21/11 Analysis ①

1) Limits and Convergence

SC Burkill Mathematical Analysis "A first Course"

M Spivak Calculus

Review from Numbers and Sets: a_n, a sequence of real numbers
Increasing $a_{n+1} \geq a_n \ \forall n$, decreasing $a_{n+1} \leq a_n \ \forall n$
"Monotone" denotes an increasing or decreasing sequence.

Strictly increasing $a_{n+1} > a_n \ \forall n$, strictly decreasing $a_{n+1} < a_n \ \forall n$

Definition $a_n \to a$ as $n \to \infty$ if given $\varepsilon > 0$, $\exists N \in \mathbb{Z}^+$, $|a_n - a| < \varepsilon$ for all $n \geq N$. Note $N = N(\varepsilon)$.

Fundamental Axiom of the Real numbers
Let a_n be an increasing sequence. Suppose there exists $A \in \mathbb{R}$ such that $a_n \leq A$ for all n. Then there exists $a \in \mathbb{R}$ such that $a_n \to a$. In other words, every increasing sequence bounded above, converges.
Equivalent Formulations of the fundamental axiom:
I. Every decreasing sequence bounded below, converges.
II Supremum: Let $S \subset \mathbb{R}$ be a nonempty subset. $k = \sup S$ if
i) $x \leq k \ \forall x \in S$
ii) Given $\varepsilon > 0$, $\exists x \in S$ such that $x > k - \varepsilon$
Every non empty subset of $\mathbb{R}$, bounded above, has a supremum.
(Bounded above: $A \in \mathbb{R}$ such that $x \leq A$ for all $x \in S$)

Lemma 1.1
i) The limit is unique. If $a_n \to a$, and $a_n \to b$, then $a = b$.
ii) If $a_n \to a$ and $n_1 < n_2 < n_3$, then $a_{n_j} \to a$ (subsequences converge to the same limit)
iii) If $a_n = c \ \forall n$, then $a_n \to c$
iv) If $a_n \to a$, $b_n \to b$, then $(a_n + b_n) \to (a+b)$
v) $a_n \to a$, $b_n \to b$, then $(a_n b_n) \to ab$
vi) If $a_n \to a$, $a_n \neq 0$, and $a \neq 0$, $\frac{1}{a_n} \to \frac{1}{a}$
vii) If $a_n \leq A \ \forall n$, $a_n \to a$, $a \leq A$.

Proof i), ii) and v) $a_n \to a$ means given $\varepsilon > 0$, $\exists N_1$ with $|a_n - a| < \varepsilon$ for all $n \geq N_1$
$a_n \to b$ means given $\varepsilon > 0$, $\exists N_2$ with $|a_n - b| < \varepsilon$ for all $n \geq N_2$
For $n \geq \max\{N_1, N_2\}$ $|a - b| \leq |a_n - a| + |a_n - b| < 2\varepsilon$
In other words $|a-b| < 2\varepsilon$. If $a \neq b$, $\varepsilon = \frac{|a-b|}{3}$
$|a-b| < \frac{2}{3}|a-b|$, absurd $\Rightarrow a = b$

$a_n \to a$ means given $\varepsilon > 0$ $\exists N$ with $|a_n - a| < \varepsilon$, $\forall n \geq N$
Note $n_j \geq j$, so if $j \geq N$, $|a_{n_j} - a| < \varepsilon$.

$a_n \to a$ means given $\varepsilon > 0$, $\exists N_1(\varepsilon)$, $|a_n - a| < \varepsilon$ $\forall n \geq N_1(\varepsilon)$
$b_n \to b$, means given $\varepsilon > 0$, $\exists N_2(\varepsilon)$ with $|b_n - b| < \varepsilon$ $\forall n \geq N_2(\varepsilon)$

$$|a_n b_n - ab| \leq |a_n b_n - a_n b| + |a_n b - ab| = |a_n||b_n - b| + |b||a_n - a|$$

Then for example $\varepsilon = 1$, $|a_n - a| < 1$ for $n \geq N_1(\varepsilon)$
$|a_n| \leq |a_n - a| + |a| < 1 + |a|$
$n \geq \max\{N_1(1), N_1(\varepsilon), N_2(\varepsilon)\}$ for this n, we have

$|a_n b_n - ab| \leq \varepsilon(1 + |a| + |b|)$ $\square$

Lemma 1·2 $\frac{1}{n} \to 0$
Proof $a_n = \frac{1}{n}$ is decreasing, and bounded below by 0, $(a_n > 0\ \forall n)$
By the fundamental axiom, there is $a \in \mathbb{R}$ with $\frac{1}{n} \to a$.

Look at $\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \to \frac{a}{2}$, by Lemma 1·1 (v)
However $\frac{1}{2n}$ is a subsequence of $\frac{1}{n}$ so $\frac{1}{2n} \to a$, $a = \frac{a}{2}$, $a = 0$
by Lemma 1·1 (ii). But the limit is unique, Lemma 1·1(i). $\square$

Remark The definition of convergence works equally well for the complex numbers $\mathbb{C}$.
$a_n \in \mathbb{C}$. $a_n \to a$ if given $\varepsilon > 0$, $\exists N$ with $|a_n - a| < \varepsilon$ for $n \geq N$
modulus in $\mathbb{C}$

Lemma 1·1 works for complex sequences except (vii) which uses the order of $\mathbb{R}$.

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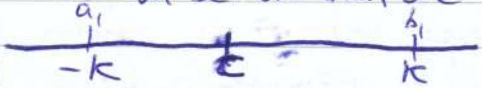
The Bolzano-Weierstrass Theorem

Theorem 1.3 If $x_n \in \mathbb{R}$, $\exists k$ such that $|x_n| \leq k \ \forall n$, then we can find $n_1 < n_2 < n_3$ and $x \in \mathbb{R}$ so that $x_{n_j} \to x$ (or in other words, any bounded sequence has a convergent subsequence.

Remark $x_n = (-1)^n$, $x_{2n+1} \to -1$, $x_{2n} \to 1$. Could be more than one such subseque

Proof Set $[a_n, b_n] = [-k, k]$

We consider the following alternatives:



Let c be the mid point of $[-k,k]$

① $x_n \in [a_1, c]$ for infinitely many values of n

② $x_n \in [b_1, c]$ for infinitely many values of n

If ① holds, set $a_2 = a_1$, $b_1 = c$

If ① fails ② holds, then $a_2 = c$ and $b_2 = b_1$

Proceed inductively to construct sequences a_n, b_n such that:

(**) $x_m \in [a_n, b_n]$ for infinitely many values of m

$a_{n-1} \leq a_n \leq b_n \leq b_{n-1}$

$$\boxed{b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} \quad *}$$

a_n is an increasing sequence bounded above

b_n is a decreasing sequence bounded below

By the Fundamental Axiom, $\exists a, b \in \mathbb{R}$ with $a_n \to a$ and $b_n \to b$

By (*) we see that $b - a = \frac{b-a}{2}$ so $b = a$

We claim that there are $n_1 \leq n_2 \leq n_3 < \ldots$ and x_{n_j} such that $a_j \leq x_{n_j} \leq b_j$ for all j.

We get this by induction: after having chosen n_j with $x_{n_j} \in [a_j, b_j]$ by (**) there are an infinite supply, so there is $n_{j+1} > n_j$ such that $x_{n_{j+1}} \in [a_{j+1}, b_{j+1}]$

$a_j \leq x_{n_j} \leq b_j$. $a_j \to a$, $b_j \to a$, must have $x_{n_j} \to a$ $\square$

Cauchy Sequences

Definition $a_n \in \mathbb{R}$ is a Cauchy sequence if given $\varepsilon > 0$, $\exists N = N(\varepsilon)$ such that $|a_m - a_n| < \varepsilon$ for all $m, n \geq N$

Lemma 1.4 A convergent sequence is a Cauchy sequence.

Proof If $a_n \to a$, given $\varepsilon > 0 \ \exists N$ such that $|a_n - a| < \varepsilon \ \forall n \geq N$

$m, n \geq N$

$|a_m - a_n| \leq |a_m - a| + |a_n - a| < 2\varepsilon$ $\square$

Theorem 1.5 Every Cauchy Sequence converges

Proof First we prove that a Cauchy sequence is bounded.

a_n being Cauchy means that given $\varepsilon > 0$, $\exists N$ such that $|a_m - a_n| < \varepsilon$ $\forall m, n \geq N$

~~$|a_m| \leq |a_m - a_n|$~~ Choose $\varepsilon = 1$, $N(1)$

$|a_m| \leq |a_m - a_{N(1)}| + |a_{N(1)}|$

$$|a_m| \leq |a_m - a_{N(1)}| + |a_{N(1)}|$$

If $m \geq N(1)$, $|a_m| \leq 1 + |a_{N(1)}|$

Take $k = \max\{1 + |a_{N(1)}|, |a_n|, n = 1, 2, \ldots, N-1\}$

For this choice of k, $|a_n| \leq k$ for all n.

$a_{n_j} \to a$

By the Bolzano - Weierstrass Theorem, a_n has a convergent subsequence

Now we show that a_n converges to a.

Since $a_{n_j} \to a$, given $\varepsilon > 0$ $\exists j_0$ such that $|a_{n_j} - a| < \varepsilon$ $\forall j \geq j_0$ and the sequence is Cauchy. Take j such that $n_j \geq \max\{n_{j_0}, N(\varepsilon)\}$

$$|a_n - a| \leq |a_n - a_{n_j}| + |a_{n_j} - a| < 2\varepsilon \quad \square$$

Remark We proved that a sequence converges IFF it is Cauchy, called "The General Principle of Convergence"

Series Definition $a_n \in \mathbb{R}$ (or $\mathbb{C}$). We say that $\sum_{j=1}^{\infty} a_j$ converges to S if the sequence of partial sums $S_N = \sum_{j=1}^{N} a_j \to S$ as $N \to \infty$

We write $\sum_{j=1}^{\infty} a_j = S$. If S_N does not converge we say that that the series $\sum_{j=1}^{\infty} a_j$ diverges.

Remark Any question on series can be turned into a question on sequences by considering partial sums.

Lemma 1.6 i) If $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ converge then so does $\sum_{j=1}^{\infty} (\lambda a_j + \mu b_j)$, $\mu, \lambda \in \mathbb{C}$

ii) Suppose $\exists N$ such that $a_j = b_j$ $\forall j \geq N$, then: either $\sum a_j$, $\sum b_j$ both converge, or both diverge

(initial terms do not affect convergence).

