23/01/12

Fluid Dynamics ①

What is a fluid? Fluids flow :)

Examples: Water, air, syrup, oil } Newtonian Fluids (simple)

Paint, toothpaste, ketchup, soup, shampoo } Non-newtonian liquids

Sand, salt } Granular fluids

Foam

} Complex Fluids

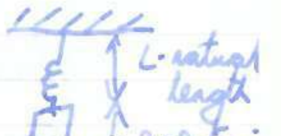
Already, Fluids > Electro

Newtonian Fluids

These have a linear relationship between stress and rate of strain.

Stress - force per unit area (e.g. pressure)

Strain - extension per unit length (e.g. in elasticity)



strain $= \frac{e}{L}$

Rate of Strain - rate of extension per unit length

↳ (is a gradient of velocity)

In part II, stress and strain rate are considered as tensor quantities, but here, we will consider situations that can be described ~~not~~ using scalar and vector fields.

We shall discuss viscous fluids. However, we shall often make an inviscid approximation.

Pressure is an example of a normal stress. The pressure force per unit area on a surface with normal $\underline{n}$, pointing into the fluid is

$\underline{\tau}_p = -p\,\underline{n}$

Gradients in pressure provide a net force.

High pressure $\xleftarrow{\nabla p}$ Low pressure

body force $\xrightarrow{-\nabla p}$
(force per unit volume)

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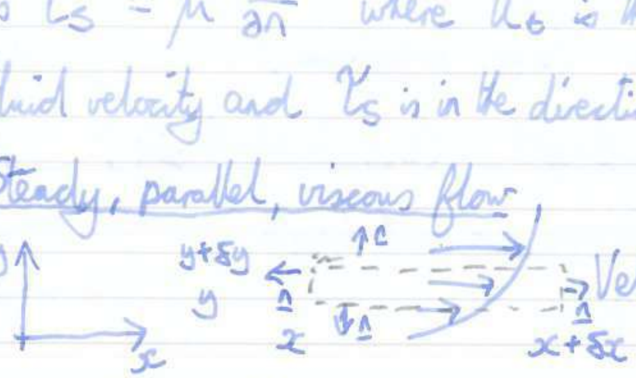
Tangential (Shear) Stress

We find experimentally that the force per unit area required to slide plates at relative speed U is $\tau_S \propto \frac{U}{h}$. We write $\tau_S = \mu \frac{U}{h}$ to define μ, the dynamic viscosity of the fluid.

V Such observations lead to the result that the tangential shear stress exerted by a fluid on a bounding surface with normal $\underline{n}$ pointing into the fluid is $\tau_S = \mu \frac{\partial u_t}{\partial n}$ where u_t is the tangential component of the fluid velocity and τ_S is in the direction of u_t

Steady, parallel, viscous flow



Velocity field $\underline{u} = (u(y), 0)$

We will start thinking about forces in the x direction, considering $p(x)$ and $p(x+\delta x)$. We will also have a shear stress, $\tau_S(y)$ and $\tau(y+\delta y)$.

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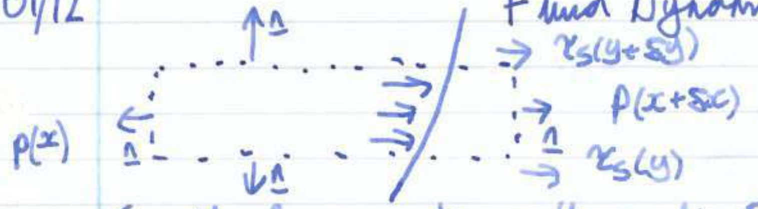
Fluid Dynamics ②



Consider forces acting in the x-direction on the dashed slab exerted by the surrounding fluid. The slab is not accelerating so forces must balance.

$$p(x)\,\delta y - p(x+\delta x)\,\delta y$$
$$+\tau_s(y)\,\delta x + \tau_s(y+\delta y)\,\delta x = 0$$

But $\tau_s = \mu \frac{\partial u}{\partial n}$, $\tau_s(y) = -\mu \frac{\partial u}{\partial y}$, $\tau_s(y+\delta y) = \mu \frac{\partial u}{\partial y}$

$$-\frac{p(x+\delta x)-p(x)}{\delta x} + \mu \frac{u_y(y+\delta y) - u_y(y)}{\delta y} = 0$$

$$\Rightarrow -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0$$

Repeat derivation in the y direction: $-\frac{\partial p}{\partial y} = 0$

Example Sheet 1: For <u>unsteady</u> parallel viscous flow, $\underline{u} = (u(y,t), 0)$ with a body force (force per unit volume) $\underline{F} = (F_x, F_y)$ acting on the fluid then

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + F_x \quad \text{(with } \rho \text{ the density)}$$

and $0 = -\frac{\partial p}{\partial y} + F_y$

Note that in a gravitational field, $\underline{F} = \rho \underline{g}$

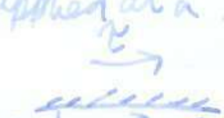
<u>Boundary Conditions</u> (for viscous fluids)

It has been verified experimentally (e.g. for water down to 2 molecular diameters ≈ 6Å) that Newtonian fluids satisfy a <u>no-slip condition</u>: that the tangential velocity of the fluid is equal to the tangential velocity of its boundary. For a stationary, rigid boundary $\underline{u}_t = 0$.

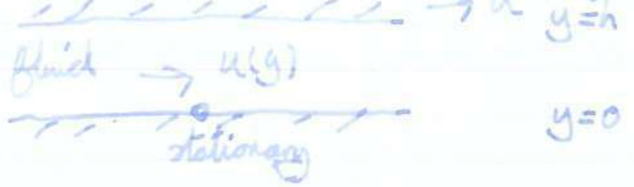
Stress condition: Sometimes the tangential stress τ_s is prescribed at the boundary rather than the velocity.

If a stress τ is applied at a boundary to the fluid then

$-\mu \frac{\partial u}{\partial n} = \tau$



Couette Flow - driven by boundary stresses or motions.



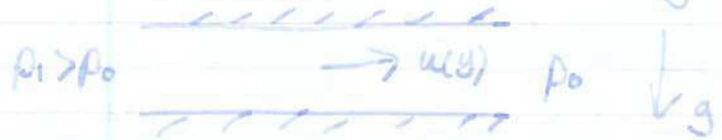
Steady, and no imposed pressure gradient, so $\frac{\partial^2 u}{\partial y^2} = 0$, $(0 < y < h)$ ①

② $u = 0$, $(y = 0)$, $u = U$, $(y = h)$ ③ ⇐ No slip

$\Rightarrow \frac{\partial u}{\partial y} = A$, $u = Ay + B$ ③ $\Rightarrow B = 0$, ② $= U = Ah$

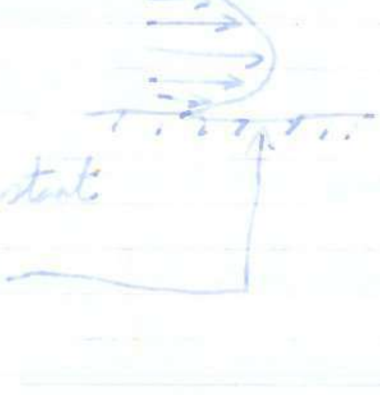
$\Rightarrow u = U \frac{y}{h}$  Linear velocity profile

Poiseuille Flow - driven by pressure gradients



$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$ ①

$0 = \frac{\partial p}{\partial y} - \rho g$ ②



No slip: $u = 0$ $(0 \leq y \leq h)$

② $\Rightarrow p = \rho g y + f(x)$

① $\Rightarrow \mu \frac{\partial^2 u}{\partial y^2} = f'(x) = -G$ constant

$\Rightarrow u = \frac{G}{2\mu} y (h - y)$

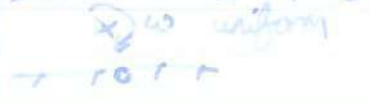
Derived properties of a flow

1. Volume flux q : volume of fluid traversing a cross-section per unit time. For parallel flow $q = \int_0^h u \, dy$ per unit transverse distance

Couette: $q = \frac{Uh}{2}$, Poiseuille $q = \frac{Gh^3}{12\mu}$

2. Vorticity $\underline{\omega}$, the curl of the velocity field. $\underline{\omega} = \nabla \times \underline{u}$

Couette flow: $\underline{\omega} = (0, 0, -\frac{U}{h})$ Poiseuille flow: $\underline{\omega} = (0, 0, \frac{G}{\mu}[\ldots]$



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Fluid Dynamics ②

3. Surface Stress

τ_s - tangential force per unit area exerted by fluid

$\tau_s = \mu \frac{\partial u}{\partial n}$, n points into the fluid

Couette flow $\tau_s = \begin{cases} \mu \frac{U}{h} & \text{for } y = 0 \\ -\mu \frac{U}{h} & y = h \end{cases}$

Poiseuille flow $\tau_s = \begin{cases} \frac{Gh}{2} & \text{as } y = 0 \\ \frac{Gh}{2} & y = h \end{cases}$

note that the Poiseuille flow has τ_s independent of μ.

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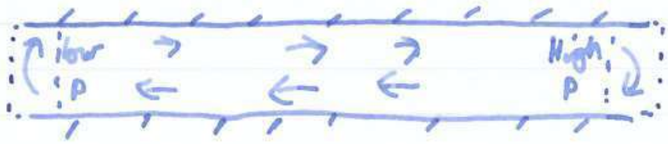
Fluid Dynamics ③

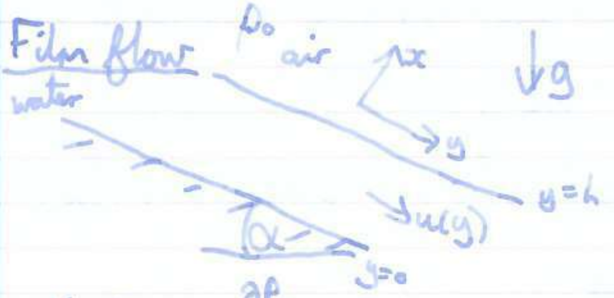
Couette - Poiseuille Mixture



Pressure gradients may develop to enforce incompressibility.

p_0 atmospheric pressure

Film flow



Assume that the air exerts no tangential stress on water ($\mu_{air} \ll \mu_{water}$

y direction: $\frac{\partial p}{\partial y} = -\rho g \cos\alpha \Rightarrow p = -\rho g \cos\alpha\,(y-h) + p(x)$, $\frac{\partial p}{\partial x} = ($

$y=h$, $p = p_0$ $y=h$, $p(x) = p_0$

x direction: $\mu \frac{\partial^2 u}{\partial y^2} = -\underbrace{\frac{\partial p}{\partial x}}_{=0} - \rho g \sin\alpha$

Boundary Conditions: No slip $\Rightarrow u(x,0) = 0$

No tangential stress $\Rightarrow \mu \frac{\partial u}{\partial y}(0,h) = 0$

$u(x,y) = \frac{\rho g}{2\mu} \sin\alpha \; y(2h-y) = \frac{g}{2\nu} \sin\alpha \; y(2h-y)$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity of the water.

