(r,\theta)\,d\theta^2$

Let γ_θ = geodesic from A with unit speed which makes an angle θ with $\overline{AB}$ $\quad \Gamma(\theta) = (f(\theta), \theta)$



$\gamma_\alpha, \gamma_\theta, \gamma_0$ are geodesics

$\psi(\theta) = \angle$ between $\overline{BC} = \Gamma(\theta)$ and γ_θ.

$g = \begin{bmatrix} 1 & 0 \\ 0 & G \end{bmatrix}$

Compute

$$\int_\triangle K\,dA = \int K \sqrt{\det g}$$

$$= \int_0^\alpha \int_0^{f(\theta)} \left(\frac{-(\sqrt{G})_{rr}}{\sqrt{G}} \right) \sqrt{G}\,dr\,d\theta = \int_0^\alpha \int_0^{f(\theta)} -(\sqrt{G})_{rr}\,dr\,d\theta$$

$$= \int_0^{\alpha} (-\sqrt{G})_r \Big|_0^{f(\theta)} d\theta = \int_0^{\alpha} (-\sqrt{G})_r \big|_{r=f(\theta)} + (\sqrt{G})_r \big|_{r=0} \, d\theta$$

<u>Lemma 1</u> $(-\sqrt{G})_r \big|_{r=f(\theta)} = \frac{d\Psi}{d\theta}$

<u>Lemma 2</u> $(\sqrt{G})_r \big|_{r=0} = 1$

$\Psi(0) = \pi - \beta, \ \Psi(\alpha) = \gamma$

$$= \int_0^{\alpha} \frac{d\Psi}{d\theta} + 1 \, d\theta = \Psi(\alpha) - \Psi(0) + \alpha = \alpha + \beta + \gamma - \pi = \delta(\triangle ABC)$$

$\square$

<u>Proof of Lemma 1</u> Let $S(\theta)$ = length of Γ between 0 and θ

$$= \int_0^{\theta} \|\Gamma'(t)\|_g \, dt = \int_0^{\theta} \sqrt{f'^2 + G} \, dt \qquad \Gamma'(t) = (f'(\theta), 1)$$

$$\frac{dS}{d\theta} = \sqrt{f'^2 + G} = h \qquad \frac{d\theta}{dS} = \frac{1}{h}$$

$$\frac{dF}{dS} = \frac{dF}{d\theta}\frac{d\theta}{dS} = \frac{1}{h}\frac{dF}{d\theta}$$

<u>Step 1</u>

If I parametrise Γ by arc-length (i.e. traverse at constant speed) then Γ will satisfy the Geodesic equation.

$$\frac{d}{dS}\left(E\frac{d\Gamma_1}{dS} + F\frac{d\Gamma_2}{dS}\right) = \frac{1}{2}\left[E_r\left(\frac{d\Gamma_1}{dS}\right)^2 + 2F_r\left(\frac{d\Gamma_1}{dS}\right)\left(\frac{d\Gamma_2}{dS}\right) + G_r\left(\frac{d\Gamma_2}{dS}\right)^2\right]$$

$$\frac{d}{dS}\left(\frac{df}{dS}\right) = \frac{1}{2}\left[G_r\left(\frac{d\Gamma_2}{dS}\right)^2\right] \qquad E = 1, \ F = 0$$

$\Gamma_1(\theta) = f(\theta), \ \Gamma_2(\theta) = \theta$

$$\frac{1}{h}\frac{d}{d\theta}\left(\frac{1}{h}\frac{df}{d\theta}\right) = \frac{1}{2}\left[G_r\left(\frac{1}{h}\frac{d\Gamma_2}{d\theta}\right)^2\right] \quad (1)$$

$$\frac{1}{h}\left(\frac{1}{h}f'\right)' = \frac{1}{2}\frac{1}{h^2}G_r$$

<u>Step 2</u> Find Ψ: $\cos\Psi = \frac{\langle \Gamma', r_\theta' \rangle_g}{\|\Gamma'\|_g \|r_\theta'\|_g} = \frac{f'}{\sqrt{f'^2 + G}} = \frac{f'}{h}$ (2)