Proof i) $S_N = \sum_{j=1}^{N} (\lambda a_j + \mu b_j) = \lambda \sum_{j=1}^{N} a_j + \mu \sum_{j=1}^{N} b_j = \lambda C_N + \mu D_N$

If $C_N \to C$, $D_N \to D$ then $S_N \to \lambda C + \mu D$ by Lemma 1.1

ii) If $n \geq N$ $\quad S_n = \sum_1^n a_j = \sum_1^{N-1} a_j + \sum_N^n a_j$

$$d_n = \sum_1^n b_j = \sum_1^{N-1} b_j + \sum_N^n b_j$$

$$S_n - d_n = \sum_1^{N-1} a_j - \sum_1^{N-1} b_j \text{ since } a_j = b_j \quad \forall j \geq N$$

So S_n converges iff d_n converges

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An Important Series

$x \in \mathbb{R}$, $a_n = x^{n-1}$

$S_n = \sum_{j=1}^{n} a_j = 1 + x + x^2 + \dots + x^{n-1}$

$xS_n = x + x^2 + \dots + x^{n-1} + x^n = S_n - 1 + x^n$

$1 - x^n = S_n(1-x)$

$x = 1$, $S_n \to \infty$, $n \to \infty$

If $x \neq 1$, $S_n = \frac{1-x^n}{1-x}$

$|x| < 1$, $x^n \to 0$, $S_n \to \frac{1}{1-x}$, $\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$

If $x > 1$, $x^n \to \infty$, $S_n \to \infty$

If $x < -1$, S_n diverges (oscillates infinitely)

If $x = -1$, $S_n = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$ does not converge

The series converges if and only if $|x| < 1$

To see for example that $x^n \to 0$ if $|x| < 1$:

Consider $0 < x < 1$. $\frac{1}{x} = 1 + \delta$ for $\delta > 0$

$x^n = \frac{1}{(1+\delta)^n}$ $\qquad (1+\delta)^n > 1 + \delta n$ (from the Binomial theorem)

$x^n < \frac{1}{1+\delta n} \to 0$

A very easy observation: Lemma 1.7

If $\sum_{j=1}^{\infty} a_j$ converges, then $\lim_{j \to \infty} a_j = 0$

Proof $S_n = \sum_{1}^{n} a_j = S_{n-1} + a_n$, so if $S_n \to S$

$(S_n - S_{n-1}) \to S - S = 0$, so $a_n \to 0$ $\square$

Remark The converse is not true.

Consider $\sum_{n=1}^{\infty} \frac{1}{n}$. $a_n = \frac{1}{n} \to 0$

$S_n = \sum_{j=1}^{n} \frac{1}{j}$, $S_{2n} = S_n + \underbrace{\frac{1}{n+1}}_{\geq \frac{1}{2n}} + \underbrace{\frac{1}{n+2}}_{\geq \frac{1}{2n}} + \dots + \frac{1}{n+n}$ etc

$S_{2n} \geq S_n + \frac{1}{2}$

If $S_{\cancel{m}} \to a$, then $S_{2n} \to a \Rightarrow a \geq a + \frac{1}{2}$

This is absurd so $\sum \frac{1}{n}$ diverges.

Series of positive (non-negative) terms (working on $\mathbb{R}$, $a_n \geq 0$

Theorem 1.8 (The Comparison Test)

or forever after finitely many n

Suppose $0 \leq b_n \leq a_n$ $\forall n$. Then if $\sum_1^\infty a_n$ converges, so does $\sum_1^\infty b_n$

Proof Let $S_N = \sum_{n=1}^{N} a_n$, $d_N = \sum_{n=1}^{N} b_n$

$b_n \leq a_n \Rightarrow d_N \leq S_N$. But S_N converges to S for example

S_N and d_N are increasing ($a_n, b_n \geq 0$). Thus $d_N \leq S_N \leq S$

Because of the fundamental axiom, d_N converges. □

Example $\sum_1^\infty \frac{1}{n^2}$ $\frac{1}{n^2} < \frac{1}{n(n-1)}$ $(n \geq 2)$

$\cancel{\sum_1^\infty \frac{1}{n^2} \leq}$ $b_n = \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$, $\sum_2^N b_n = 1 - \frac{1}{2} + \frac{1}{2} - \dots - \frac{1}{N}$

$\sum_1^N b_n \to 1$. By the comparison test $\sum_1^\infty \frac{1}{n^2}$ converges.

In fact, $\sum_1^\infty \frac{1}{n^2} < 2$

Some consequences:

Theorem 1.9 (Root test) ~~Assume~~

Assume $a_n \geq 0$ and $a_n^{\frac{1}{n}} \to a$ as $n \to \infty$

Then if $a < 1$, $\sum a_n$ converges. If $a > 1$, $\sum a_n$ diverges.

~~Re~~ Remark Nothing can be said for $a = 1$.

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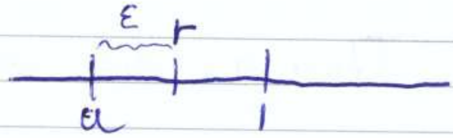
Analysis ③



__Proof__ Assume first $a < 1$

Pick r with $a < r < 1$

By definition of limit $\exists N$ such that $\forall n \geq N$, $a_n^{\frac{1}{n}} < r$

$a_n < r^n \quad \forall n \geq N$

Since $r < 1$, $\sum r^n$ converges, so by Theorem 1·8, $\sum_1^\infty a_n$ converges

If $a > 1$, take r with $1 < r < a$.

Then there is N such that $n \geq N$, $a^{\frac{1}{n}} > r$, that is

$a_n > r^n \quad \forall n \geq N$

$\Rightarrow a_n$ does not tend to zero as $n \to \infty$ and hence $\sum a_n$ must diverge

__Theorem 1·10 (Ratio test)__

Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to L$

If $L < 1$, $\sum a_n$ converges; if $L > 1$, then $\sum a_n$ diverges.

__Remark__ Nothing can be said for $L = 1$.

__Proof__ If $L < 1$, choose $L < r < 1$.

By definition of limit, $\exists N$ such that $\forall n \geq N$, $\frac{a_{n+1}}{a_n} < r$

$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}$

In other words, $a_n < kr^n$ where k is independent of n.

But $\sum kr^n$ converges (as $r < 1$) so by the comparison test,

$\sum a_n$ converges.

If $L > 1$, choose $1 < r < L$

Then $\exists N$ such that $\forall n \geq N$, $\frac{a_{n+1}}{a_n} > r \quad \forall n \geq N$

Similarly $a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N}$

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Thus a_n does not tend to zero and $\sum_1^\infty a_n$ diverges.

$n^{\frac{1}{n}} = 1 + \delta_n \quad , \delta_n > 0$

$n = (1+\delta_n)^n \doteq 1 + n\delta_n + \frac{n(n-1)}{2}\delta_n^2 + \dots$

$n = (1+\delta_n)^n \geq \frac{n(n-1)}{2}\delta_n^2$

$\Rightarrow \delta_n^2 \leq \frac{2}{n-1}$

$n \to \infty \Rightarrow \delta_n \to 0 \, , \; n^{\frac{1}{n}} \to 1$

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Analysis (4)

For $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$

Root test: If $a_n^{\frac{1}{n}} \to a$ then $\sum_{n=1}^{\infty} a_n$ converges for $a < 1$ and diverges for $a > 1$.

Ratio test: If $\frac{a_{n+1}}{a_n} \to a$ then $\sum_{n=1}^{\infty}$ converges for $a < 1$ and diverges for $a > 1$.

Examples $\sum_{n=1}^{\infty} \frac{n}{2^n}$ Ratio: $\frac{a_{n+1}}{a_n} = \frac{1}{2}\frac{n+1}{n} \to \frac{1}{2}$ convergence

Root: $\left(\frac{n}{2^n}\right)^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{2}$, $\lim n^{\frac{1}{n}} = 1$

$n^{\frac{1}{n}} = 1 + \delta_n$ $\delta_n > 0$

$n = (1+\delta_n)^n \geq \frac{n(n-1)}{2}\delta_n^2$ (Binomial Expansion)

$\delta_n^2 \leq \frac{2}{n-1} \to 0$ as $n \to \infty$

The tests are inconclusive if $a = 1$.

$\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ With both these series, the ratio and root tests give $a = 1$, but $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^2}$ converges

One further useful test for the case $a_n \geq 0$:

Theorem 1.11 Cauchy's Condensation Test

a_n is a decreasing sequence of positive terms. Then a_n converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$

Proof $a_{2^n} \leq a_{2^{n-1}+i} \leq a_{2^{n-1}}$ (*) $1 \leq i \leq 2^{n-1}$, $n \geq 1$

Suppose first $\sum_1^{\infty} a_n$ converges. We wish to show that $\sum_1^{\infty} 2^n a_{2^n}$ converges. $2^{n-1} a_{2^n} = \overbrace{a_{2^n} + a_{2^n} + \dots + a_{2^n}}^{2^{n-1}\text{ terms}}$

$\leq a_{2^{n-1}+1} + a_{2^{n-1}+2} + \dots + a_{2^n} = \sum_{j=2^{n-1}+1}^{2^n} a_j$

Thus $\sum_{n=1}^{N} 2^{n-1} a_{2^n} \leq \sum_{n=1}^{N} \sum_{j=2^{n-1}+1}^{2^n} a_j = \sum_{n=2}^{2^N} a_n$

$$\sum_{n=1}^{N} 2^n a_{2^n} \leq 2 \sum_{n=2}^{2^N} a_m \leq 2A \text{ , where } A = \sum_2^{\infty} a_m$$

Then by the fundamental axiom, $\sum 2^n a_{2^n}$ converges.

Conversely, assume now $\sum 2^n a_{2^n}$ converges. (*)

Then $\sum_{m=2^{n-1}+1}^{2^n} a_m = a_{2^{n-1}+1} + a_{2^{n-1}+2} + \dots + a_{2^n} \leq 2^{n-1} a_{2^{n-1}}$

$\Rightarrow \sum_{n=2}^{2^N} a_n = \sum_{n=1}^{N} \sum_{m=2^{n-1}+1}^{2^n} a_m \leq \sum_{n=1}^{N} 2^{n-1} a_{2^{n-1}} \leq B$

$\Rightarrow \sum_{m=2}^{N} a_m$ is bounded and therefore the series converges.

<u>Example/Application</u> $\sum_1^{\infty} \frac{1}{n^k}$ $(k > 0)$

$a_n = \frac{1}{n^k}$ IMPORTANT: Check that the a_n are positive and decreasing

$2^n a_{2^n} = 2^n \left(\frac{1}{2^n}\right)^k = 2^{n-nk} = (2^{1-k})^n$

$r = 2^{1-k}$, $2^n a_{2^n} = r^n$, geometric series

From Cauchy's Condensation test, $\sum \frac{1}{n^k}$ converges if and only if $r < 1$, i.e. $k > 1$.

<u>Alternating Series</u>

<u>Theorem 1.12 (The alternating series test)</u>

If a_n decreases and tends to zero as $n \to \infty$ then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

<u>Example</u> $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, $a_n = \frac{1}{n} \to 0$. Converges by the above test.

<u>Proof</u>

$S_n = \sum_1^n (-1)^{n+1} a_n = a_1 - a_2 + \dots + (-1)^{n+1} a_n$

as a_n is decreasing

$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2n-1} - a_{2n}) \geq S_{2n-2}$

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Analysis ④

S_{2n} is an increasing sequence

$S_n = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2n-2} - a_{2n-1}) - a_{2n}$
$\leq a_1$

as a_n is a decreasing sequence.