Unsteady Parallel Viscous Flow

Consider a semi-infinite domain $y > 0$ initially at rest. At time $t=0$ the boundary $y=0$ is set into motion with speed U.

y direction: $\frac{\partial p}{\partial y} = \rho g$, $p = \rho g y + f(x)$

Assume there is no imposed pressure gradient

x direction: $\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} - \cancel{f'(x)}$

$u(0,t) = U$ for $t > 0$, $u \to 0$ as $y \to \infty$, $u(y,0) = 0$ for $y > 0$

$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$. The velocity satisfies the diffusion equation and the kinematic viscosity $\nu = \frac{\mu}{\rho}$ can be thought of as a diffusivity for momentum (or vorticity, see later)

The diffusion equation can be solved by Fourier transform in time (Q4), Fourier series in y (Q5, separation of variables), Laplace transforms etc.

<u>Similarity solution</u> - see IA Differential Equations

In the infinite domain, the diffusion equation has a similarity solution.

$u(y,t) = U f(\eta)$, $\eta = \frac{y}{\sqrt{\nu t}}$

$\Rightarrow -\frac{1}{2}\eta f' = f''$, $f = \mathrm{erfc}\left(\frac{\eta}{2}\right)$, $u = U\,\mathrm{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)$

<u>Kinematic vs Dynamic Viscosity</u>

	μ / kg m^{-1}s^{-1}	ρ / kg m^{-3}	ν / m^{2}s^{-1}	
Water	10^{-3}	10^{3}	10^{-6}	(rtp, 20° etc)
Air	2×10^{-5}	1	2×10^{-5}	

i) $\nu_{air} \approx 20\,\nu_{water}$ so the motion is induced further into the air.

ii) Tangential Stress on $y=0$ is $\tau_s = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \mu \frac{U}{2\sqrt{\nu t}}\left(-\frac{2}{\sqrt{\pi}}\right)e^{-\eta^2}\Big|_{y=0}$

$$\tau_s = -\mu \frac{U}{\sqrt{\pi \nu t}} \propto \frac{\mu}{\sqrt{\nu}}$$

$\left(\frac{\mu}{\sqrt{\nu}}\right)_{water} \approx 1$ (SI units), $\left(\frac{\mu}{\sqrt{\nu}}\right)_{air} \approx 4\cdot5\times10^{-3}$

So water exert a much greater shear stress on the boundary for the same boundary motion.

viscous diffusion / dissipation

inertia

<u>Dimensional Analysis</u>

Governing equation $\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$. Let U be a characteristic magnitude (scale) of u. "T" for t, "δ" intrinsic length scale.

Note that the current problem has no extrinsic length scale.

There is a balance between inertia and viscous dissipation when

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Fluid Dynamics (3)

$\rho \frac{U}{T} \sim \mu \frac{U}{\delta^2} \quad \Rightarrow \delta \sim \sqrt{\nu T}$

We can thus determine characteristic scales of the flow without solving the differential equation.

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Fluid Dynamics (4)

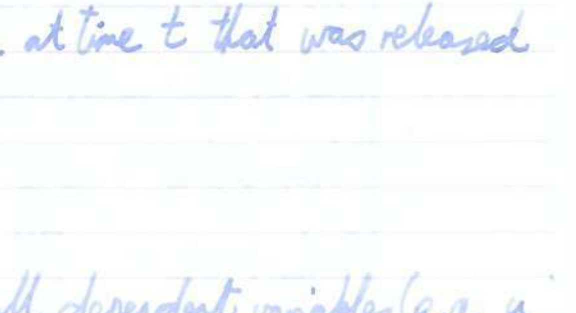
Characterisation / Visualisation of Flow

Lagrangian Picture

Mark (dye) a fluid particle and follow its trajectory. Trajectories can cross and it is difficult to formulate differential equations. Properties (e.g. density) would be written as $\rho(t; \underline{x}_0)$ for a particle at time t that was released from $\underline{x}_0$ at time $t_0 = 0$ say.

Eulerian Picture

Sit still and watch the world go by. Write all dependent variables (e.g. $\underline{u}$) as functions of fixed location $\underline{x}$ at time t.



Material time derivative (Eulerian Picture)

Consider a time dependent field $f(\underline{x}, t)$

Along a path $\underline{x} = \underline{x}(t)$

$$\frac{df}{dt}(\underline{x}(t), t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t} = \dot{\underline{x}} \cdot \nabla f + \frac{\partial f}{\partial t}$$

If $\underline{x}(t)$ is the Lagrangian path followed by a fluid particle then $\dot{\underline{x}}(t) = \underline{u}$, the local fluid velocity. We write $\frac{df}{dt} = \frac{Df}{Dt}$ for the time derivative along a path (trajectory) of a fluid particle.

Material time derivative : $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \underline{u} \cdot \nabla f$ ← Advective time derivative

Lagrangian time derivative — Eulerian time derivative

Conservation of Mass

Consider a fixed region of space $\mathcal{D}$ with boundary $\partial\mathcal{D}$ and outward normal $\underline{n}$.

Mass is not created or destroyed so the mass inside $\mathcal{D}$ can only change by a net flow across $\partial\mathcal{D}$

$$\frac{d}{dt}\int_{\mathcal{D}} \rho \, dV = -\int_{\partial\mathcal{D}} \rho \, \underline{u}\cdot\underline{n} \, dS \quad \Rightarrow \int_{\mathcal{D}} \frac{\partial\rho}{\partial t} \, dV = -\int_{\mathcal{D}} \nabla\cdot(\rho\underline{u}) \, dV$$

using the divergence theorem. $\Rightarrow \int_{\mathcal{D}} \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{u}) \, dV = 0$

This is true for arbitrary $\mathcal{D}$ so $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0$ pointwise.

This has the general form of a conservation law:

rate of change of stuff + divergence of the stuff flux = 0

Mass flux = $\rho\underline{u}$

Product rule: $\frac{\partial\rho}{\partial t} + \underline{u}\cdot\nabla\rho + \rho\nabla\cdot\underline{u} = 0$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0$$

If the fluid is incompressible then the density of a fluid particle cannot change

i.e. $\frac{D\rho}{Dt} = 0 \quad \Rightarrow \underline{\nabla\cdot\underline{u} = 0}$ continuity equation.

Note that in parallel flow, $\underline{u} = (u(y,z), 0, 0) \Rightarrow \nabla\cdot\underline{u} = \frac{\partial u}{\partial x} = 0$

N.B. an unconfined fluid can be treated as incompressible provided $|\underline{u}| \ll c$, speed of sound. $c_{air} \approx 340 \, ms^{-1}$, $c_{water} \approx 1500 \, ms^{-1}$

<u>Kinematic Boundary Condition</u>

Consider a material boundary moving with velocity $\underline{U}$.

In a local frame of reference moving with velocity $\underline{U}$, the fluid velocity has relative value $\underline{u}' = \underline{u} - \underline{U}$ and the boundary is stationary.

Fluid cannot cross the boundary so $\underline{n}\cdot\underline{u}' = 0$

$\Rightarrow \underline{n}\cdot\underline{u} = \underline{n}\cdot\underline{U}$ at boundaries.

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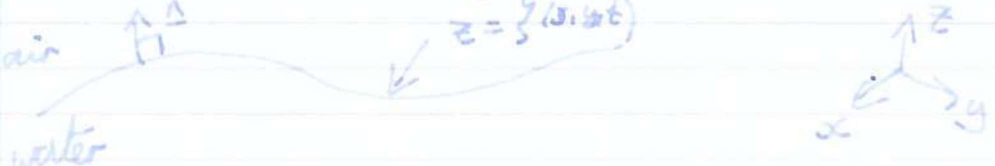
Fluid Dynamics (4)

i) At a ~~rigid~~ stationary boundary, $\underline{U} = \underline{0} \Rightarrow \underline{n} \cdot \underline{u} = 0$

ii) At a free material boundary, such as the surface of a water wave,



ζ Think of the surface as a contour:

$$F(x, y, z, t) = z - \zeta(x, y, t) \qquad (= 0, \text{ a } 0 \text{ contour})$$

The normal to the surface is the normal to the contour of F

$$\underline{n} = \nabla F = \left(-\frac{\partial \zeta}{\partial x}, -\frac{\partial \zeta}{\partial y}, 1\right)$$

$$\underline{U} = \left(0, 0, \frac{\partial \zeta}{\partial t}\right), \quad \underline{u} = (u, v, w)$$

$$\Rightarrow -u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y} + w = \frac{\partial \zeta}{\partial t}$$

$$\Rightarrow w = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}$$

$$w = \frac{D\zeta}{Dt}$$

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Fluid Dynamics ⑤

Stream functions for 2-D incompressible flow

Incompressible flow $\Rightarrow \nabla \cdot \underline{u} = 0 \quad \Rightarrow \underline{u} = \nabla \times \underline{A}$

for some vector potential $\underline{A}$.

If $\underline{u} = (u, v, 0)$ in Cartesian components, we can find $A = (0, 0, \psi(x,y))$

Note : $u = u(x,y)$, $v = v(x,y)$

Putting this together : $\underline{u} = \left(\frac{\partial \psi}{\partial y}, - \frac{\partial \psi}{\partial x}, 0 \right)$

The scalar function $\psi(x,y)$ is called the stream function for the flow.

Properties of the Stream function

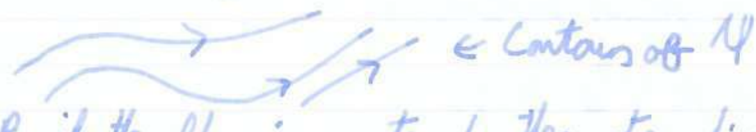
The contours of $\psi(x,y)$ have normals $\underline{n} = \nabla \psi = \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, 0 \right)$

So $\underline{n} \cdot \underline{u} = \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} = 0$

So $\underline{u}$ is orthogonal to the normals of the contours of ψ

$\Rightarrow \underline{u}$ is tangent to the contours of ψ.

(i) The contours of ψ are <u>streamlines</u>



← Contours of ψ

N.B. if the flow is unsteady then streamlines are <u>not</u> particle paths.

The way to visualise streamlines is to seed the fluid with lots of particles and take a photograph with an open shutter of short duration

e.g. $\underline{u} = (t, 1, 0)$ (Lagrangian) particle paths are given

 $t = 0$  $t = 1$ by $\dot{\underline{x}} = \underline{u}$, $\underline{x}(0) = \underline{x}_0$

$\dot{x} = t$, $\dot{y} = 1$, $x = \frac{1}{2}t^2 + x_0$, $y = t + y_0$

where t can be considered as a parameter for the path. Particle paths can be found by eliminating $t \Rightarrow (x - x_0) = \frac{1}{2}(y - y_0)^2$

(ii) The flow is faster where streamlines are closer.