$\sin^2\Psi = 1 - \cos^2\Psi = 1 - \frac{f'^2}{f'^2 + G} = \frac{G}{f'^2 + G} \Rightarrow \sin\Psi = \frac{\sqrt{G}}{h}$ (3)

<u>Step 3</u>

Differentiate (2) $-\Psi'\sin\Psi = \left(\frac{f'}{h}\right)'$ (use (1) on LHS, (3) on RHS)

$$-\Psi'\frac{\sqrt{G}}{h} = \frac{1}{2h}G_r$$

$$\Psi' = -\frac{1}{2}\frac{G_r}{\sqrt{G}} = -(\sqrt{G})_r$$

$\square$

<u>Proof of Lemma 2</u> We claim that $G(r,\theta) = r^2\alpha(r,\theta)$, $\alpha \to 1$ as $r \to 0$ so that $(\sqrt{G})_r = \sqrt{\alpha(r,\theta)} + r\frac{\partial}{\partial r}(\sqrt{\alpha}) \to 1$ as $r \to 0$.

<u>Proof of Claim</u> $S: (0,\Sigma) \times (0, 2\pi) \to \mathcal{U}$, ~~$(r,\theta)$~~ $T_p = r_p(\mathcal{U})$

$(r,\theta) \xmapsto{\rho} r\cos\theta \, \underline{e_1} + r\sin\theta \, \underline{e_2} \xmapsto{r} \mathcal{U}$

$T_p\mathcal{U}$. Then at O, $\underline{e_1}, \underline{e_2}$ are an orthonormal basis for $T_p\mathcal{U}$

If we pull back any metric on $T_p\mathcal{U}$ by this map, we will see that it has this form as $r \to 0$, where $\underline{e_1}, \underline{e_2}$ are an orthonormal basis of $\mathcal{U}$. $\square$

14/03/12 Geometry (16)

B) Abstract Surfaces

Definition An abstract surface is a metric space Σ together with:

1. For every $p \in \Sigma$, there is an open set $U_p \subset \mathbb{R}^2$, $V_p \subset \Sigma$, $p \in V_p$
2. A homeomorphism $\varphi_p : U_p \to V_p$
3. The composition $\varphi_p^{-1} \circ \varphi_{p'}$ is a diffeomorphism onto its image where it is defined.

$U_{p'} \to \Sigma \to U_p$

φ_p are charts for Σ.

Example 1 Σ is an embedded surface.

Example 2 $\Sigma = T^2 = S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z}^2$

$\varphi_1 : A \times A \to S^1 \times S^1$

$\varphi_2 : A \times B \to S^1 \times S^1$ $\quad A = \left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right)$ $\quad \varphi_i(\alpha, \beta) = (e^{i\alpha}, e^{i\beta})$

$\varphi_3 : B \times A \to S^1 \times S^1$ $\quad B = \left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$

$\varphi_4 : B \times B \to S^1 \times S^1$

A Riemannian metric on Σ is a choice of Riemannian metrics g_p on U_p which is consistent in the sense that $(\underbrace{\varphi_p^{-1} \circ \varphi_{p'}}_{\text{transition functions}})^*(g_p) = g_{p'}$

Example T^2 as before, with the Euclidean metric $dx^2 + dy^2$ on all U.
This does not embed into $\mathbb{R}^3$. This metric has $K = 0$

C) Global Gauss-Bonnet

Theorem If Σ is a compact (abstract) surface

$$\int K \, dA = 2\pi \chi(\Sigma)$$

$\delta(\Delta_i) = a_{i,1} + a_{i,2} + a_{i,3} - \pi$

Proof Cut Σ into geodesic triangles contained in charts.

$$\int_\Sigma K \, dA = \sum_{\Delta s} \int_{\Delta_i} K \, dA = \sum_{\Delta s} \delta(\Delta_i) = \sum_{\substack{\text{interior angles} \\ \text{of all } \Delta s}} a_i - \pi \# \Delta s$$

$$= 2\pi \cdot V - \pi F$$

$V = \#$ vertices, $F = \#$ faces. $3F = 2E$ (triangulation)

$$= 2\pi \left(V - \tfrac{1}{2} F\right) = 2\pi (V - E + F) = 2\pi \chi(\Sigma)$$

$\square$