S_{2n} is bounded above, so by the fundamental axiom, $S_{2n} \to S$

$S_{2n+1} = S_{2n} + a_{2n+1}$

In the limit $S_{2n+1} \to S + 0$ (as $a_n \to 0$, $S_{2n} \to S$)

So, S_{2n} and S_{2n+1} both converge to the same limit S. This implies

$S_n \to S$, because given $\varepsilon > 0$,

$\exists N_1$ with $|S_{2n} - S| < \varepsilon \quad \forall n \geqslant N_1$

$\exists N_2$ with $|S_{2n+1} - S| < \varepsilon \ \forall n \geqslant N_2$

$N = 2\max\{N_1, N_2\}$, then for $k \geqslant N$, $|S_k - S| < \varepsilon$

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Analysis ⑤

Definition $a_n \in \mathbb{C}$

We say that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Example $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges to $\ln 2$ but is not absolutely convergent as $\sum \frac{1}{n}$ diverges.

Theorem 1.13

If $\sum a_n$ is absolutely convergent, then it is convergent.

Proof

First assume $a_n \in \mathbb{R}$ $\forall n$.

Let $v_n = \frac{|a_n| + a_n}{2}$, $w_n = \frac{|a_n| - a_n}{2}$

$v_n + w_n = |a_n|$, $v_n - w_n = a_n$

$v_n, w_n \geq 0$ $\qquad |a_n| \geq v_n, w_n$

Since $\sum |a_n|$ converges, by the comparison test $(v_n, w_n \geq 0)$ $\sum v_n$ and $\sum w_n$ converge. But since $a_n = v_n - w_n$, $\sum a_n$ must also converge

If $a_n \in \mathbb{C}$, $a_n = x_n + iy_n$, $x_n, y_n \in \mathbb{R}$

Note $|x_n|, |y_n| \leq |a_n|$

By the comparison test, $\sum |x_n|, \sum |y_n|$ converge.

By proof for $a_n \in \mathbb{R}$, $\sum x_n$ and $\sum y_n$ converge so $\sum a_n$ converges as $a_n = x_n + iy_n$ $\square$

2^{nd} Proof (using Cauchy Sequences) $\qquad S_n = \sum_{j=1}^{n} a_j$

It is enough to show that S_n is a Cauchy sequence.

$q \geq 1 \quad |S_{n+q} - S_n| = |\sum_{n+1}^{n+q} a_j| \leq \sum_{n+1}^{n+q} |a_j|$

$d_n = \sum_{j=1}^{n} |a_j| \qquad d_n$ converges $\Rightarrow d_n$ is Cauchy

But d_n being Cauchy means $\forall \varepsilon > 0 \ \exists N$ with

$\sum_{n+1}^{n+q} |a_j| = d_{n+q} - d_n < \varepsilon \quad \forall n \geq N, q \geq 1$

$\Rightarrow S_n$ is Cauchy. $\square$

The Check that a sequence converges if and only if it is Cauchy is also true over $\mathbb{C}$.

Example

$\sum_{n=1}^{\infty} \frac{z^n}{2^n} \quad z \in \mathbb{C} \qquad$ We wish to classify for $\forall z \in \mathbb{C}$.

Look at $\sum_{n=1}^{\infty} \frac{|z|^n}{2^n} \qquad$ Root test $\frac{|z|}{2} < 1 \Rightarrow$ Convergence

If $|z| \geq 2$, $|a_n| = \frac{|z|^n}{2^n} \geq 1$

Thus a_n does not tend to zero $\Rightarrow \sum \frac{z^n}{2^n}$ diverges for $|z| \geq 2$

Terminology

Sometimes a series which converges but is not absolutely convergent, like $\sum (-1)^{n+1} \frac{1}{n}$, is called "conditionally convergent" (or "converges, but not absolutely")

"Conditional" because the sum to which the series converges is conditional on the order in which the terms occur. If the terms are rearranged, the sum is altered.

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Analysis (5)

Rieman

One could rearrange series that converge, but not absolutely, so that the sum is anything we choose (not lectured).

Sanity is restored for absolutely convergent series.

Theorem 1.14

If $\sum a_n$ is absolutely convergent, every series consisting of the same terms in any order has the same sum. More formally:

$a_n \in \mathbb{C}$ $\qquad \sigma : \mathbb{N} \to \mathbb{N}$, a bijection $\quad a'_n = a_{\sigma(n)}$, a rearrangement

Proof

We do it first for $a_n \in \mathbb{R}$. Let $\sum a_n'$ be a rearrangement of $\sum a_n$. Let $S_n = \sum_{j=1}^{n} a_j$, $t_n = \sum_{j=1}^{n} a_j'$

$S = \sum_1^\infty a_n$.

Assume further that $a_n \geq 0$. Given n we can find q such that S_q contains every term of t_n.

If $a_n \geq 0$ then $t_n \leq S_q \leq S \Rightarrow t_n$ is an increasing sequence bounded above so it converges to t. Moreover, $t \leq S$.

By symmetry, $S \leq t$, so $t = S$.

If a_n has any sign, consider v_n and w_n from the proof of Theorem 1.13. Consider $\sum a_n'$, $\sum v_n'$, $\sum w_n'$.

Since $\sum |a_n|$ converges, both $\sum v_n$ and $\sum w_n$ converge as $v_n, w_n \leq |a_n|$

Now use that $v_n, w_n \geq 0$ to deduce that
$\sum v_n' = \sum v_n$ and $\sum w_n' = \sum w_n$
But $a_n' = v_n' - w_n'$ so $\sum a_n = \sum a_n'$

Finally, if $a_n \in \mathbb{C}$, $a_n = x_n + i y_n$. Since $|x_n|, |y_n| \leq |a_n|$, $\sum x_n$ and $\sum y_n$ converge absolutely and by the above rearrangements
$\sum x_n' = \sum x_n$, $\sum y_n' = \sum y_n$.
Since $a_n' = x_n' + i y_n'$ the result follows. $\square$

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2 Continuity

Setting $E \subseteq \mathbb{C}$, a non empty set

$f: E \to \mathbb{C}$ (this includes real valued functions)

Definition 1 f is continuous at $a \in E$ if for any sequence $z_n \in E$ with $z_n \to a$ we have $f(z_n) \to f(a)$

Definition 2 f is continuous at $a \in E$ if:

Given $\varepsilon > 0$, $\exists \delta > 0$ such that when

$|z - a| < \delta$, $z \in E$, then $|f(z) - f(a)| < \varepsilon$

The two definitions are equivalent.

Definition 2 $\Rightarrow$ Definition 1

We assume that given $\varepsilon > 0$, $\exists \delta > 0$ such that if

$|z - a| < \delta$, $z \in E$, then $|f(z) - f(a)| < \varepsilon$

Take $z_n \in E$, $z_n \to a$ by the definition of limit. Given $\delta > 0$,

$\exists n_0$ such that $\forall n \geq n_0$, $|z_n - a| < \delta$

$\Rightarrow |f(z_n) - f(a)| < \varepsilon$ i.e. $f(z_n) \to f(a)$

Definition 1 $\Rightarrow$ Definition 2:

Suppose Definition 2 is false. Then

$\exists \varepsilon > 0$ such that $\forall \delta$ $\exists z \in E$ with $|z - a| < \delta$,

and $|f(z) - f(a)| \geq \varepsilon$

Take $\delta = \frac{1}{n}$ then there is a $z_n \in E$ such that $|z_n - a| < \frac{1}{n}$

but $|f(z_n) - f(a)| \geq \varepsilon$

But $|z_n - a| < \frac{1}{n} \Rightarrow z_n \to a$

Since $|f(z_n) - f(a)| \geq \epsilon$, $f(z_n)$ does not tend to $f(a)$, which contradicts Definition 1.

Proposition 2.1

$a \in E$, $g, f : E \to \mathbb{C}$ which are continuous at a. Then so are the functions $f(z) + g(z)$, $f(z)g(z)$, $\lambda f(z)$ for any $\lambda \in \mathbb{C}$. In addition, if $f(z) \neq 0$ $\forall z \in E$ then $\frac{1}{f}$ is also continuous at a.

Proof: Consider $f+g$. Suppose $z_n \to a$ ($z_n \in E$). Since f and g are continuous at a, $f(z_n) \to f(a)$, $g(z_n) \to g(a)$ and by Lemma 1.1, $f(z_n) + g(z_n) \to f(a) + g(a)$ thus $f+g$ is continuous at a. The other claims also follow from Lemma 1.1

Consequence:

Any polynomial is a continuous function. By noting that $f(z) = z$ and $f(z) = \lambda$ are continuous we can construct any polynomial by Proposition 2.1.

Definition:

We say f is continuous on E if it is continuous at every point.

Compositions

Theorem 2.2 $f : A \to \mathbb{C}$, $g : B \to \mathbb{C}$ are two functions that can be composed, i.e. $f(A) \subset B$. Suppose f is continuous at $a \in A$, and g is continuous at $f(a)$. Then $g \circ f$ is continuous at a.

Proof

Take $z_n \to a$, $z_n \in A$. We are required to prove that $g(f(z_n)) \to g(f(a))$

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Analysis ⑥

Since f is continuous at a, $f(z_n) \to f(a)$.

Since g is continuous at $f(a)$, $g(f(z_n)) \to g(f(a))$

Examples

1. $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Assuming $\sin(x)$ is continuous, Proposition 2.1 and Theorem 2.2 imply that f is continuous for every $x \neq 0$

Discontinuous at 0: take the sequence $\frac{1}{x_n} = (2n+\frac{1}{2})\pi$, $f(x_n) = 1$ and $x_n \to 0$, but $f(0) = 0$ so f is not continuous at 0.

2. $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Proposition 2.1 and Theorem 2.2 imply continuity of f at every $x \neq 0$.

Let $x_n \to 0$, $|f(x_n)| \leq |x_n|$ because $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$

$\Rightarrow f(x_n) \to 0 = f(0) \Rightarrow$ continuity at 0.

3. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \text{ (rationals)} \\ 0 & x \notin \mathbb{Q} \end{cases}$ not continuous at any point.

03/02/11 Analysis (7)

Limits of functions

$f: E \subseteq \mathbb{C} \to \mathbb{C}$

We would like to define $\lim_{z \to a} f(z)$ even when a may not be in E. For example, $\lim_{z \to 0} \frac{\sin z}{z}$ for $f(z) = \frac{\sin z}{z}$, $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$

Definition

$E \subset \mathbb{C}$, $a \in \mathbb{C}$ and assume there exists a sequence $z_n \in E$ such that $z_n \to a$ and $z_n \neq a$ for all n. (Note that a may be, but may not be, in E). We say that $\lim_{z \to a} f(z) = L$ if given $\varepsilon > 0$, $\exists \delta > 0$ such that whenever $z \in E$, $0 < |z - a| < \delta$ then $|f(z) - L| < \varepsilon$ ("f tends to L as z tends to a")

A point a as in the definition is usually called a limit point of E.

Example

$E = \{0\} \cup [1,2]$. 0 is NOT a limit point.

If $a \in E$ and is not a limit point it is called isolated.

Remarks

1. $\lim_{z \to a} f(z) = L$ if and only if for every sequence $z_n \in E$, $z_n \neq a$ $(\forall n)$ and $z_n \to a$ we have $f(z_n) \to L$.

   This is proved exactly the same way as the equivalence of Definition (1) and Definition (2) for continuity of f.