(iii) Volume flux (per unit length in the z direction) crossing any curve from $\underline{x}_0$ to $\underline{x}_1$ $= q = \int_\Gamma \underline{u} \cdot \underline{n} \, dl = \psi(\underline{x}_1) - \psi(\underline{x}_0)$

$\psi_1 = \psi(\underline{x}_1), \quad \psi_0 = \psi(\underline{x}_0)$

iv) ψ is constant on a stationary, rigid boundary. We are always free to choose $\psi = 0$ on one of the stationary boundaries of a flow.

<u>Plane-polar coordinates</u> (r, θ)

For 2-D flow expressed in plane polars, embedded in cylindrical polars (r, θ, z) and write $\underline{u} = \nabla \times (0, 0, \psi) = \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$

$$\underline{u} = \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}, 0 \right)$$

<u>Navier-Stokes Equation</u>

Newton's 2nd law for a fluid particle is

$$\rho \frac{D\underline{u}}{Dt} = \underbrace{-\nabla p + \mu \nabla^2 \underline{u}}_{\text{surface forces}} + \underline{F}$$

mass × acceleration = surface forces + body forces

Note that the acceleration is a Lagrangian (material) derivative of the velocity. Derivation of the viscous term $\mu \nabla^2 \underline{u}$ requires consideration of the stress tensor, see part II fluids.

[Note: $\nabla^2 \underline{u} = \nabla(\nabla \cdot \underline{u}) - \nabla \times (\nabla \times \underline{u})$

$\qquad = -\nabla \times (\nabla \times \underline{u})$ for incompressible flow

In <u>Cartesians</u> we are straightforwardly $\nabla^2 \underline{u} = (\nabla^2 u, \nabla^2 v, \nabla^2 w)$

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Fluid Dynamics ⑤

<u>Exercise</u> Show that if $\underline{u} = (u(y,t), 0, 0)$ then the Navier Stokes equations reduce to the earlier parallel flow equations derived.

In a gravitational field, $\underline{F} = \rho \underline{g}$

Navier-Stokes equations:

$$\rho \left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right) = -\nabla p + \mu \nabla^2 \underline{u} + \underline{F}$$

The term $\underline{u} \cdot \nabla \underline{u}$ is <u>strongly</u> non-linear and gives rise to all the interesting behaviour of fluids.

08/02/12 Fluids (6)

Hydrostatic Pressure

If $\underline{u} \equiv 0$, write $p = p_H$ $\qquad 0 = -\nabla p_H + \rho \underline{g}$

$\Rightarrow p_H = \rho \underline{g} \cdot \underline{x} + p_0 = p_0 - \rho g z$

Pressure decreases upwards.

Archimedes Static force on a submerged body

fluid density ρ

$\underline{F} = \int_{\partial D} -p_H \underline{n} \, dS = -\int_D \nabla p_H dV$ (By the divergence theorem)

$= -\int_D \rho \underline{g} \, dV = -\rho g \int_D dV = -\rho g V = -M_f g$

where $M_f = \rho g$ is the mass of fluid displaced by the body.

Upthrust = weight of fluid displaced

Dynamic Pressure - causing or resulting from fluid flow.

Write $p = p_H + p'$ ← dynamic pressure

$\Rightarrow \rho \frac{D\underline{u}}{Dt} = -\nabla p' + \mu \nabla^2 \underline{u}$

1. We usually drop the prime so that p is referred to as the dynamic pressure.
2. Every fluid particle is neutrally buoyant.
3. If there is a free surface (e.g. water/air) then we do need to consider gravity.
4. The dynamic pressure is often determined internally and is always sufficient to maintain the incompressibility constraint, $\nabla \cdot \underline{u} = 0$

Reynolds Number

Suppose the flow has a characteristic (typical) magnitude $\mathcal{U}$ and extrinsic length scale, externally imposed by geometry.

These two scales define a timescale $T = \frac{L}{U}$

Suppose that pressure <u>differences</u> have characteristic magnitude P.
What is the relative importance of the different terms in the Navier-Stokes equations?

$$\frac{\partial \underline{u}}{\partial t} \quad + \quad \underline{u}\cdot\nabla\underline{u} \quad = \quad -\frac{1}{\rho}\nabla p \quad + \quad \nu\nabla^2\underline{u}$$

$$\frac{U}{L/U} \quad : \quad U\frac{U}{L} \quad : \quad \frac{1}{\rho}\frac{P}{L} \quad : \quad \nu\frac{U}{L^2}$$

$$\Rightarrow 1 \quad : \quad 1 \quad : \quad \frac{P}{\rho U^2} \quad : \quad \frac{\nu}{UL} = \frac{1}{Re}$$

The Reynolds number $Re = \frac{UL}{\nu}$ gives the relative magnitude of the ~~inert~~ inertial terms to the viscous terms.

N.B. Pressure must always scale to balance the dominant term in the equation so that we can impose $\nabla\cdot\underline{u} = 0$

<u>Small Re</u> $Re \ll 1$

Small lengths (e.g. cells), Slow flows (slow running tap)

Large viscosity (syrup, lava, oil)

Inertial terms are negligible $P \sim \rho U^2 \frac{\nu}{UL} = \mu \frac{U}{L}$

We can approximate by the Stokes Equations $0 = -\nabla p + \mu\nabla^2\underline{u}$ and $\nabla\cdot\underline{u} = 0$. Note that $\underline{u} \propto \nabla p$

These will be studied in detail next year

<u>Large Re</u> $Re \gg 1$

Viscous terms are negligible on extrinsic length scales. $P \sim \rho U^2$, scales with the momentum flux. On extrinsic length scales, we can approximate the NS equations with the Euler equations: $\rho\frac{D\underline{u}}{Dt} = -\nabla p$, $\nabla\cdot\underline{u} = 0$

Note that pressure gradients give rise to <u>accelerations</u>.

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Intrinsic length scales on which inertial terms balance viscous terms so δ where $\frac{U^2}{\delta} \sim \nu \frac{U}{\delta^2} \Rightarrow \delta \sim \frac{\nu}{U} \Rightarrow \frac{\delta}{L} = \frac{\nu}{UL} = \frac{1}{Re} \ll 1$

At large Reynolds numbers, viscosity acts on small length scales (e.g. Rigid Boundarys)

$$\underline{u} \cdot \nabla \underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_3}{\partial x_3} \end{pmatrix} = u_1 \frac{\partial u_1}{\partial x_1} + \ldots + u_3 \frac{\partial u_3}{\partial x_3}$$

$$(\underline{u} \cdot \nabla) \underline{u} = \left(u_1 \frac{\partial}{\partial x_1} + \ldots + u_3 \frac{\partial}{\partial x_3} \right) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

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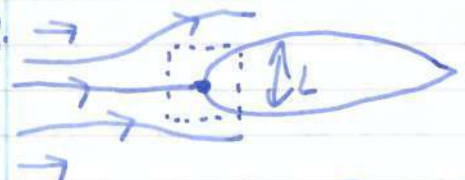
$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u} \qquad \text{NS}$$

At large Re, viscous terms are unimportant on the (extrinsic) length scales of the flow.

e.g.



Inset of dashed region

A Case Study - Stagnation Point Flow

There is an exact solution of NS in which

$\underline{u} \sim (Ex, -Ey, 0)$ as $y \to \infty$, $\underline{u} = 0$ on $y = 0$ ← No slip

The solution has the form $\underline{u} = (Ex\,g'(\eta), -\sqrt{\nu E}\,g(\eta), 0)$

where $\eta = \frac{y}{\delta} = \sqrt{\frac{E}{\nu}}\,y$. The (intrinsic) length scale $\delta = \sqrt{\frac{\nu}{E}}$

Exercises

i) Show that $\nabla\cdot\underline{u} = 0$ and that the stream function

$\psi = \sqrt{\nu E}\,x\,g(\eta)$ with $\psi \sim Exy$ as $y \to \infty$

ii) Show that the NS equations give

$$Ex(g'^2 - gg'') = -\frac{1}{\rho}p_x + E^2 x\,g''' \quad (1)$$

$$E\sqrt{\nu E}\,gg' = -\frac{1}{\rho}p_y - E\sqrt{\nu E}\,g'' \quad (2)$$

Differentiate:

$\frac{\partial}{\partial x}(2)$ $\Rightarrow p_{xy} = 0$

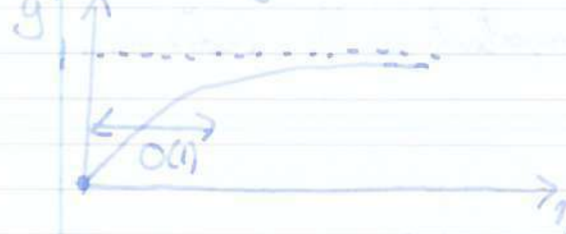
$\frac{\partial}{\partial y}(1)$ $\Rightarrow (g'^2 - gg'')' = g^{(4)}$

iii) Show that the boundary conditions give

$g' \to 1$, $g \sim \eta$ as $\eta \to \infty$, $g = g' = 0$ when $\eta = 0$

All dimensional parameters have been absorbed into the scaled variables g and η.

So we only need to solve the ODE once to find



The far field velocity $\underline{u} = (Ex, -Ey, 0)$ is reached to an excellent approximation when $\eta \gtrsim 1 \Rightarrow y \gtrsim \delta = \sqrt{\frac{\nu}{E}}$

<u>Overall Picture</u>

Recall that $\underline{u} \sim (Ex, -Ey, 0)$ in the far field.

If we're interested in the flow on scales much larger than δ then solve the equations in $y > \delta$, ignoring the viscous terms. Formally, we will take the limit $\delta \to 0$ and solve the Euler equations in $y > 0$. BUT we can't then use the no-slip condition.

<u>Inviscid Approximation</u>

If $Re \gg 1$ then solve the Euler equations

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p \quad (+\underline{f})$$

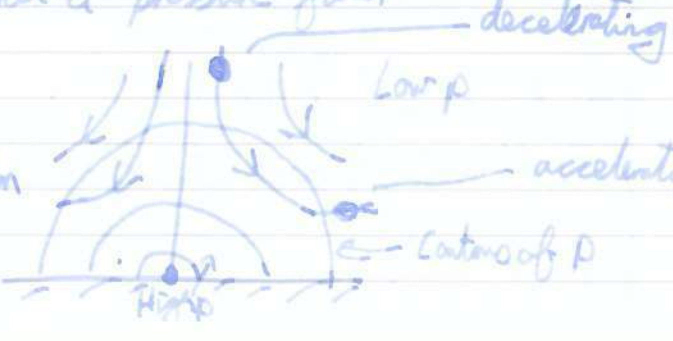
$$\nabla \cdot \underline{u} = 0 \quad , \quad \underline{u} \cdot \underline{n} = 0 \text{ at stationary, rigid boundaries.}$$

The no-slip condition <u>cannot</u> be imposed on solutions of the Euler equation.

<u>Exercise</u> Show that the flow

$\underline{u} = (Ex, -Ey, 0)$ satisfies the Euler equations in $y > 0$ with a rigid boundary at $y = 0$ and a pressure field

$$p = p_0 - \frac{\rho E^2}{2}(x^2 + y^2)$$



The pressure field acts as an internal reaction force imposing the constraint of incompressibility.

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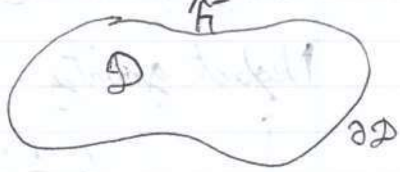
Momentum Equation

For incompressible, inviscid flow. The momentum of fluid inside a region $\mathcal{D}$ with boundary $\partial\mathcal{D}$ can change owing to



i) Momentum flowing across the boundary.

ii) Volume (body) forces

iii) Surface pressure forces

iv) Surface viscous forces : $\underbrace{\text{tangential shear stress, normal viscous stress}}_{\text{Ignore}}$

(*) $$\frac{d}{dt}\int_{\mathcal{D}} \rho \underline{u}\, dV = -\int_{\partial\mathcal{D}} \underbrace{\rho \underline{u}\, \underline{u}\cdot\underline{n}\, dS}_{(i)} + \int_{\mathcal{D}} \underbrace{\underline{f}\, dV}_{(ii)} - \int_{\partial\mathcal{D}} \underbrace{p\underline{n}\, dS}_{(iii)}$$

In components:

$$\frac{d}{dt}\int_{\mathcal{D}} \rho u_i\, dV = -\int_{\partial\mathcal{D}} \rho u_i u_j n_j\, dS + \int_{\mathcal{D}} f_i\, dV - \int_{\partial\mathcal{D}} p n_i\, dS$$

$$\Rightarrow \int_{\mathcal{D}} \rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial}{\partial x_j}(u_i u_j)\, dV = \int_{\mathcal{D}} -\frac{\partial p}{\partial x_i} + f_i\, dV$$

This is true for arbitrary regions $\mathcal{D}$, so

$$\rho \frac{\partial u_i}{\partial t} + \rho\left(u_j \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial u_j}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + f_i$$

$$\rho \frac{\partial \underline{u}}{\partial t} + \rho(\underline{u}\cdot\nabla)\underline{u} + \underbrace{(\nabla\cdot\underline{u})}_{=0 \text{ for incompressible flow}}\underline{u} = -\nabla p + \underline{f}$$

$$\Rightarrow \rho\frac{D\underline{u}}{Dt} = -\nabla p + \underline{f}$$ Euler Momentum Equation

$\frac{Du}{Dt}$ is the acceleration of a fluid particle, $\frac{\partial u}{\partial t}$ is the local rate of change of the velocity at a point

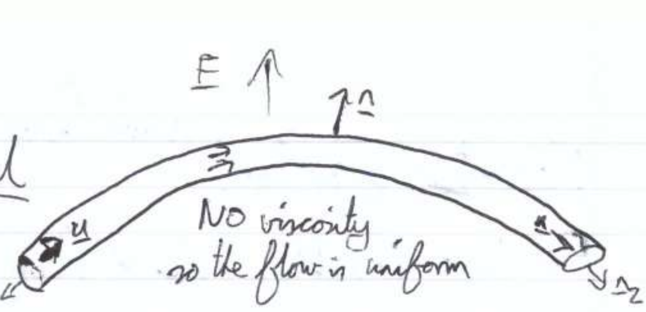
Consenative body forces

$\underline{f} = -\nabla\chi$, e.g. gravity, $f = \rho\underline{g} = \nabla(\rho\,\underline{g}\cdot\underline{x})$, $\chi = -\rho\underline{g}\cdot\underline{x} = \rho g z$

Momentum Integral for steady flow

From (*), $0 = -\int_{\partial\mathcal{D}} \rho\underline{u}(\underline{u}\cdot\underline{n})\, dS - \int_{\mathcal{D}} \nabla\chi\, dV - \int_{\partial\mathcal{D}} p\underline{n}\, dS$

$$\Rightarrow \int_{\partial\mathcal{D}} \rho\underline{u}(\underline{u}\cdot\underline{n}) + p\underline{n} + \chi\underline{n}\, dS = 0$$



Application of Momentum Integral

Curved hose-pipe

Steady flow. Conservation of mass $\Rightarrow |\underline{u}| = \text{constant} = u$

Neglect gravity. No acceleration along the pipe. $\Rightarrow p$ is constant

$$\int_{\text{curved surface}} \rho\, \underline{u}(\underline{u}\cdot\underline{n}) + p\underline{n}\, dS + \underbrace{A}_{\text{Cross sectional Area}}\rho \underbrace{(-u\underline{n_1})}_{\underline{u}}\underbrace{(-u)}_{\underline{u}\cdot\underline{n}} + A\rho\, u\underline{n_2}\, u = 0$$
$$+ Ap\underline{n_1} + Ap\underline{n_2}$$

Force on a pipe:

$$\underline{F} \equiv \int p\underline{n}\, dS = -A(\rho u^2 + p)(\underline{n_1} + \underline{n_2})$$

Bernoulli's Equation

Vector identity: $\underline{u} \times (\nabla \times \underline{u}) = \nabla(\tfrac{1}{2}|u|^2) - (\underline{u}\cdot\nabla)\underline{u}$

From Euler's Equation $\quad \rho\frac{\partial u}{\partial t} + \nabla(\tfrac{1}{2}\rho|\underline{u}|^2) - \rho\,\underline{u}\times(\nabla\times\underline{u}) = -\nabla p - \nabla\chi \quad (**)$

Dot with $\underline{u}$:

① Steady Flow $\quad \underline{u}\cdot\nabla(\tfrac{1}{2}\rho|\underline{u}|^2 + p + \chi) = 0$

Directional derivative with respect to time along a streamline

$$\Rightarrow \underbrace{H = \tfrac{1}{2}\rho|\underline{u}|^2 + p + \chi}_{\text{Bernoulli's Theorem.}} = \text{constant along streamline}$$

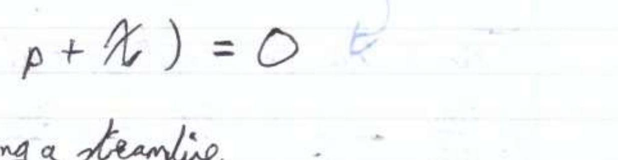
Use the divergence theorem recalling $\nabla\cdot\underline{u} = 0$

② Integrate (**) $\quad \frac{d}{dt}\int_{\mathcal{D}} \tfrac{1}{2}\rho|\underline{u}|^2 dV = -\int_{\partial\mathcal{D}} H\,\underline{u}\cdot\underline{n}\; dS$

So the rate of change of the kinetic energy inside $\mathcal{D}$ is equal to the flux of H across the boundary of $\mathcal{D}$. So H is the transportable energy of the fluid.

Applications of Bernoulli's Equation

i) Pitot tube for measuring air speed.



Along stagnation point streamline $\qquad$ Stagnation Point

$$\underset{①}{\tfrac{1}{2}\rho u^2 + p_0} = \underset{②}{p_1} \quad \Rightarrow \quad u = \left(2\frac{p_1 - p_0}{\rho}\right)^{\frac{1}{2}}$$

Measuring p_1 gives u. In the barometer $p_1 - p_0 = \rho_W gh$

$$u = (2\rho_W gh)^{\frac{1}{2}}$$

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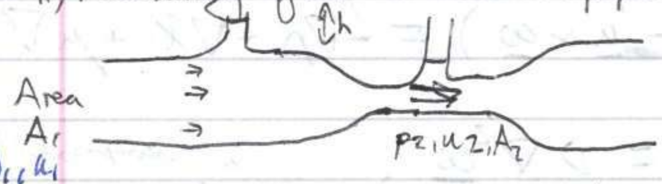
<u>Bernoulli's Theorem</u>

$H \equiv \frac{1}{2}\rho|u|^2 + p + \chi$ = constant along streamlines

ii) Measuring flow rate in a pipe



Conservation of mass : Volume flow rate $q = A_1 u_1 = A_2 u_2$

Bernoulli : $H_1 = H_2$

$$\frac{1}{2}\rho u_1^2 + p_1 = \frac{1}{2}\rho u_2^2 + p_2$$

$$\frac{1}{2}\rho\left(\frac{q}{A_1}\right)^2 + p_1 = \frac{1}{2}\rho\left(\frac{q}{A_2}\right)^2 + p_2$$

$$\Rightarrow \left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right) q^2 = \frac{2}{\rho}(p_1 - p_2) = \frac{2\rho g h}{\rho} = 2gh$$

$$\Rightarrow q = \sqrt{2gh}\,\frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}}$$

<u>Linear flows</u>

Consider the flow in the neighbourhood of a fixed point $\underline{x_0}$

$$\underline{u}(\underline{x}) = \underline{u}(\underline{x_0}) + (\underline{x} - \underline{x_0})\cdot\nabla\underline{u}(\underline{x_0}) + \ldots$$

$$\simeq \underline{u_0} + \underline{r}\cdot\nabla\underline{u} \quad \text{- linear approximation}, \quad \underline{r} = \underline{x} - \underline{x_0}$$

where $\underline{u}(\underline{x_0}) = \underline{u_0}$ is a constant vector

$$\nabla\underline{u} = \frac{\partial u_i}{\partial x_j} = E_{ij} + \Omega_{ij} = \underline{\underline{E}} + \underline{\underline{\Omega}}$$

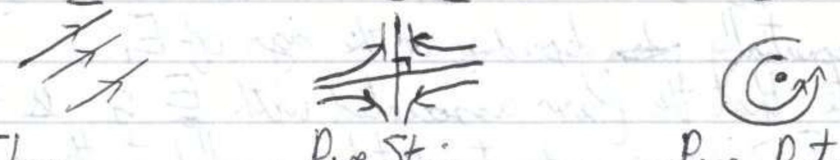
$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, symmetric. $\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$, antisymmetric

Recall : vorticity $\underline{\omega} = \nabla \times \underline{u}$

Note : $\underline{\omega} \times \underline{r} = (\nabla \times \underline{u}) \times \underline{r} = \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) r_j = 2\Omega_{ij} r_j = 2\underline{\underline{\Omega}}\,\underline{r}$

Therefore

$$\underline{u} \simeq \underline{u_0} \quad + \underline{\underline{E}}\cdot\underline{r} \quad + \frac{1}{2}\underline{\omega}\times\underline{r}$$