2. If $a \in E$ and is a limit point then $\lim_{z \to a} = f(a)$ if and only if f is continuous at a. (Straight from the definitions)

This limit has very similar properties to the limits of sequences:

1. It is unique, $\lim_{z\to a} f(z) = A$, $\lim_{z\to a} f(z) = B \Rightarrow A = B$

$|A-B| \leq |f(z) - A| + |f(z) - B|$

$\lim_{z\to a} f(z) = A$, given $\varepsilon > 0$, $\exists\, \delta_1$ such that if $0 < |z-a| < \delta_1$, then $|f(z) - A| < \varepsilon$

$\lim_{z\to a} f(z) = B$ given $\varepsilon > 0$, $\exists\, \delta_2$ such that if $0 < |z-a| < \delta_1$ then $|f(z) - B| < \varepsilon$

Take $z \in E$ such that $0 < |z-a| < \delta_1, \delta_2$.

Such a z exists because a is a limit point. Then $|A-B| < 2\varepsilon$ for all $\varepsilon > 0 \Rightarrow A = B$

2. $f(z) + g(z) \to A + B$ if $f(z) \to A$, $g(z) \to B$ as $z \to a$
3. $f(z)\, g(z) = AB$
4. If $B \neq 0$, $\frac{f(z)}{g(z)} \to \frac{A}{B}$

all proved in the same way as before

Later on, $\frac{f(z) - f(a)}{z - a}$

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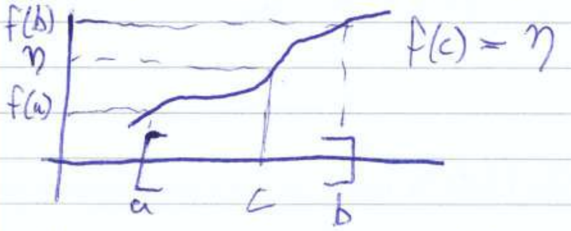
Analysis (7)

The Intermediate Value Theorem



Theorem 2·3 $f : [a,b] \to \mathbb{R}$, continuous

Suppose $f(a) \neq f(b)$. Then f takes every value that lies between $f(a)$ and $f(b)$.

Proof Without loss of generality, assume $f(a) < f(b)$ and take η such that $f(a) < \eta < f(b)$

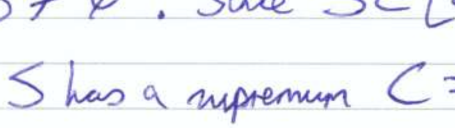
Consider the set $S = \{ x \in [a,b] : f(x) < \eta \}$

Note: $a \in S$ so $S \neq \emptyset$. Since $S \subset [a,b]$, b is an upper bound for S. Thus S has a supremum $C = \sup S$. By definition of supremum, given n, there is $x_n \in S$ such that $C - \frac{1}{n} < x_n < C$.

Note: $x_n \to C$

Since $x_n \in S$, $f(x_n) < \eta$. But f is continuous so $f(x_n) \to f(c)$

$\Rightarrow f(c) \leq \eta$.



Note $c \neq b$ because if $c = b$, $f(b) \leq \eta$ but $f(b) > \eta$.

Then for n large enough, $C + \frac{1}{n} \in [a,b]$

$C + \frac{1}{n} \to C$. Since $C + \frac{1}{n} > C$ then $f(C + \frac{1}{n}) \geq \eta$

By continuity, $f(C + \frac{1}{n}) \to f(C) \Rightarrow f(C) \geq \eta$

$\Rightarrow f(C) = \eta$ $\square$

Application : Existence of N-root of a positive number

$y > 0$, $y \in \mathbb{R}$, $c^N = y$, $f(x) = x^N$

on $[0, 1+y]$

$0 = f(0) < y < f(1+y) = (1+y)^N$

By the Intermediate Value Theorem $\exists\ c \in (0, 1+y)$ such that $c^N = y$.

c is unique. If d is another positive number with $d^N = y$.

If $d \neq c$ and $d < c \Rightarrow d^N < c^N \Rightarrow y < y$, absurd

$$a = b$$
$$a^2 = ab$$
$$a^2 + (a^2 - 2ab) = ab + (a^2 - 2ab)$$
$$2a^2 - 2ab = a^2 - ab$$
$$2(a^2 - ab) = (a^2 - ab)$$
$$2 = 1$$

$a > b \Rightarrow a = b + c$

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Analysis (8)

Bounds of Continuous Functions

Theorem 2·4 $f:[a,b]\to\mathbb{R}$ continuous. There exists $k>0$ such that $|f(x)|\le k$ for all $x\in[a,b]$. (is bounded)

Note $f(x)=\frac{1}{x}$ on $(0,1]$ is continuous but not bounded.

Proof By contradiction, suppose the statement is not true. Given $k>0$, $\exists x\in[a,b]$ such that $|f(x)|>k$. Choose $k=n$ (a positive integer) to get $x_n\in[a,b]$ such that $|f(x_n)|>n$

By Bolzano-Weierstrass, x_n has a convergent subsequence $x_{n_j}\to x$

Note that $a\le x_{n_j}\le b$ so $a\le x\le b$. f is continuous, so $f(x_{n_j})\to f(x)$

However, $|f(x_{n_j})|>n_j\to\infty$, which is absurd $\square$

Theorem 2·5 $f:[a,b]\to\mathbb{R}$, continuous (and attains its bounds *)

Then there exist $x_1,x_2\in[a,b]$ such that $f(x_1)\le f(x)\le f(x_2)$ for all $x\in[a,b]$

Proof 1 Consider $A=f([a,b])=\{f(x):x\in[a,b]\}$

$A\neq\emptyset$. By Theorem 2·4 the set A is bounded. Therefore A has a supremum $M=\sup A$. By definition of the supremum,

By definition of the supremum, given $n\in\mathbb{Z}^+$ $\exists x_n\in[a,b]$ such that $M-\frac{1}{n}<f(x_n)\le M$ (*). By Bolzano Weierstrass x_n has a convergent subsequence, $x_{n_j}\to x\in[a,b]$

f is continuous so $f(x_{n_j})\to f(x)$. But $f(x_{n_j})\to M$ by * so $f(x)=M$. So take $x_2=x$. An analogous argument gives $x_1\in[a,b]$ such that $f(x_1)\le f(x)$ $\forall x\in[a,b]$ $\square$

Proof 2 As in proof 1, consider A. $M = \sup A$

Suppose there is no $x \in [a,b]$ for which $f(x) = M$

Consider $g : [a,b] \to \mathbb{R}$, $g(x) = \frac{1}{M - f(x)}$ which is continuous on $[a,b]$. By Theorem 2.4, g is bounded so there is $k > 0$ such that $g(x) = \frac{1}{M-f(x)} \leq k$

$\Rightarrow f(x) \leq M - \frac{1}{k} < M$, absurd because $M = \sup A$ $\square$

Inverse Functions

Definition A function $f : [a,b] \to \mathbb{R}$ is said to be increasing if for any $x_1, x_2 \in [a,b]$ with $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$. Strictly increasing is $f(x_1) < f(x_2)$ (similar for decreasing)

Theorem 2.6 $f : [a,b] \to \mathbb{R}$ continuous and strictly increasing. Set $c = f(a)$ and $d = f(b)$. Then $f : [a,b] \to [c,d]$ is a bijection and its inverse $g = f^{-1} [c,d] \to [a,b]$ is continuous and strictly increasing.

Proof Take $k \in [c,d]$. By the intermediate value theorem, $\exists h \in [a,b]$ with $f(h) = k$ and since f is strictly increasing, h is uniquely determined by k.

Define $g : [c,d] \to [a,b]$ by $g(k) = h$ ($g(c) = a$, $g(d) = b$).

g is strictly increasing: $y_1 < y_2$ in $[c,d]$. But $y_i = f(x_i)$, $i = 1,2$ x_1, x_2 are uniquely defined. If $x_1 \geq x_2$, since f is strictly increasing, $f(x_1) \geq f(x_2) \Rightarrow y_1 \geq y_2$ which is clearly not true.

g is continuous:

Consider $k \in (c,d)$

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Analysis (9)

From last time: $f : [a,b] \to [c,d]$ $\quad c = f(a)$, $d = f(b)$

f is continuous and strictly increasing. It is <u>bijective</u>.

$g = f^{-1}[c,d] \to [a,b]$, g must also be continuous and strictly increasing

Proof that g is continuous:

Take $k \in (c,d)$, let $h = g(k)$. Let $\varepsilon > 0$ be given and small enough so that $h+\varepsilon, h-\varepsilon \in [a,b]$. Let $k_1 = f(h-\varepsilon)$, $k_2 = f(h+\varepsilon)$

f strictly increasing $\Rightarrow k_1 < k < k_2$

Take y such that $k_1 < y < k_2$. g is strictly increasing ~~$\Rightarrow g(k_1) < g(y) < ...$~~

$\Rightarrow g(k_1) < g(y) < g(k_2)$, $\quad h - \varepsilon < g(y) < h + \varepsilon$

i.e. $|g(y) - h| < \varepsilon$. Take $\delta = \min\{k_2 - k, k - k_1\}$

$\Rightarrow g$ is continuous. At the end points (c,d) the argument is very similar (check it!) $\square$

3 Differentiability

$f : E \subset \mathbb{C} \to \mathbb{C}$

<u>Definition</u>

Let $x \in E$ such that $x_n \in E$ with $x_n \neq x$ $\forall n$ and $x_n \to x$, a limit point. f is said to be differentiable at x, with derivative $f'(x)$ if $\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'(x)$. If f is differentiable at each point $x \in E$, we say f is differentiable on E.

Most of the time, $E = (a,b)$, $[a,b]$, some interval, or even a closed disc in $\mathbb{C}$.

<u>Remark 1</u> Terminology: $f'(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$ where $y = f(x)$

<u>Remark 2</u> Equivalently: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$, $y = x+h$

<u>Remark 3</u> Let $\mathcal{E}(h) = f(x+h) - f(x) - hf'(x)$

$\lim_{h \to 0} \frac{\mathcal{E}(h)}{h} = 0$

So we could have defined differentiability as ~~f~~follows: f is differentiable at x if there are $f'(x)$, $\mathcal{E}$ such that

$f(x+h) = f(x) + hf'(x) + \mathcal{E}(h)$ where $\lim_{h \to 0} \frac{\mathcal{E}(h)}{h} = 0$

$h \mapsto hf'(x)$, $\mathbb{R} \to \mathbb{R}$

<u>Remark 4</u> Minor Variations

- $f(x+h) = f(x) + hf'(x) + h\,\mathcal{E}(h)$ where $\lim_{h \to 0} \mathcal{E}(h) = 0$
- f is differentiable at $a \in E$ if $f(x) = f(a) + (x-a)f'(a) + (x-a)\mathcal{E}_f(x)$ where $\lim_{x \to a} \mathcal{E}_f(x) = 0$

<u>Remark 5</u>

If f is differentiable at $x \Rightarrow f$ is continuous at x

$f(x+h) = f(x) + \underbrace{hf'(x) + h\mathcal{E}(h)}_{0}$

Let $h \to 0$ $\qquad \lim_{h \to 0} f(x+h) = f(x)$

So f is continuous at x.

<u>Example</u> $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$

f is continuous $\forall x \in \mathbb{R}$

If $x > 0$, $f(x) = x$, $\frac{f(y) - f(x)}{y - x}$, y near x

$= \frac{y-x}{y-x} = 1$, $f'(x) = 1$

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Analysis (9)

if $x<0$, $f(x)=-x$, $f'(x)=-1$

f is not differentiable at 0.

$$\frac{f(y)-f(0)}{y-0} = \begin{cases} y/y = 1, & y>0 \\ -y/y = -1, & y<0 \end{cases} \quad \text{so } \lim_{y\to 0} \frac{f(y)-f(0)}{y-0} \text{ does not exist.}$$

<u>Proposition 3·1</u>

i) If $f(x)=c$ $\forall x \in E$ then f is differentiable at x with $f'(x)=0$

ii) If f,g are differentiable at x then so is $f+g$, and

$$(f+g)'(x) = f'(x)+g'(x)$$

iii) If f,g are differentiable at x, then so is fg.

$$(fg)'(x) = f'(x)g(x)+f(x)g'(x)$$

iv) If f is differentiable at x and $f(t)\neq 0$ $\forall t\in E$, then $\frac{1}{f}$ is differentiable at x and $\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{[f(x)]^2}$

<u>Proof</u> i) $\frac{f(y)-f(x)}{y-x} = \frac{c-c}{y-x} = 0$

$$\Rightarrow \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = f'(x) = 0$$

ii) $$\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} = \underbrace{\frac{f(x+h)-f(x)}{h}}_{\to f'(x)} + \underbrace{\frac{g(x+h)-g(x)}{h}}_{\to g'(x)}$$

let $h\to 0$, now the properties of limits, and differentiability of f and g at x gives $\lim_{h\to 0}\frac{1}{h}[f(x+h)+g(x+h)-f(x)-g(x)] = f'(x)+g'(x)$

iii) $\frac{1}{h}[f(x+h)g(x+h) - f(x)g(x)]$

$$= \underbrace{\frac{1}{h}f(x+h)}_{\to f(x)}\underbrace{[g(x+h)-g(x)]}_{\to g'(x)} + \frac{1}{h}g(x)\underbrace{[f(x+h)-f(x)]}_{\to f'(x)}$$

because f is continuous at x, since it is differentiable at x.

By the properties of limits this tends to $f'(x)g(x)+f(x)g'(x)$

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Analysis ⑩

<u>Proposition 3·1</u> iv)

If f is differentiable at x and $f(t) \neq 0 \;\forall t \in E$, then $\frac{1}{f}$ is differentiable at x and $\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{[f(x)]^2}$

Proof: $\dfrac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \dfrac{f(x) - f(x+h)}{h\, f(x+h) f(x)} \to -\dfrac{f'(x)}{[f(x)]^2}$

<u>Remark</u> If f and g are differentiable at x and g does not vanish, then f/g is differentiable at x and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$ from iii) and iv)

<u>Example</u> $f(x) = x^n \qquad n \in \mathbb{Z},\ n > 0$

$n = 1$, $f(x) = x$ is differentiable with $f'(x) = 1$

<u>Claim</u> $f'(x) = nx^{n-1}$ $\qquad$ ~~$f(x) = x^{n+1}$~~

<u>Induction</u> By proposition 3·1 $\quad f'(x) = (x\,x^n)' = 1x^n + x\,n\,x^{n-1}$
$= (n+1)x^n$

Any polynomial is therefore a differentiable function (Check $(x^{-n})' = -n\frac{1}{x^{n+1}}$

<u>Theorem 3·2 (The chain rule)</u>

$f: U \to \mathbb{C}$ such that $f(x) \in V \;\forall x \in U$. If f is differentiable at $a \in U$ and $g: V \to \mathbb{C}$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and $(g \circ f)' = g'(f(a))\,f'(a)$

<u>Proof</u>

f differentiable at a means that we can write

$f(x) = f(a) + (x-a)f'(a) + (x-a)\,\varepsilon_f(x)$ where

$\varepsilon_f \to 0$ as $x \to a$

Let $b = f(a)$. g differentiable at b means

$$g(y) = g(b) + (y-b)g'(b) + (y-b)\varepsilon_g(y) \text{ where}$$

$\varepsilon_g(y) \to 0$ as $y \to b$

Define $\varepsilon_f(a) = 0$, $\varepsilon_g(b) = 0$ so that they are continuous at $x = a$ and $y = b$ respectively.

$$\begin{aligned} g(f(x)) &= g(b) + [f(x) - b][g'(b) + \varepsilon_g(f(x))] \\ &= g(f(a)) + (x-a)[f'(a)g'(b)] \\ &\quad + \underbrace{(x-a)\left[\varepsilon_f(x)g'(b) + \varepsilon_g(f(x))\left(f'(a) + \varepsilon_f(x)\right)\right]}_{\sigma(x)} \end{aligned}$$

All we need to do is to check that $\lim_{x \to a} \sigma(x) = 0$

But σ is continuous at a since it is given as products, sums and compositions of continuous functions (ε_f, ε_g and f are continuous at the appropriate points).

Examples

1) $\sin(x^2)$ $\quad (\sin x)' = \cos x \quad (\sin x^2)' = 2x\cos x^2$

2) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases} \quad x \in \mathbb{R}$

We saw that f is continuous everywhere. If $f \neq 0$ f is differentiable because it is the product and composition of differentiable functions. (Prop 3·1 + Thm 3·2). What happens at $x = 0$?

$$\frac{f(x) - f(0)}{x - 0} = \frac{x\sin(\frac{1}{x}) - 0}{x} = \sin(\tfrac{1}{x})$$

But we know that the limit of $\sin(\frac{1}{x})$ as $x \to 0$ does not exist. $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist so f is not differentiable at 0

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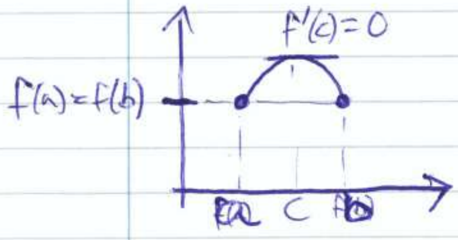
Analysis (10)

Now we look in detail at differentiable functions with $f : [a, b] \to \mathbb{R}$. We start with the following basic existence result.



Theorem 3·3 Rolle's Theorem

$f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$ and differentiable on (a, b).
If $f(a) = f(b)$ $\exists\, c$ such that $f'(c) = 0$.

Proof

Let $M = \max_{x \in [a,b]} f(x)$, $m = \min_{x \in [a,b]} f(x)$. We know from Theorem 2·5 that these values are achieved.

Let $k = f(a) = f(b)$. Note that $m \leq k \leq M$.

If $m = M = k$ then f is constant so $f'(c) = 0$ $\forall\, c \in (a, b)$

If f is not constant then $M > k$ or $m < k$.

Suppose $M > k$ (if $m < k$ the proof is similar). We know there exists $c \in (a, b)$ such that $f(c) = M$

If $f'(c) > 0$ then write $f(c+h) - f(c) = hf'(c) + h\mathcal{E}(h)$ because f is differentiable at c. $\mathcal{E}(h) \to 0$ as $h \to 0$.

Since $\mathcal{E}_h \to 0$ as $h \to 0$, $f'(c) + \mathcal{E}(h) > 0$ for small h.

But if h is small and positive then $f(c+h) > f(c) = M$, which is absurd. Similarly if $f'(c) < 0$ there is a point left of c

which gives a similar contradiction. Thus $f'(c) = 0$.

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<u>Theorem 3·3 Rolle's Theorem</u>

Let $f : [a,b] \to \mathbb{R}$ which is differentiable on (a,b).

If $f(a) = f(b)$ then $\exists\, c \in (a,b)$ with $f'(c) = 0$.

<u>Theorem 3·4 - The mean value theorem</u>

Let $f : [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b). Then $\exists\, c \in (a,b)$.

$$f(b) - f(a) = f'(c)(b-a)$$

we wish to find k so that $\phi(a) = \phi(b)$

<u>Proof</u> $\phi(x) = f(x) - kx$, where k is a constant. Note that ϕ is continuous on $[a,b]$ and differentiable on (a,b)

$\phi(a) = \phi(b) \Rightarrow f(b) - kb = f(a) - ka$ so

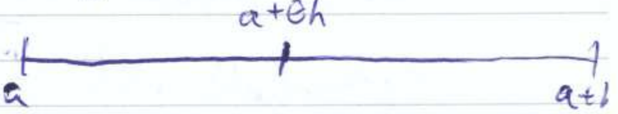
$k = \frac{f(b) - f(a)}{b-a}$. For this k, ϕ satisfies the hypothesis of Rolles Theorem, thus $\exists\, c \in (a,b)$ such that $\phi'(c) = 0$.

But $\phi'(x) = f'(x) - k \Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$ $\square$

<u>Remark</u>

One often rewrites the mean value theorem as follows:

$f(a+h) = f(a) + hf'(a+\theta h)$



where $\theta \in (0,1)$ $(b = a+h)$

<u>¡¡Warning!!</u> θ depends on h.

<u>Corollary</u> $f : [a,b] \to \mathbb{R}$ continuous, differentiable on (a,b)

i) If $f'(x) > 0$ $\forall x \in (a,b)$ then f is strictly increasing on $[a,b]$ $(b \geq y > x \geq a \Rightarrow f(y) > f(x))$

ii) $f'(x) \geqslant 0 \ \forall x \in (a,b)$ then f is increasing in $[a,b]$ $(b \geqslant y > x \geqslant a \Rightarrow f(y) \geqslant f(x))$

iii) If $f'(x) = 0 \quad \forall x \in (a,b)$, then f is constant on $[a,b]$

Proof

i) By the mean value theorem applied to f on $[x,y]$, $\exists c \in (x,y)$ such that $f(y) - f(x) = f'(c)(y-x) > 0$

ii) Same as above: $f'(c) \geqslant 0 \Rightarrow f(y) - f(x) = f'(c)(y-x) \geqslant 0$

iii) Take $x \in [a,b]$. Apply the mean value theorem to f on the interval $[a,x]$:

$\exists c \in (a,x)$ such that $f(x) - f(a) = f'(c)(x-a)$

but $f'(c) = 0 \Rightarrow f(x) = f(a)$, any point.

Inverse rule (Inverse Function Theorem)

Theorem 3.6

$f: [a,b] \to \mathbb{R}$ continuous which is differentiable on (a,b) and $f'(x) > 0 \ \forall x \in (a,b)$. Let $f(a) = c$, $f(b) = d$. Then the function $f[a,b] \to [c,d]$ is bijective and f^{-1} is differentiable with $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$, on (c,d)

Proof

By Corollary 3.5 f is strictly increasing on $[a,b]$.
By Theorem 2.6, $\exists g: [c,d] \to [a,b]$ which is a continuous, strictly increasing inverse for f.