Uniform Flow   Pure Strain   Pure Rotation

Note that the local rotation rate = $\frac{1}{2}$ the vorticity

Note also that the Strain Rate tensor $\underline{\underline{E}}$ is symmetric, and hence diagonalisab[le] with orthogonal eigenvectors, and $\underline{\underline{E}}$ is traceless, because $\nabla\cdot\underline{u} = 0$

$$E_{kk} = \frac{\partial u_k}{\partial x_k} = \nabla\cdot\underline{u} = 0$$

With respect to principle axes, $\underline{\underline{E}} = \begin{pmatrix} E_1 & & \\ & E_2 & \\ & & E_3 \end{pmatrix}$ where $\overbrace{E_1 + E_2 + E_3}^{\text{real}} =$

Vorticity Equation

The Navier-Stokes equations for a viscous fluid gives

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p - \nabla\chi + \mu\nabla^2\underline{u}$$

$$\Rightarrow \rho\left(\frac{\partial \underline{u}}{\partial t} + \nabla\left(\tfrac{1}{2}|u|^2\right) - \underline{u}\times\underline{\omega}\right) = -\nabla p - \nabla\chi + \mu\nabla^2\underline{u}$$

Take the curl:

$$\frac{\partial\underline{\omega}}{\partial t} - \nabla\times(\underline{u}\times\underline{\omega}) = \nu\nabla^2\underline{\omega}$$

But $\nabla\times(\underline{u}\times\underline{\omega}) = \cancel{(\nabla\cdot\underline{\omega})}\underline{u} + (\underline{\omega}\cdot\nabla)\underline{u} - \overbrace{\cancel{(\nabla\cdot\underline{u})}}^{=0 \text{ incompressibility}}\underline{\omega} - (\underline{u}\cdot\nabla)\underline{\omega}$

So $\dfrac{\partial\underline{\omega}}{\partial t} + \underline{u}\cdot\nabla\underline{\omega} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega}$

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega}$$

$\frac{\partial\omega}{\partial t}$ is the local rate of change of vorticity

$\underline{u}\cdot\nabla\underline{\omega}$ represents advection of vorticity

$\underline{\omega}\cdot\nabla\underline{u}$ represents amplification (or reduction) of vorticity by stretching (or compressi

$\nu\nabla^2\underline{\omega}$ represents dissipation of vorticity by the action of viscosity, and also allows for the generation of vorticity by the no-slip condition at rigid boundaries.

Vortex amplification by stretching

The vorticity equation in an inviscid fluid is

$$\underline{\omega}\cdot\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u}\cdot\underline{\omega}$$

$$\Rightarrow \frac{D}{Dt}\left(\tfrac{1}{2}\omega^2\right) = \underline{\omega}\cdot(\underline{\underline{E}} + \underline{\underline{\Omega}})\cdot\underline{\omega} = \underline{\omega}\cdot\underline{\underline{E}}\cdot\underline{\omega}$$

wrt principal axes of $\underline{\underline{E}}$

$$\frac{D}{Dt}\left(\tfrac{1}{2}\omega^2\right) = E_1\omega_1^2 + E_2\omega_2^2 + E_3\omega_3^2$$

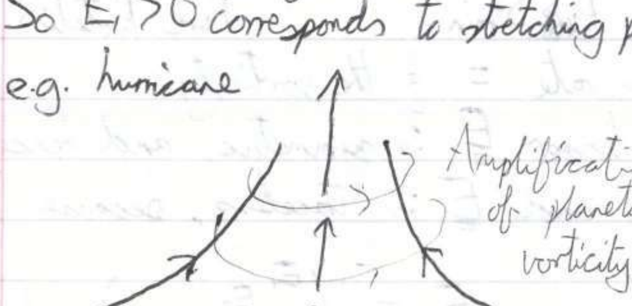
Suppose $\underline{\omega} = (\omega_1, 0, 0)$.

Then $\frac{D}{Dt}\left(\frac{1}{2}\omega_1^2\right) = E_1\omega_1^2$ and the magnitude of vorticity grows or decays exponentially ~~when~~ depending on the sign of E_1.

Note that the flow associated with $\underline{\underline{E}}$ is $\underline{u} = (E_1x, E_2y, E_3z)$

So $E_1 > 0$ corresponds to stretching parallel to the x axis.

e.g. hurricane



The hurricane circulates cyclonically (in the direction of Earth's rotation)

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Fluid Dynamics ⑩

Vorticity Equation

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega} \cdot \nabla \underline{u} \quad \left(+ \nu \nabla^2 \underline{u}\right)$$ ← not in this course

Vortex Stretching

Considering two neighbouring (Lagrangian) fluid particles.

$$\frac{Dx_2}{Dt} = \underline{u}(x_2), \quad \frac{Dx_1}{Dt} = \underline{u}(x_1)$$

$$\Rightarrow \frac{D}{Dt}\underline{\delta l} = \underline{u}(x_2) - \underline{u}(x_1) = \underline{u}(\underline{x_1}) + \underline{\delta l} \cdot \nabla \underline{u}(\underline{x_1}) + \ldots - u(\underline{x_1})$$

$$= \underline{\delta l} \cdot \nabla \underline{u} \quad \text{to leading order}$$

$\underline{\delta l}$ satisfies the same equation as $\underline{\omega}$, so vorticity is stretched as fluid elements are stretched.

Inviscid (homogeneous) fluids

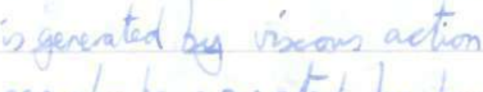
$$\frac{\partial \underline{\omega}}{\partial t} = \underline{\omega} \cdot \nabla \underline{u} - \underline{u} \cdot \nabla \underline{\omega}.$$

Vorticity can be amplified, and advected, but cannot be generated.

Aside

i) Vorticity is generated by viscous action near rigid boundaries. ↘ Inviscid rules this out

ii) Vorticity can also be generated by buoyancy ↘ Homogeneity rules this out



Inviscid, irrotational flow

If $\nabla \times \underline{u} \equiv 0$ at $t = 0$, and the fluid is inviscid, then $\nabla \times \underline{u} \equiv 0$ for all time

$\Rightarrow \underline{u} = \nabla \varphi$ for some scalar "velocity potential"

Incompressibility : $\nabla \cdot \underline{u} = 0 \Rightarrow \nabla^2 \varphi = 0$, Laplace's Equation

Three-dimensional Flows

i) Spherically symmetric flow: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) = 0$

$$\frac{\partial \varphi}{\partial r} = \frac{A}{r^2} \Rightarrow \varphi = -\frac{A}{r} + B \quad (B = 0 \text{ WLOG})$$

Velocity field $\underline{u} = \nabla \varphi = \frac{\partial \varphi}{\partial r}\underline{e_r} = \frac{A}{r^2}\underline{e_r}$

independent of A

Volume flux across the surface of the sphere $r = a$ is

$q_1 = \int_S \underline{u} \cdot \underline{n}\, dS = \int_S \frac{\partial \phi}{\partial r}\, dS = \int_S \frac{A}{a^2}\, dS = \frac{A}{a^2} 4\pi a^2 = 4\pi A$

$\Rightarrow \phi = -\frac{q_1}{4\pi r}$

Represents a point source of fluid of strength (volume flux) q_1 at the origin

Note: $\nabla^2 \phi = q_1 \delta(\underline{x})$, so ϕ is the 3D Green's function for potential flows.

ii) <u>Axisymmetric Solutions</u>

In spherical polar coordinates $\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right)$

$\Rightarrow \phi = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n-1}) P_n(\cos\theta)$ with P_n the n^{th} Legendre polynomial

e.g. uniform flow past a sphere.

In Cartesian coordinates, $\underline{u}_\infty = (0, 0, U)$

$\phi_\infty = Uz = Ur\cos\theta$

① $\nabla^2 \phi = 0 \quad (r > a)$ ② $\phi \sim Ur\cos\theta$ as $r \to \infty$)

③ $\underline{u} \cdot \underline{n} = \frac{\partial \phi}{\partial r} = 0$ at $r = a$ (fluid cannot penetrate a rigid boundary)

Notes: equation is <u>linear</u>.

forcing $\propto \cos\theta \; (= P_1(\cos\theta))$

$P_n(\cos\theta)$ is an eigenfunction of ∇^2 in spherical polars.

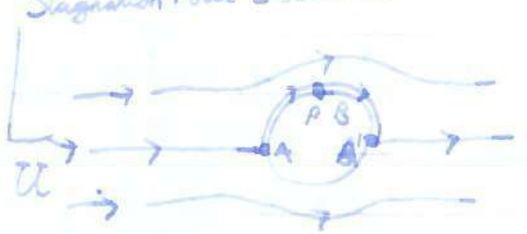
So the solution is $\phi = f(r) P_1(\cos\theta) = \left(Ar + \frac{B}{r^2}\right) P_1(\cos\theta)$

② $\Rightarrow A = U$ ③ $\Rightarrow A - \frac{2B}{a^3} = 0$,

$\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta$

uniform flow → ← dipole field

Stagnation Point Streamline



<u>Velocity and Pressure</u>

$u_r = \frac{\partial \phi}{\partial r} = U\left(1 - \frac{a^3}{r^3}\right)\cos\theta$

$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta$

Note that $u_\theta \neq 0$ on $r = a$.

Use Bernoulli's Theorem on the stagnation point streamline

$\frac{1}{2}\rho U^2 + p_\infty = p + \frac{1}{2}\rho U^2 \frac{9}{4}\sin^2\theta$, $p = p_\infty + \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right)$

Note: $p_A = p_{A'} = p_\infty + \frac{1}{2}\rho U^2$ (high)

$p_B = p_\infty - \frac{5}{8}\rho U^2$ (low)

Pressure is symmetric fore and aft (around the equator), so the net force on the sphere is zero! (D'Alembert's Paradox)

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Fluid Dynamics (11)

Solid Sphere (non-examinable)

Empirically, $F = \rho U^2 \pi a^2 \times \frac{1}{2} C_D$, where C_D is a measured drag coefficient.
$C_D = C_D(Re) \simeq 0.4$ for large Re

Bubble

Potential flow solution is reasonable in several circumstances.
Exercise K.E. of fluid $= \int_{r>a} \frac{1}{2}\rho |\underline{u}|^2 dV = \frac{\pi}{3} a^3 \rho U^2$
$= \frac{1}{2} M_A U^2$
where M_A is the "added mass" (better terminology might be "added inertia")
$M_A = \frac{1}{2}\left(\frac{4}{3}\pi a^3\right)\rho = \frac{1}{2} M_D$ where M_D is the mass of fluid displaced.
Change in P.E. $= - M_D g h$ (missing mass)
So $\frac{1}{2} M_A U^2 - M_D g h = E$ (constant)
$\Rightarrow M_A U \dot{U} - M_D g U = 0 \quad (\dot{h} = U)$
$\Rightarrow \dot{U} = 2g$, so the bubble accelerates (upwards) at $2g$.