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Analysis (11)

We are required to prove that g is differentiable and

$g'(y) = \frac{1}{f'(x)}$ ~~and~~ where $y = f(x)$, $x \in (a, b)$

Let $k \neq 0$, small, we need to look at $\frac{g(y+k) - g(y)}{k}$

Let h be given by $y + k = f(x+h)$ [or $x + h = g(y+k)$]

$$\frac{g(y+k) - g(y)}{k} = \frac{\cancel{x}+h-\cancel{x}}{f(x+h) - f(x)} = \frac{1}{\frac{1}{h}[f(x+h) - f(x)]} \to \frac{1}{f'(x)}$$

Let $k \to 0$, then $h \to 0$ because g is continuous $\square$

<u>Example</u>

Let $q \in \mathbb{Z}^+$, $f(x) = x^q$. $g(x) = x^{1/q}$ $(x > 0)$

Since f is differentiable and $f'(x) = q\,x^{q-1}$

$\Rightarrow g$ is differentiable and $g'(x) = \frac{1}{q\,(x^{\frac{1}{q}})^{q-1}}$ by the Inverse Function Theorem.

$g'(x) = \frac{1}{q} x^{\frac{1}{q} - 1}$

Now check that if $f(x) = x^r$, r rational, then $f'(x) = r x^{r-1}$

Later on we'll see how to define x^r for irrational r and $f'(x) = r x^{r-1}$ is still true.

<u>Next</u>

$f, g : [a, b] \to \mathbb{R}$, continuous and differentiable on (a, b)

with $g(b) \neq g(a)$. By the mean value theorem $\exists t \in (a, b)$ such that

$g(b) - g(a) = g'(t)(b - a)$

Also $\exists s \in (a, b)$ such that $f(b) - f(a) = f'(s)(b-a)$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(s)}{g'(t)}$$

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x} = \frac{e^x - e^0}{\sin x - \sin 0} \qquad [a,b] = [0,x]$$

$$\text{If } s = t \quad \frac{e^t}{\cos t}$$

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Theorem 3·7 (Cauchy's Mean Value Theorem)

Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions which are differentiable on (a, b). Then $\exists\, t \in (a, b)$ such that

$$[f(b) - f(a)]\, g'(t) = f'(t)\,[g(b) - g(a)]$$

$\lim_{x \to 0} \frac{e^x - 1}{\sin x}$ Application!

Use $[0, x]$, $f(x) = e^x$, $g(x) = \sin x$

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x} = \lim_{x \to 0} \frac{f(x) - f(0)}{g(x) - g(0)} = \lim_{x \to 0} \frac{f'(t)}{g'(t)} = \lim_{t \to 0} \frac{e^t}{\cos t} = 1$$

$t \in (0, x)$, $x \to 0 \Rightarrow t \to 0$

Proof

Let $h(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}$ $h : [a, b] \to \mathbb{R}$ continuous, differentiable on (a, b)

$h(a) = 0$, $h(b) = 0$

Rolle's Theorem $\Rightarrow \exists\, t \in (a, b)$ such that $h'(t) = 0$

$h'(x) = f'(x) g(b) - g'(x) f(b) + g'(x) f(a) - g(a) f'(x)$

so $h'(t) = 0$ gives the result $\square$

We wish to extend the Mean Value Theorem to include higher order derivatives.

Theorem 3·8 (Taylor's Theorem with Lagrange's Remainder)

Suppose f and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exists for $x \in (a, a+h)$

Then:

$$f(a+h) = f(a) + h f'(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \underbrace{\frac{h^n}{n!} f^{(n)}(a + \theta h)}_{R_n}$$

where $\theta \in (0, 1)$

Remarks

1. The case $n=1$ is exactly the Mean Value Theorem
2. R_n is called the Lagrange form of the remainder.

Proof (Strategy is always the same: Choose a suitable auxiliary function and apply Rolle's Theorem)

$0 \leq t \leq h$

(*) Let $\phi(t) = f(a+t) - f(a) - tf'(a) - \ldots - \frac{t^{n-1}}{(n-1)!} f^{(n-1)}(a) - B\frac{t^n}{n!}$

where B is a constant.

Now choose B so that $\phi(h) = 0$. Note that $\phi(0) = 0$ and $\phi'(0) = 0$, and all derivatives up to $\phi^{(n-1)}(0) = 0$

By Rolle's Theorem applied to ϕ, $\exists h_1 \in (0, h)$ such that $\phi'(h_1) = 0$. Now apply Rolle's Theorem to ϕ' in the interval $[0, h_1]$ to get $h_2 \in (0, h_1)$ for which $\phi''(h_2) = 0$. Continue applying Rolle's Theorem to obtain $0 < h_n < h_{n-1} < \ldots < h_1 < h$ such that $\phi^{(j)}(h_j) = 0$ for $j \in \{1, 2, \ldots, n\}$

$\phi^{(n)}(t) = f^{(n)}(a+t) - B$. But $\phi^{(n)}(h_n) = 0$ so if we set $h_n = \theta h$ with $\theta \in (0,1)$ $\Rightarrow B = f^{(n)}(a + \theta h)$

Go back to (*), set $t=h$ and use this value of B $\square$

Remark

The case $a=0$ is sometimes called Maclaurin's Theorem.

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Theorem 3.9 (Taylor's Theorem with Cauchy's form of remainder)

With the same hypothesis as Theorem 3.8, and $a = 0$ (to simplify)

we have:

Lagrange's Remainder $\boxed{\dfrac{f^{(n)}(\theta h)h^n}{n!}}$

$$f(h) = f(0) + hf'(0) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

where $R_n = \dfrac{(1-\theta)^{n-1}f^{(n)}(\theta h)h^n}{(n-1)!}$, $\theta \in (0,1)$

Proof: For $0 \le t \le h$

Define $F(t) = f(h) - f(t) - (h-t)f'(t) - \dots \dfrac{(h-t)^{n-1}f^{(n-1)}(t)}{(n-1)!}$

$$F'(t) = \cancel{-f'(t)} + \cancel{f'(t)} - \cancel{(h-t)f''(t)} + \cancel{(h-t)f''(t)} - \cancel{\frac{(h-t)^2}{2}f'''(t)} - \dots - \frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$$

Note!! $F(h) = 0$

$$= -\frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$$

$$\phi(t) = F(t) - \left(\frac{h-t}{h}\right)^p F(0) \qquad p \in \{1, 2, \dots, n\}$$

$$\phi(0) = F(0) - F(0) = 0$$

$$\phi(h) = F(h) - 0 = 0$$

Apply Rolle to ϕ to obtain $\theta \in (0,1)$ such that $\phi'(\theta h) = 0$

$$\phi'(t) = F'(t) + \frac{p}{h}\left(\frac{h-t}{h}\right)^{p-1}F(0)$$

$$\phi'(\theta h) = 0 \Rightarrow F'(\theta h) + \frac{p}{h}(1-\theta)^{p-1}F(0) = 0$$

$$\Rightarrow 0 = -\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta h) + \frac{p(1-\theta)^{p-1}}{h}\left[f(h) - f(0) - hf'(0) - \dots - \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0)\right]$$

$$\Rightarrow f(h) = f(0) + hf'(0) + \dots + \frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!}$$

$$+ \frac{h^n(1-\theta)^{n-1}}{(n-1)!\,p(1-\theta)^{p-1}}f^{(n)}(\theta h) \quad \leftarrow R_n$$

$p = 1$ We get Cauchy's Form of remainder

$p = n$ We get Lagrange's Form

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Taylor's Theorem:

If $f, f', \ldots, f^{(n-1)}$ exist and are continuous on $[0, h]$ and $f^{(n)}(x)$ exists $\forall x \in (0, h)$ then

$$f(h) = \sum_{i=0}^{n-1} \frac{f^{(i)}(0)h^i}{i!} + R_n$$

where $R_n = \begin{cases} \frac{h^n f^{(n)}(\theta h)}{n!} & \text{Lagrange} \\ \frac{(1-\theta)^{n-1} f^{(n)}(\theta h) h^n}{(n-1)!} & \text{Cauchy} \end{cases}$

¡NOTE! θ depends on n

The same result holds on an interval $[h, 0]$ with $h < 0$

The Taylor series: Requires proof that $R_n \to 0$ as $n \to \infty$

This requires estimates (meaning effort!).

Example The Binomial Series $(r \in \mathbb{Q})$

$$f(x) = (1+x)^r$$

Claim $f(x)$ can be written in the form: $f(x) = 1 + \binom{r}{1}x + \binom{r}{2}x^2 + \ldots$

where $\binom{r}{n} = \frac{r(r-1)\ldots(r-n+1)}{n!}$ where $|x| < 1$

If $r \in \mathbb{Z}^+$, then $f^{(n+1)} = 0$ and we have a polynomial of degree r.

First we look at absolute convergence of $\sum_{i=0}^{\infty} \binom{r}{i} x^i$

Note: can clearly assume $x \neq 0$

$$\sum_{i=0}^{\infty} \left|\binom{r}{i} x^i\right| = \sum_{i=0}^{\infty} a_i$$

Ratio test $\frac{a_{n+1}}{a_n} = \left|\frac{r(r-1)\ldots(r-n+1)(r-n)}{(n+1)!} x^{n+1}\right| \left|\frac{n!}{r(r-1)\ldots(r-n+1)x^n}\right|$

$$\frac{a_{n+1}}{a_n} = \left|\frac{r-n}{n+1} x\right| \to |x| \text{ as } n \to \infty$$

So if $|x| < 1$, the ratio test $\Rightarrow$ convergence. In particular,

$a_n \to 0$, $\binom{r}{n} x^n \to 0$ as $n \to \infty$ provided $|x| < 1$

We estimate R_n

$$f^{(n)}(x) = r(r-1)\ldots(r-n+1)(1+x)^{r-n}$$

Lagrange: $R_n = \frac{x^n r(r-1)\dots(r-n+1)(1+\theta x)^{r-n}}{n!}$

$R_n = \binom{r}{n} x^n (1+\theta x)^{r-n}$

For $0 < x < 1$, $1+\theta x > 1$ so if $n > r$, $(1+\theta x)^{r-n} < 1$

$|R_n| \leq \left|\binom{r}{n} x^n\right|$ for $n > r$, so if $n \to \infty$, $R_n \to 0$

But for $-1 < x < 0$ this argument fails.

So we use Cauchy: $R_n = \frac{(1-\theta)^{n-1} r(r-1)\dots(r-n+1)(1+\theta x)^{r-n} x^n}{(n-1)!}$

$R_n = r\binom{r-1}{n-1} x^n (1+\theta x)^{r-1} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$

$\frac{1-\theta}{1+\theta x} < 1$ $\qquad r\binom{r-1}{n-1} x^n \to 0$ as $n \to \infty$

We deal with $(1+\theta x)^{r-1}$. Note that $1+\theta x < 1$ if $-1 < x < 0$

if $r-1 \geq 0$, $(1+\theta x)^{r-1} \leq 1$

if $r-1 < 0$, $\frac{1}{(1+\theta x)^{1-r}} < \frac{1}{(1+x)^{1-r}}$

In any case, $(1+\theta x)^{r-1} \leq \max\left\{1, \frac{1}{(1+x)^{1-r}}\right\}$

$\Rightarrow |R_n| \leq k_r \left|\binom{r-1}{n-1} x^n\right|$, where $k_r = r\max\left\{1, \frac{1}{(1+x)^{1-r}}\right\}$

So k_r is independent of $\underline{\underline{n}}$. (but not r of course).