Two dimensional Potential Flows

(i) Point Source Solve $\nabla^2 \varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \varphi}{\partial r}\right) = 0$ for antisymmetric flow
or

From mass conservation:
$2\pi r u = q$ - constant source strength
$\Rightarrow u = \frac{q}{2\pi r} = \frac{\partial \varphi}{\partial r} \Rightarrow$ WLOG, $\varphi = \frac{q}{2\pi} \ln r$

ii) General Solution in Plane Polar coordinates
$\varphi = \frac{q}{2\pi} \ln r + \frac{\kappa}{2\pi}\theta + \sum_{n=1}^{\infty} \left(A_n r^n + B_n r^{-n}\right)\begin{Bmatrix}\cos n\theta \\ \text{or} \\ \sin n\theta\end{Bmatrix}$

iii) Point Vortex $\varphi = \frac{\kappa}{2\pi}\theta \Rightarrow \underline{u} = \nabla\varphi = \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\underline{e}_\theta = \frac{\kappa}{2\pi r}\underline{e}_\theta$

Exercise: Show that $\nabla \times \underline{u} = \underline{0}$ if $r \neq 0$
and that $\oint_C \underline{u}\cdot d\underline{l} = \begin{cases} \kappa & \text{if the origin is inside } C \\ 0 & \text{otherwise}\end{cases}$

The circulation $\oint_C \underline{u}\cdot d\underline{l} = \int_S \underline{\omega}\cdot\underline{n}\, dS$ $[\underline{\omega} = \nabla \times \underline{u}]$ is a measure of the vorticity enclosed within C.

(iv) Uniform Flow past a circle (or cylinder)

$\nabla^2 \varphi = 0 \quad (r > a)$

$\varphi \sim U r \cos\theta \quad (r \to \infty)$

$u_r = \frac{\partial \varphi}{\partial r} = 0 \quad (r = a)$, $\quad \varphi = U\left(r + \frac{a^2}{r}\right)\cos\theta + \frac{k}{2\pi}\theta$

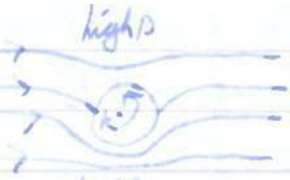
No net source of fluid, so $q = 0$ but we must allow a non-zero k to account for any vorticity in the viscous boundary layer near the surface of the circle.

Velocity $u_r = \frac{\partial \varphi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$, $\quad u_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{k}{2\pi r}$

Streamfunction (c.f. Sheet 1, Q8)

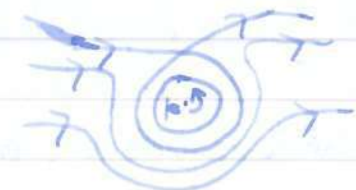
$\psi = U r \sin\theta\left(1 - \frac{a^2}{r^2}\right) - \frac{k}{2\pi}\ln\left(\frac{r}{a}\right)$

$k = 0$



$0 < k < 4\pi a U$

$4\pi a U < k$



Note : For steady potential flow, Bernoulli works everywhere, not just along streamlines.

So $p_\infty + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho\left(\frac{k}{2\pi a} - 2U\sin\theta\right)^2$ on the surface of the circle.

$\Rightarrow p = p_\infty + \frac{1}{2}\rho U^2 - \frac{\rho k^2}{8\pi^2 a^2} + \frac{\rho U k}{\pi a}\sin\theta - 2\rho U^2 \sin^2\theta$

Symmetric fore and aft, so that there is no net force in the flow direction.

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Fluid Dynamics (12)

Force in the y direction:

$$F_y = -\int_0^{2\pi} p\sin\theta \, a \, d\theta = -\int_0^{2\pi} \frac{\rho U k}{\pi a}\sin^2\theta \, a \, d\theta$$
$$= -\rho U k$$

In general, in "magnus force" resulting from the interaction between a flow $\underline{U}$ and a vortex $\underline{k}$ is

$\underline{F} = \rho \, \underline{U} \times \underline{k}$

Aerofoils generate circulation around the wing by having a sharp trailing edge, controlling separation.

Kutta Condition - the circulation is just sufficient to cause separation at the leading edge.

Pressure Field in (time dependent) potential flows

Euler Equation:

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \nabla\left(\tfrac{1}{2}|\underline{u}|^2\right) - \underline{u}\times\underline{\omega}\right) = -\nabla p - \nabla \chi$$

If $\underline{u} = \nabla\varphi$, $\underline{\omega} = \nabla\times\nabla\varphi = 0$

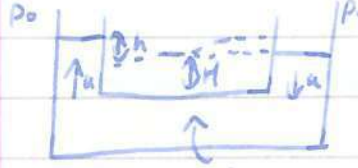
and we have

$$\nabla\left[\rho\frac{\partial\varphi}{\partial t} + \tfrac{1}{2}\rho|\underline{u}|^2 + p + \chi\right] = 0$$
$$\Rightarrow \rho\frac{\partial\varphi}{\partial t} + \tfrac{1}{2}\rho|\underline{u}|^2 + p + \chi = f(t) \text{, independent of position}$$

In particular, if the flow is steady, then

$H = \frac{1}{2}\rho|\underline{u}|^2 + p + \chi$ is constant throughout the fluid domain.

Oscillations in a manometer



Side arms have equal areas.

$u = \dot{h} \qquad \dot{u} = \ddot{h}$

Large cross section $\Rightarrow$ Slow flow $\Rightarrow \varphi \simeq$ constant $= 0$ WLOG

LHS $\varphi \simeq uy$, $\frac{\partial\varphi}{\partial t} = \dot{u}y = \ddot{h}y$

RHS $\varphi \simeq -uy$, $\frac{\partial\varphi}{\partial t} = -\dot{u}y = -\ddot{h}y$

Consider pressure on the two free surfaces.

$$\rho\ddot{h}\underbrace{y}_{H+h} + \cancel{\tfrac{1}{2}\rho\dot{h}^2} + \cancel{p_0} + \rho g h = -\rho\ddot{h}\underbrace{y}_{H-h} + \cancel{\tfrac{1}{2}\rho\dot{h}^2} + \cancel{p_0} - \rho g h$$

$$\Rightarrow 2\rho H\ddot{h} + 2\rho g h = 0 \Rightarrow \ddot{h} + \frac{g}{H}h = 0$$

This is the equation for SHM with frequency $\sqrt{\frac{g}{H}}$

Oscillations of a bubble

Consider the flow of fluid external to a bubble of radius $a(t)$.

$\nabla^2 \varphi = 0 \quad (r > a) \quad , \varphi \to 0 \quad (r \to \infty)$

$\frac{\partial \varphi}{\partial r} = \dot{a} \quad (r = a)$

Then $\varphi = \frac{A}{r} \Rightarrow -\frac{A}{a^2} = \dot{a} \Rightarrow \varphi = -\frac{a^2\dot{a}}{r}$

Ignore gravity and compare a point on the surface of the bubble with the far field

Note $\frac{\partial \varphi}{\partial t}\Big|_{r=a} = -\frac{a^2\ddot{a} + 2a\dot{a}^2}{r}\Big|_{r=a} = -(a\ddot{a} + 2\dot{a}^2)$

$\left[\neq \frac{d}{dt}\left[\varphi(r=a)\right] \right]$

$\Rightarrow \underbrace{-\rho(a\ddot{a} + 2\dot{a}^2)}_{\rho \frac{\partial \varphi}{\partial t}} + \underbrace{\tfrac{1}{2}\rho\dot{a}^2}_{\frac{1}{2}\rho u^2} + p = p_\infty(t)$

$\rho\left(a\ddot{a} + \tfrac{3}{2}\dot{a}^2\right) = p(a,t) - p_\infty(t)$

Small Oscillations of gas bubbles

$a = a_0 + \eta(t), \quad \eta \ll a$

$\rho\left((a_0+\eta)\ddot{\eta} + \tfrac{3}{2}\dot{\eta}^2\right) = p(a_0, t) + \eta \frac{\partial p}{\partial r}\Big|_{a_0} + \ldots - p_\infty$

If oscillations are adiabatic, then $pV^\gamma = $ constant (where γ is the ratio of specific heats) $\Rightarrow \frac{\delta p}{p} = -\gamma \frac{\delta V}{V} \approx -3\gamma \frac{\eta}{a_0} \quad (V \propto a^3)$

Therefore $\rho a_0 \ddot{\eta} \approx -3\gamma p_\infty \frac{\eta}{a_0}$

$\Rightarrow \ddot{\eta} + \frac{3\gamma p_\infty}{\rho a_0^2}\eta = 0$

SHM with frequency $\omega = \sqrt{\frac{3\gamma p_\infty}{\rho a_0^2}}$

Frequency $\approx 2\times10^{3}\ s^{-1}$ for a 1 cm air bubble in water.

High frequency, so the adiabatic assumption is valid.

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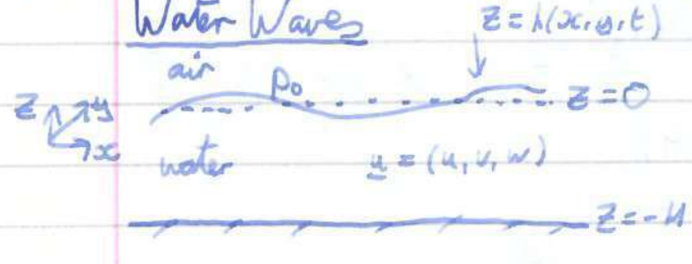
Fluid Dynamics (13)

Water Waves



Assume that the water is inviscid, and motion is started from rest. Then the flow is irrotational, $\underline{u} = \nabla\varphi$, and $\nabla^2\varphi = 0$ $(-H<z<$ (1)

Kinematic Boundary conditions:

At the rigid base $w \equiv \dfrac{\partial\varphi}{\partial z} = 0 \quad (z = -H)$ (2)

At the free surface $\dfrac{Dh}{Dt} \equiv \dfrac{\partial h}{\partial t} + u\dfrac{\partial h}{\partial x} + v\dfrac{\partial h}{\partial y} = w \quad (z = h)$

$$= \frac{\partial h}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial h}{\partial y} = \frac{\partial\varphi}{\partial z} \quad (3)$$

Dynamic Boundary Conditions: $p = p_0 \quad (z = h)$ (4)

Use expression for pressure in time-dependent potential flow:

$$\rho\frac{\partial\varphi}{\partial t} + \frac{1}{2}\rho|\nabla\varphi|^2 + p_0 + \rho g h = f(t) \quad (z = h) \quad (5)$$