So when $n \to \infty$, $R_n \to 0$ $\square$

Note: $f(x) = (1+x)^r$, $r \in \mathbb{Q}$. If we had defined x^r for $r \in \mathbb{R}$, which we will, we get the same binomial theorem.

<u>Remarks on differentiation of functions</u> $f: E \subseteq \mathbb{C} \to \mathbb{C}$

Complex Differentiable functions are very special.

All the results like sums, products and compositions of differentiable functions still hold in this case.

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Analysis (13)

Example $f: \mathbb{C} \to \mathbb{C}$

$f(z) = \bar{z}$. Fix $z \in \mathbb{C}$. $z_n = z + \frac{1}{n}$

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} + \frac{1}{n} - \bar{z}}{z + \frac{1}{n} - z} = 1$$

Now choose $z_n = z + \frac{i}{n}$

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} - \frac{i}{n} - \bar{z}}{z + \frac{i}{n} - z} = -1$$

$\Rightarrow \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$ does not exist.

$\Rightarrow f(z) = \bar{z}$ is nowhere differentiable on $\mathbb{C}$.

$f: \mathbb{R}^2 \to \mathbb{R}^2$, $z = x + iy$, $f(x,y) = (x, -y)$

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Analysis (14)

4 Power Series

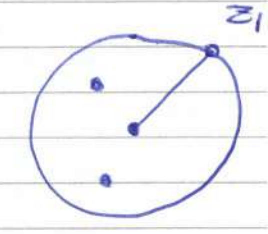
$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots, \quad a_n, z \in \mathbb{C}$$

The more general case $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, for fixed $z_0 \in \mathbb{C}$, follows from the case $z=0$ by translation, i.e. $w = z - z_0$.

Question: For which values of z do we have convergence?

Lemma 4.1

If $\sum_{n=0}^{\infty} a_n z_1^n$ converges, and $|z| < |z_1|$, then $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent.



Proof

Since $\sum a_n z_1^n$ converges, we know that $a_n z_1^n \to 0$ as $n \to \infty$. In particular, $\exists k > 0$ such that $|a_n z_1^n| \le k \ \forall n$

$$|a_n z^n| = |a_n z_1^n| \frac{|z|^n}{|z_1|^n} \le k \left(\frac{|z|}{|z_1|}\right)^n$$

$|z| < |z_1| \Rightarrow \sum \left(\frac{|z|}{|z_1|}\right)^n$ is a convergent geometric series.

By comparison, $\sum |a_n z^n|$ converges. $\square$

Theorem 4.2

A power series either

(i) Converges absolutely for all z, or

(ii) Converges absolutely for all z inside a circle $|z| = R$ and diverges for all z outside, or

(iii) Converges only for $z = 0$

Definition

The circle $|z| = R$ is called the circle of convergence and R the radius of convergence. In (i) we agree that $R = \infty$ and in (iii) $R = 0$.

Proof of 4·2

$S = \{x \in \mathbb{R} : x \geq 0, \sum a_n x^n \text{ converges}\}$

Clearly, $0 \in S$, so $S \neq \emptyset$. By Lemma 4·1, if $x_1 \in S$, then $[0, x_1] \subset S$. If S is unbounded, then it must be the whole real line, $[0, \infty)$, so we have case (i).

If S is bounded, then it has a supremum, $R = \sup S \geq 0$.

If $R > 0$, we show that if $|z_1| < R$, then the series is absolutely convergent:

Choose R_0 such that $|z_1| < R_0 < R$. Then $R_0 \in S$, and the series converges for $z = R_0$, and by Lemma 4·1, $\sum a_n z_1^n$ is absolutely convergent.

Finally, we show that if $|z_2| > R \geq 0$ then the series does not converge for z_2. Take R_0 such that $R < R_0 < |z_2|$.

If $\sum a_n z^n$ converges, then by Lemma 4·1, $\sum a_n R_0^n$ would be convergent, which contradicts $R = \sup S$ $\square$

Lemma 4·3 (a way of computing R)

If $\left|\frac{a_{n+1}}{a_n}\right| \to L$ as $n \to \infty$, then $R = 1/L$

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Analysis (14)

~~Lemma~~ Proof

Applying the ratio test to the series $\sum |a_n z^n|$ gives

$\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \left|\frac{a_{n+1}}{a_n}\right||z| \to L|z|$. So if $L|z| < 1$, we have absolute convergence. Also, if $L|z| \geq 1$, then $|a_n z^n|$ does not tend to zero so we have divergence, $\Rightarrow R = 1/L$ $\square$

Remark

Using the root test, one shows in a similar way that if $|a_n|^{1/n} \to L$ then $R = 1/L$.

Examples

1. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0$ as $n \to \infty$ so $R = \infty$

2. $\sum_{n=0}^{\infty} z^n$, Geometric series, $R = 1$. Note that at the boundary, $|z| = 1$, it diverges, because $|z|^n \not\to 0$.

3. $\sum_{n=0}^{\infty} n! z^n$, $\left|\frac{a_{n+1}}{a_n}\right| = n+1 \to \infty$ as $n \to \infty$, so $R = 0$ (only converges for $z = 0$)

4. $\sum_{n=1}^{\infty} \frac{z^n}{n}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} \to 1$, so $R = 1$.

For $z = 1$, the series diverges. What about other z with $|z| = 1$?

$\sum_{n=1}^{\infty} \frac{(1-z)z^n}{n}$. If $S_N = \sum_{n=1}^{N} \frac{(1-z)z^n}{n} = \sum_{n=1}^{N} \frac{z^n}{n} - \sum_{n=1}^{N} \frac{z^{n+1}}{n}\frac{(n+1)}{(n+1)}$

$$= \sum_{1}^{N} \frac{z^n}{n} - \sum_{2}^{N+1} \frac{z^n}{n}\frac{n}{n-1} = z - \frac{z^{N+1}}{N} + \sum_{n=1}^{N} \frac{z^n}{n}\left(1 - \frac{n}{n-1}\right)$$

$$= z - \frac{z^{N+1}}{N} + \underbrace{\sum_{n=2}^{N} \frac{z^n}{n(n-1)}}_{\text{Converges as } n \to \infty}$$

So S_N converges for $|z| = 1$

Conclusion

In general, nothing can be said in general about convergence on $|z| = R$

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Analysis (15)

Theorem 4.4

$f(z) = \sum_0^\infty a_n z^n$ has radius of convergence R. Then f is differentiable at all points z with $|z| < R$ and

$$f'(z) = \sum_{n=1}^\infty n a_n z^{n-1}$$

Proof (Not Examinable)

We need two auxiliary lemmas:

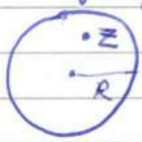
Lemma 4.5

If $\sum_0^\infty a_n z^n$ has radius of convergence R, then so do $\sum_1^\infty n a_n z^{n-1}$ and $\sum_2^\infty n(n-1) a_n z^{n-2}$

Lemma 4.6

i) $\binom{n}{r} \leq n(n-1)\binom{n-2}{r-2}$, $2 \leq r \leq n$

ii) $|(z+h)^n - z^n - nhz^{n-1}| \leq n(n-1)(|z| + |h|)^{n-2}|h|^2 \quad \forall h, z \in \mathbb{C}$



Proof of 4.4

By Lemma 4.5, for $|z| < R$

$\sum_1^\infty n a_n z^{n-1}$ converges absolutely, so it defines $C = \sum_1^\infty n a_n z^{n-1}$

we must prove $\Rightarrow \lim_{h \to 0} \frac{1}{h}\left[f(z+h) - f(z) - Ch\right] = 0$

$$\frac{1}{h}\left[f(z+h) - f(z) - Ch\right] = \frac{1}{h}\left[\sum_0^\infty a_n (z+h)^n - \sum_0^\infty a_n z^n - h\sum_1^\infty n a_n z^{n-1}\right]$$

$$= \frac{1}{h}\sum_0^\infty a_n\left[(z+h)^n - z^n - hnz^{n-1}\right] = I$$

We must prove that $\lim_{h \to 0} I = 0$

$$|I| = \frac{1}{|h|}\left|\lim_{N \to \infty}\sum_0^N a_n\left[(z+h)^n - z^n - hnz^{n-1}\right]\right|$$

$$|I| = \frac{1}{|h|}\lim_{N \to \infty}\left|\sum_0^N a_n\left[(z+h)^n - z^n - hnz^{n-1}\right]\right| \quad (*)$$

$$\left|\sum_0^N a_n\left[(z+h)^n - z^n - hnz^{n-1}\right]\right| \leq \sum_0^N |a_n|\,|(z+h)^n - z^n - hnz^{n-1}|$$

(Lemma 4.6) $\leq \sum_0^N |a_n|\, n(n-1)(|z|+|h|)^{n-2}|h|^2$

Take r such that $|z|+r<R$, then for $|h|<r$,

$$\sum_0^N |a_n|\, n(n-1)(|z|+|h|)^{n-2}|h|^2 \leq \sum_0^N |a_n|\, n(n-1)(|z|+r)^{n-2}|h|^2$$

By Lemma 4.5, $\lim_{N\to\infty} \sum_0^N |a_n|\, n(n-1)(|z|+r)^{n-2} = A$

We go back to $(*)$ to get:

$|I| \leq \frac{1}{|h|} A |h|^2 \to 0$ as $h \to 0$ $\square$

Now we prove the two lemmas:

<u>Proof of 4.5</u>

$\sum_1^\infty n a_n z^{n-1}$ has radius of convergence R. Take $|z|<R$

Choose R_0 such that $|z|<R_0<R$. Since $\sum_0^\infty a_n R_0^n$ converges, $a_n R_0^n \to 0$ as $n\to\infty$. In particular, $\exists\, k>0$ such that $|a_n R_0^n| \leq k \quad \forall n \geq 0$

$$n|a_n||z^{n-1}| = \frac{n|a_n||z|^n}{|z|}\frac{R_0^n}{R_0^n} \leq \frac{n}{|z|}\left(\frac{|z|}{R_0}\right)^n k \quad , \left(\frac{|z|}{R_0}<1\right)$$

<u>Claim</u> $\sum_1^\infty n\left|\frac{z}{R_0}\right|^n$ converges.

Indeed, by the ratio test $\frac{(n+1)\left|\frac{z}{R_0}\right|^{n+1}}{n\left|\frac{z}{R_0}\right|^n} = \frac{n+1}{n}\left|\frac{z}{R_0}\right| \xrightarrow{n\to\infty} \left|\frac{z}{R_0}\right|<1$

By comparison $\sum_1^\infty n|a_n||z|^{n-1}$ converges

$|a_n z^n| \leq n|z||a_n z^{n-1}|$ implies by comparison that the radius of convergence of $\sum n a_n z^{n-1}$ can't be larger than that of $\sum a_n z^n$.