Linearisation Assume $h \ll H$, $\underbrace{\dfrac{\partial h}{\partial x} \ll 1}$

wave heigh << wave length

Also assume that $|\underline{u}|$ is small. In consequence of these conditions:

a) Ignore terms of quadratic order or higher in disturbance quantities.

b) Use Taylor expansions to write (for example)

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=h} = \left.\frac{\partial\varphi}{\partial z}\right|_{z=0} + \underbrace{h\left.\frac{\partial^2\varphi}{\partial z^2}\right|_{z=0}}_{\text{quadratic}} + \ldots$$

Linear Water Waves

$$\nabla^2\varphi = 0 \quad (-H < z < 0) \quad (1)$$

$$\frac{\partial\varphi}{\partial z} = 0 \quad (z = -H) \quad (2)$$

$$\frac{\partial h}{\partial t} = \frac{\partial\varphi}{\partial z} \quad (z = 0) \quad (3)$$

$$\rho\frac{\partial\varphi}{\partial t} + p_0 + \rho g h = f(t) \quad (z = 0) \quad (4,5)$$

Look for 2D solutions (3D is not much harder) in the form of a single Fourier Mode (travelling sine wave)

$$h = h(x, t) = h_0 e^{i(kx - \omega t)}$$

(ω here is the angular frequency, and k is the horizontal wave number)

$$\varphi = \varphi(x, z, t) = \hat{\varphi}(z)\, e^{i(kx - \omega t)}$$

Substitute into Laplace's equation:

$-k^2\hat{\phi} + \hat{\phi}'' = 0$ ① $\quad \hat{\phi}' = 0$ at $z=-H$ ②

$\Rightarrow \hat{\phi} = \phi_0 \cosh k(z+H)$

Boundary conditions at $z=0$ give

$-i\omega h_0 = k\phi_0 \sinh kH$ ③

$e^{i(kx-\omega t)}\left(-i\omega \phi_0 \cosh kH + g h_0\right) = 0$ (4,5)

Combining the last two equations:

(Note: these are homogeneous eigenvalue equations for $\omega(k)$. They do not determine amplitudes ϕ_0, h_0)

$-\omega^2 \phi_0 \cosh kH + gk\phi_0 \sinh kH = 0$

$\Rightarrow \boxed{\omega^2 = gk \tanh kH}$ Dispersion relation

Note that waves of different wavenumbers k have different frequencies ω.

<u>Phase Speed</u> - the speed at which crests travel, for example

$c = \frac{\omega}{k} \quad \left[h \propto e^{i(kx-\omega t)} = e^{ik(x-ct)}\right]$

So $c^2 = \frac{\omega^2}{k^2} = \frac{g}{k}\tanh kH$

and c depends on wavelength.

Short waves (deep water)

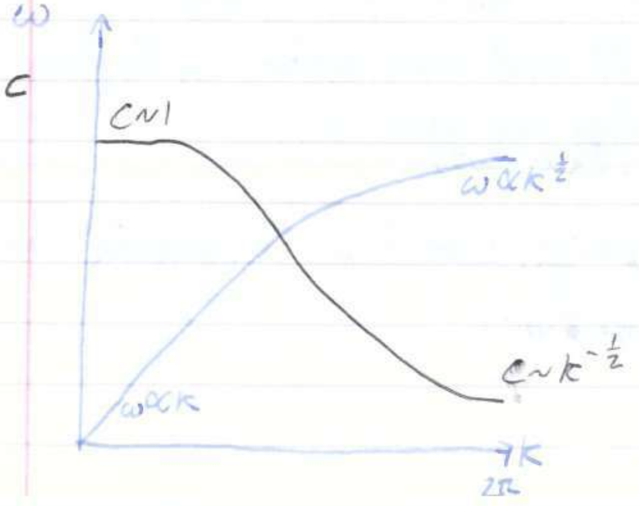
wavelength $\lambda = \frac{2\pi}{k} \ll H \Rightarrow kH \gg 1$

$\tanh kH \simeq 1 \Rightarrow \omega \sim \sqrt{gk}$, $c \sim \sqrt{\frac{g}{k}}$

Long waves (shallow water)

$\lambda \gg H$, $kH \ll 1$, $\tanh kH \simeq kH$

$\Rightarrow \omega \sim \sqrt{gH}\, k$, $c \sim \sqrt{gH}$, constant



07/03/12

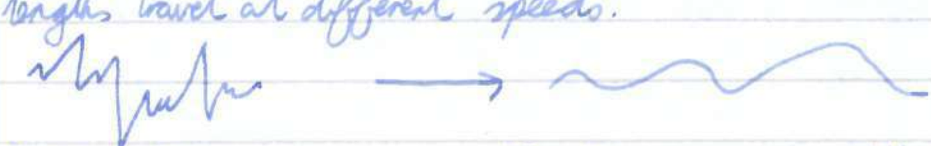
Fluid Dynamics (14)

Water Waves

$\omega^2 = gk \tanh kH$

Note:

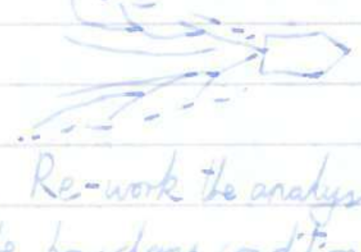
- Unlike light and sound, water waves are dispersive; waves of different lengths travel at different speeds.



Distant Storm — Smooth waves arrive at the beach, long waves first.

- Shallow-water waves (long waves) are approximately non-dispersive
e.g. Tsunamis: $H \approx 4\text{km}$, $\lambda \approx 500\text{km}$, shallow water waves
The constant wave speed $c \approx \sqrt{gH} \approx \sqrt{\underset{ms^{-2}}{10} \times \underset{m}{4 \times 10^3}} = 200\,ms^{-1}$
This is roughly the speed of a commercial aeroplane.

** NON EXAMINABLE
- The group velocity (velocity of energy propagation) is $c_g = \frac{\partial \omega}{\partial k}$
(See Part II Waves, or Asymptotic methods for proofs)
So for deep water waves
$\omega \sim \sqrt{gk}$, $c \sim \sqrt{\frac{g}{k}}$, $c_g \sim \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c$
The difference between the ~~wave~~ phase speed and the group velocity gives rise to the pattern of ship waves. For example:



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Exercise Re-work the analysis specifically for deep water waves using the boundary condition $\varphi \to 0$ as $z \to -\infty$
(Strictly, $\frac{\partial \varphi}{\partial z} \to 0$ as $z \to -\infty$)

Bound Waves e.g. in a square-cylinder

Linearised equations:

$\nabla^2 \varphi = 0 \qquad (z < 0)$

$\frac{\partial \varphi}{\partial z} \to 0 \qquad (z \to -\infty)$

$\frac{\partial \varphi}{\partial x} = 0 \qquad (x = 0, a)$

$\frac{\partial \varphi}{\partial y} = 0 \qquad (y = 0, a)$

$\frac{\partial h}{\partial t} = \frac{\partial \varphi}{\partial z} \qquad (z = 0)$

$\rho \frac{\partial \varphi}{\partial t} + \rho g h = P(t) \qquad (z = 0)$

Separation of Variables :
$\varphi = \varphi_0 \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right) e^{kz} e^{i\omega t}$
$h = h_0 \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right) e^{i\omega t}$
$k^2 = \frac{\pi^2}{a^2}(n^2+m^2)$, $\omega^2 = gk$ (as before)
The only difference is that k is quantised by the side walls. These are standing waves.

Rayleigh - Taylor Instability
The solution (in water) is exactly as for water waves but with $g \mapsto -g$
Therefore $\omega^2 = -gk$, $\omega = \pm i\sqrt{gk} \Rightarrow h \propto \underbrace{Ae^{\sqrt{gk}\,t}}_{\text{grows exponentially}} + \underbrace{Be^{-\sqrt{gk}\,t}}_{\text{decays exponentially}}$
Random Perturbation (initial condition) will have $A \neq 0$, so the disturbance will grow in time.

** Non Examinable

Rayleigh Taylor Instability with surface tension
Consider water waves first. $p = p_0 + \underset{\text{surface tension}}{\gamma} \times \text{curvature}$
$p = p_0 + \gamma \times \text{curvature} \simeq p_0 - \gamma \frac{\partial^2 h}{\partial x^2}$
if $\frac{\partial h}{\partial x} \ll 1$. So the linearised dynamic boundary condition becomes
$\rho \frac{\partial \varphi}{\partial t} + \rho g h - \gamma \frac{\partial^2 h}{\partial x^2} = f(t) \quad (z=0)$
In deep water : $\omega^2 = gk + \frac{\gamma}{\rho} k^3$
This is the dispersion relation for capillary - gravity waves.
For the Rayleigh Taylor instability, $\omega^2 = \frac{\gamma}{\rho} k^3 - gk$
Instability if $\omega^2 < 0 \Rightarrow k^2 < \frac{\rho g}{\gamma}$
$\Rightarrow (n^2+m^2)\pi^2 < \frac{\rho g a^2}{\gamma}$
Most unstable if $n=1$, $m=0$ (say)
$\Rightarrow a^2 > \frac{\gamma \pi^2}{\rho g}$, $a > \sqrt{\frac{\gamma\pi^2}{\rho g}} \approx 9\,\text{mm}$ for water over air

12/03/12

Fluid Dynamics (15)

Fluid Dynamics in a Rotating Frame

The Lagrangian (particle) acceleration in a rotating frame of reference is $\frac{D\underline{u}}{Dt} + 2\underline{\Omega}\times\underline{u} + \underline{\Omega}\times(\underline{\Omega}\times\underline{x})$ (See Dynamics)

So $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} + 2\underline{\Omega}\times\underline{u}\right) = -\nabla p - \rho\underline{\Omega}\times(\underline{\Omega}\times\underline{x}) + \rho\underline{g}$

where $\underline{\Omega}$ is the rotation rate of the frame.

Note that $\rho\underline{\Omega}\times(\underline{\Omega}\times\underline{x})$ is a constant (in time) and conservative body force

So the centrifugal force just adds to the hydrostatic pressure, so globally, the sea surface, for example, is not spherical.

For Earth, $\Omega \approx \frac{2\pi}{10^5}\,s^{-1}$, Largest scale $\approx 10^4$ km

So $\frac{\rho|\underline{\Omega}\times(\underline{\Omega}\times\underline{x})|}{\rho g} \le \frac{(2\pi)^2\times10^{-10}\times10^7}{10} \approx 4\times10^{-3} \ll 1$

So we will ignore the centrifugal term.

Consider motions for which $|\underline{u}\cdot\nabla\underline{u}| \ll |\underline{\Omega}\times\underline{u}|$

$\Rightarrow |\underline{\omega}| \ll \Omega$

relative vorticity $\ll$ planetary vorticity

e.g. an atmospheric weather system, $u = 10\,ms^{-1}$, $L \approx 10^3$ km

$\Rightarrow \frac{|\omega|}{|\Omega|} = \frac{10/10^6}{2\pi\times10^{-5}} \approx \frac{1}{2\pi}$, fairly small.

It is a reasonable approximation to ignore the advection term

With these two approximations, $\frac{\partial\underline{u}}{\partial t} + 2\underline{\Omega}\times\underline{u} = -\frac{1}{\rho}\nabla p + \underline{g}$

This is the Euler equation at small Rossby number $R_o = \frac{|\omega|}{|\Omega|} = \frac{u}{L\Omega}$ in a rotating frame. This approximation relates to

- strong rotation
- low fluid speeds
- large length scales

It is conventional to write $2\underline{\Omega} = \underline{f}$, $\underline{f}$ called the "planetary vorticity" or the "Coriolis Parameter". It is also conventional in this subject to use ζ for the relative vorticity $\nabla\times\underline{u}$.

1

Shallow Water Equations

Consider a layer of fluid of depth $h(x,y)$ with $p = p_0$ on $z = h(x,y)$, surface of the ocean or a constant pressure surface defining the top of the atmosphere

$$\nabla \cdot \underline{u} = 0 \Rightarrow \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \qquad \frac{w}{h} \sim \frac{u}{L}, \frac{v}{L}$$

$$\Rightarrow w \sim u\frac{h}{L}, v\frac{h}{L} \Rightarrow w << u, v$$

Consider $\underline{u} = (u, v, 0)$, $\underline{f} = (0, 0, f)$

$$\Rightarrow \rho\frac{\partial u}{\partial t} - \rho f v = -\frac{\partial p}{\partial x} \quad (1)$$

$$\rho\frac{\partial v}{\partial t} + \rho f u = -\frac{\partial p}{\partial y} \quad (2)$$

$$0 = -\frac{\partial p}{\partial z} - \rho g \quad (3)$$

From (3), $p = p_0 + \rho g\left[h(x,y) - z\right]$ applying $p = p_0$ on $z = h(x,y)$

Horizontal momentum equations

(1) $\rho\frac{\partial u}{\partial t} - \rho f v = -\rho g \frac{\partial h}{\partial x}$, Note that with our shallow-water

(2) $\rho\frac{\partial v}{\partial t} + \rho f u = -\rho g \frac{\partial h}{\partial y}$ approximation, pressure is hydrostatic

- Note also that horizontal accelerations are independent of z.
  Initial Conditions are usually such that u, v are also independent of z.
- Horizontal pressure gradients are proportional to horizontal variations in h.

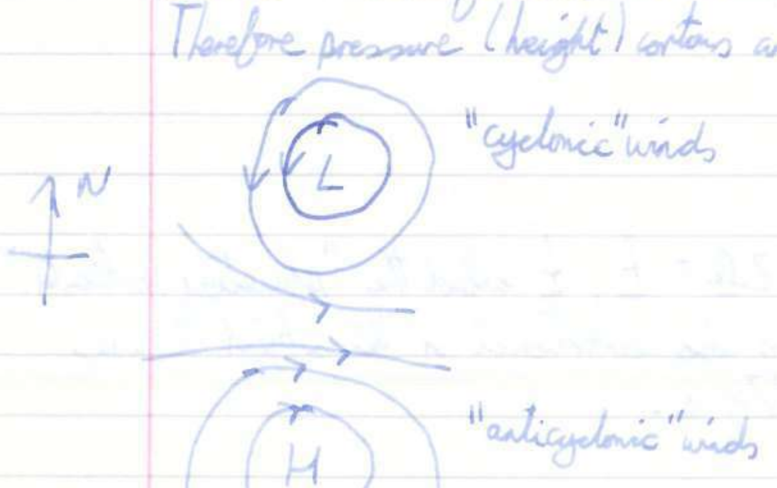
Geostrophic Balance

In steady state, (2) $u = \frac{\partial}{\partial y}\left(-\frac{gh}{f}\right) = \frac{\partial}{\partial y}\left(-\frac{p_g}{\rho f}\right)$ where p_g is the pressure on the base of the layer.

(1) $v = -\frac{\partial}{\partial x}\left(-\frac{gh}{f}\right) = -\frac{\partial}{\partial x}\left(-\frac{p_g}{\rho f}\right)$

So the 2D streamfunction $\psi = -\frac{gh}{f} = -\frac{p_g}{\rho f}$

Therefore pressure (height) contours are streamlines.



In the Northern hemisphere, with the wind on your back
Low is on your left
High is on your right

2

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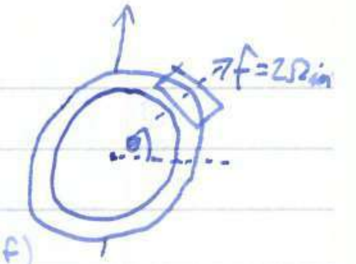
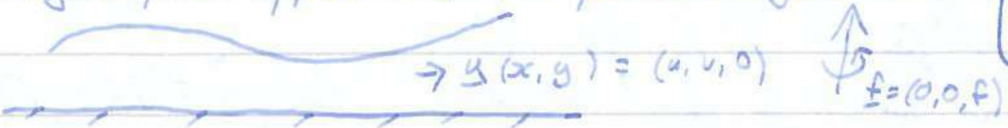
Fluid Dynamics (16)

Tangent-plane approximation / f-plane analysis





Mass Conservation

Consider a cylinder with horizontal cross-section $\mathcal{D}$

$$\frac{d}{dt}\int_{\mathcal{D}} \rho h \, dV = -\int_{\partial\mathcal{D}} h \rho \underline{u}_H \cdot \underline{n} \, dS$$

$$\int_{\mathcal{D}} \frac{\partial}{\partial t}(\rho h)\, dV = -\int_{\mathcal{D}} \nabla_H \cdot (\rho h \underline{u}_H)\, dV \qquad \left(\nabla_H = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right)$$

This is true for arbitrary domains, so $\frac{\partial h}{\partial t} + \nabla_H \cdot (h\underline{u}_H) = 0$

Note that we are still assuming ρ = constant.

In Cartesian components $\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$. Note $\nabla_H \cdot \underline{u}_H \neq 0$

Linearised equations of motion

Suppose $h = h_0 + \eta(x,y,t)$, $\eta \ll h$

① Then $\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$, $\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$ $\Big\rangle$ $\frac{\partial \underline{u}}{\partial t} + \underline{f}\times\underline{u} = -g\nabla\eta$

② $\frac{\partial \eta}{\partial t} + h_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$ $\qquad \frac{\partial \eta}{\partial t} + h_0 \nabla\cdot\underline{u} = 0$

(dropping the subscript H)

1. Exercise: For non-rotating, shallow-water waves ($f = 0$), show that $\frac{\partial^2 \eta}{\partial t^2} - g h_0\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) = 0$

   Wave equation with waves of speed $c = \sqrt{g h_0}$

2. Eliminate η from ① by taking the curl:

   $\frac{\partial \underline{\zeta}}{\partial t} + (\nabla\cdot\underline{u})\underline{f} = 0$ where $\underline{\zeta} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\underline{e}_z$ is the relative vorticity

   Substitute for $\nabla\cdot\underline{u}$ from ②

   $\Rightarrow \frac{\partial}{\partial t}\left(\underline{\zeta} - \frac{\eta}{h_0}\underline{f}\right) = 0$

   $\Rightarrow$ Potential Vorticity $\underline{Q} = \underline{\zeta} - \frac{\eta}{h_0}\underline{f}$ is constant in time at each location in space

   So $\underline{Q} = \underline{Q}_0(x,y)$ from initial conditions.

Aside In general (not linearised, or rapidly rotating)

$$\frac{D}{Dt}\left(\frac{1}{h}(\underline{\zeta} + \underline{f})\right) = 0$$

The quantity $\underline{\zeta} - \frac{\eta}{h_0}\underline{f}$ is the linearised form of the total potential vorticity, $\frac{1}{h_0}$ times $\frac{1}{h}(\underline{\zeta} + \underline{f})$

$\nearrow$ $\nabla\cdot\underline{u}$ can be non-zero

③ Take the divergence of ①: $\frac{\partial}{\partial t}(\nabla\cdot\underline{u}) - \underline{f}\cdot\nabla\times\underline{u} = -g\nabla^2\eta$

Substitute for $\nabla\cdot\underline{u}$ from ②

$-\frac{1}{h_0}\frac{\partial^2\eta}{\partial t^2} - \underline{f}\cdot\underline{\zeta} = -g\nabla^2\eta$ where $\underline{\zeta} = \nabla\times\underline{u}$ is the relative vorticity

Use conservation of potential vorticity to write $\underline{\zeta} = \underline{Q_0} + \frac{\eta}{h_0}\underline{f}$

$\Rightarrow \frac{\partial^2\eta}{\partial t^2} - gh_0\nabla^2\eta + f^2\eta = -h_0\underline{f}\cdot\underline{Q_0}$, independent of time

Note that the unforced equation ($Q_0 = 0$) supports waves with $\eta \propto e^{i\omega t}$

<u>Example</u> Suppose that there is a region of high pressure next to a region of low pressure.

In the non-rotating case, we get waves travelling in both directions.

In the rotating case, there is a non-trivial, geo-strophically balanced steady flow. Note that

$$\underline{Q_0} = \underline{\zeta}_0 - \frac{\eta}{h_0}\underline{f} = \mp\frac{\eta_0}{h_0}\underline{f} \quad \text{in } x \gtrless 0$$

$$\frac{\partial^2\eta}{\partial t^2} - gh_0\nabla^2\eta + f^2\eta = \pm f^2\eta_0 \quad \text{in } x \gtrless 0$$

Steady (Geo-strophically balanced) flow has $\eta = \eta(x)$

$$\eta'' - \frac{1}{R^2}\eta = \mp\frac{1}{R^2}\eta_0$$

where $R = \frac{\sqrt{gh_0}}{f}$, is the <u>Rossby radius</u> (of deformation).

Solve with $\eta \to 0$ as $x \to \pm\infty$

η, η' continuous at $x = 0$

$\hookrightarrow$ Continuity of u

$\hookrightarrow$ This gives continuity of pressure.

14/03/12

Fluid Dynamics (16)

$$\Rightarrow \eta = \begin{cases} \eta_0 \left(1 - e^{-x/R}\right) & x > 0 \\ -\eta_0 \left(1 - e^{x/R}\right) & x > 0 \end{cases}$$

$$u = \frac{\partial}{\partial y}\left(-\frac{g\eta}{f}\right) = 0 \quad , \quad v = -\frac{\partial}{\partial x}\left(-\frac{g\eta}{f}\right) = \eta_0 \sqrt{\frac{g}{H_0}}\, e^{-x/R}$$

The Rossby radius gives the characteristic length scale for balanced flow in the atmosphere and oceans.

In plan-view :

This is like a section through