Similarly, one shows that $\sum_2^\infty n(n-1)a_n z^{n-2}$ also has radius of convergence R. $\square$

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Analysis (15)

Proof of Lemma 4·6

i) $\binom{n}{r} \leq n(n-1)\binom{n-2}{r-2}$

$$\binom{n}{r} \Big/ \binom{n-2}{r-2} = \frac{n!}{r!(n-r)!} \frac{(r-2)!(n-r)!}{(n-2)!} = \frac{n(n-1)}{r(r-1)} \leq n(n-1)$$

ii) $|(z+h)^n - z^n - nhz^{n-1}| \leq n(n-1)(|z|+|h|)^{n-2}|h|^2$

$$(z+h)^n - z^n - nhz^{n-1} = \sum_2^n \binom{n}{r} z^{n-r} h^r$$

$$|(z+h)^n - z^n - nhz^{n-1}| \leq \sum_2^n \binom{n}{r} |z|^{n-r}|h|^r$$

$$\leq n(n-1)\left(\sum_2^n \binom{n-2}{r-2} |z|^{n-r}|h|^{r-2}\right)|h|^2 = n(n-1)(|z|+|h|)^{n-2}|h|^2$$

Now note that $(|z|+|h|)^{n-2} = \sum_2^n \binom{n-2}{r-2}|z|^{n-r}|h|^{r-2}$ $\square$

We saw last time that $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has $R = \infty$

Thus we can define $e : \mathbb{C} \to \mathbb{C}$

$$e(z) = \sum_0^{\infty} \frac{z^n}{n!}$$

Theorem 4·4 $\Rightarrow$ e is differentiable and

$$e'(z) = \sum_1^{\infty} \frac{nz^{n-1}}{n!} = \sum_0^{\infty} \frac{z^n}{n!} = e(z) \qquad z \in \mathbb{C}$$

$$e(a+b) = e(a)e(b)$$

Consider $F(z) = e(a+b-z)e(z)$ $\quad F : \mathbb{C} \to \mathbb{C}$

$$F'(z) = -e(a+b-z)e(z) + e(a+b-z)e(z) = 0$$

Using a previous lemma, F is constant, $z=b$

$$F(b) = e(a)e(b) = F(0) = e(a+b)\underbrace{e(0)}_{=1}$$

3

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Analysis (16)

The Standard Functions (exp, log, trigonometric, etc)

$e : \mathbb{C} \to \mathbb{C} \quad e(z) = \sum_0^\infty \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \dots$

Theorem 4.4 gives $e'(z) = e(z)$ so e is infinitely differentiable

$e(0) = 1 \qquad e(a+b) = e(a)\,e(b)$

We used $F : \mathbb{C} \to \mathbb{C}$, $F(z) = e(a+b-z)\,e(z)$, $F'(z) = 0$

$\Rightarrow F$ is constant. Take $z = b$, $F(b) = e(a)e(b) = F(0) = e(a+$

Now we prove that $F'(z) = 0$ for all $z \Rightarrow F$ constant.

$g(t) = F(tz)$

Chain Rule $g'(t) = F'(tz)\,z = 0$

$g(t) = u(t) + iv(t)$ (where $u, v \in \mathbb{R}$)

$g'(t) = u'(t) + iv'(t)$ (Check, follows from definition of derivative)

$g'(t) = 0 \Rightarrow u', v' = 0 \Rightarrow u, v$ are constant

By Corollary 3.5 $\Rightarrow g(0) = g(1) \Rightarrow F(z) = F(0)$ $\square$

We now restrict e to the real line to get $e : \mathbb{R} \to \mathbb{R}$

Theorem 4.7 i) $e : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable, $e'(x) = e(x)$

ii) $e(x+y) = e(x)\,e(y)$

iii) $e(x) > 0 \quad \forall x \in \mathbb{R}$

iv) e is strictly increasing

v) $e(x) \to \infty$ as $x \to \infty$, and $e(x) \to 0$ as $x \to -\infty$

vi) $e : \mathbb{R} \to (0, \infty)$ is a bijection

i) and ii) have already been proved

iii) From the definition, if $x \geq 0$, $e(x) > 0$

$x > 0 \quad 1 = e(0) = e(x - x) = e(x) \cdot e(-x)$

$e(-x) = \frac{1}{e(x)} > 0$

iv) $e'(x) = e(x) > 0 \Rightarrow e$ is strictly increasing

v) $e(x) > 1 + x$ for $x > 0$

$\Rightarrow e(x) \to \infty$ as $x \to \infty$

$e(-x) = \frac{1}{e(x)} \Rightarrow e(x) \to 0$ as $x \to -\infty$

vi) Strictly Increasing $\Rightarrow e$ is injective

Take $y \in (0, \infty)$

$e(x) \to \infty$ as $x \to \infty$, $e(x) \to 0$ as $x \to -\infty$

$\Rightarrow \exists\, a, b \in \mathbb{R}$ such that $e(a) < y < e(b)$

Now by the intermediate value theorem $\exists x \in \mathbb{R}$ such that $e(x) = y$ □

Remark

$e : (\mathbb{R}, +) \to ((0, \infty), \times)$, an isomorphism

We introduce $L : (0, \infty) \to \mathbb{R}$, the inverse of e

$\therefore e[L(t)] = t, \quad L[e(x)] = x$

Theorem 4.8

i) $L : (0, \infty) \to \mathbb{R}$ is a bijection, $L[e(x)] = x \quad \forall x \in \mathbb{R}$

and $e[L(t)] = t \quad \forall t \in (0, \infty)$

ii) L is differentiable and $L'(t) = \frac{1}{t}$

iii) $L(xy) = L(x) + L(y)$

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Proof

i) Already done

ii) Inverse rule ⇒ L is differentiable

$$L'(t) = \frac{1}{e'[L(t)]} = \frac{1}{e[L(t)]} = \frac{1}{t}$$

iii) From IA Groups L is a homomorphism ⇒ $L(xy) = L(x) + L(y)$ □

Suppose $x > 0$, $\alpha \in \mathbb{R}$

$$r_\alpha(x) = e[\alpha L(x)] \qquad r_\alpha : (0,\infty) \to (0,\infty)$$

Clearly this is differentiable $\quad r_\alpha'(x) = e'[\alpha L(x)]\frac{\alpha}{x} = \frac{\alpha r_\alpha(x)}{x}$

Theorem 4.9

Suppose $x, y > 0$, $\alpha, \beta \in \mathbb{R}$. Then

i) $r_\alpha(xy) = r_\alpha(x) r_\alpha(y)$

ii) $r_{\alpha+\beta}(x) = r_\alpha(x) r_\beta(x)$

iii) $r_\alpha[r_\beta(x)] = r_{\alpha\beta}(x)$

iv) $r_1(x) = x$

Proof (ii) and iii) left as exercises)

i) $r_\alpha(xy) = e[\alpha L(xy)] = e[\alpha L(x) + \alpha L(y)]$

$\qquad = e[\alpha L(x)]\, e[\alpha L(y)] = r_\alpha(x) r_\alpha(y)$

iv) $r_1(x) = e[L(x)] = x$ □

$(n \in \mathbb{Z}^+)$ $r_n(x) = r_{1+1+1+\dots+1}(x) = r_1(x)\dots r_1(x) = x^n$

$r_1(x) r_{-1}(x) = r_0(x) = 1, \quad r_{-1}(x) = \frac{1}{r_1(x)} = \frac{1}{x}$

$\Rightarrow r_{-n}(x) = x^{-n}$

$[r_{\frac{1}{q}}(x)]^q = r_{q/q}(x) = x \Rightarrow r_{\frac{1}{q}}(x) = x^{\frac{1}{q}}$

$\Rightarrow r_{p/q}(x) = x^{p/q}$

$\Rightarrow r_\alpha(x)$, with α rational, coincides with x^α

If $\alpha \in \mathbb{R}$, we define $x^\alpha = r_\alpha(x)$

$\log(x) := L(x) \qquad \exp(x) := e(x)$

Define $e = \sum_0^\infty \frac{1}{n!} = e(1) \Rightarrow \log e = 1$

$e(x) = e(x \log e) = r_x(e) = e^x$

$(x^\alpha)' = \frac{x^\alpha \alpha}{x} = \alpha x^{\alpha-1}$

$f(x) = a^x, \ a > 0$

$f(x) = e^{x\log a}, \ f'(x) = e^{x \log a} \log a = a^x \log a$

"Exponentials beat powers"

$\lim_{x\to\infty} \frac{e^x}{x^k} \qquad e^x = \sum_0^\infty \frac{x^n}{n!}$

$e^x > \frac{x^n}{n!}$ for large n

$(n > k) \quad \frac{e^x}{x^k} > \frac{x^{n-k}}{n!} \to \infty$ as $x \to \infty$

02/11

Analysis (17)

$\sin z, \cos z : \mathbb{C} \to \mathbb{C}$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$

These two power series have an infinite radius of convergence

Theorem 4·4 $\Rightarrow$ $\sin z, \cos z$ are differentiable, and

$(\cos z)' = -\sin z \qquad (\sin z)' = \cos z$

$$e^{iz} = \sum_0^\infty \frac{(iz)^n}{n!} = \sum_0^\infty \frac{(iz)^{2k}}{(2k)!} + \sum_0^\infty \frac{(iz)^{2k+1}}{(2k+1)!}$$

$$e^{iz} = \sum_0^\infty (-1)^k \frac{z^{2k}}{(2k)!} + i \sum_0^\infty (-1)^k \frac{z^{2k+1}}{(2k+1)!} = \cos z + i \sin z$$

From the definition: $\cos(0) = 1$, $\sin(0) = 0$, $\cos(-z) = \cos(z)$, $\sin(-z) = -\sin z$

$$e^{-iz} = \cos(-z) + i\sin(-z) = \cos z - i \sin z$$

$(*)$ $\Rightarrow \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

From this, it is very easy to check the following:

1. $\sin(z + w) = \sin z \cos w + \cos z \sin w$
2. $\cos(z + w) = \cos z \cos w - \sin z \sin w$
3. $\sin^2 z + \cos^2 z = 1$

$\forall z, w \in \mathbb{C}$

To show 3, just use $(*)$, and for 1, $(*)$ with $e^{a+b} = e^a e^b$

<u>Note:</u> if $x \in \mathbb{R}$, $\sin x, \cos x \in \mathbb{R}$

$3 \Rightarrow |\sin x|, |\cos x| \leq 1$

<u>Warning:</u> $\cos(iy) = \frac{1}{2}\{e^{-y} + e^{y}\}$, $y \in \mathbb{R}$

which tends to infinity at $y \to \infty$

Periodicity of trigonometric functions

Proposition 4.10 There is a smallest positive number ω, where $\sqrt{2} < \frac{\omega}{2} < \sqrt{3}$ such that $\cos\frac{\omega}{2} = 0$ (for real numbers)

Proof If $0 < x < 2$

$$\sin x = \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \dots \quad > 0$$

$$\frac{x^{2n-1}}{(2n-1)!} > \frac{x^{2n+1}}{(2n+1)!} \iff x^2 < (2n)(2n+1) \text{, true for } 0<x<2$$

$$(\cos x)' = -\sin x < 0 \quad \text{for } 0 < x < 2$$

$\cos x$ is strictly decreasing in $(0,2)$

Claim $\cos\sqrt{2} > 0$, $\cos\sqrt{3} < 0$

This claim gives our proposition because by the intermediate value theorem, there is ω such that $\cos\frac{\omega}{2} = 0$, $\sqrt{2} < \frac{\omega}{2} < \sqrt{3}$ and is the smallest

$$\cos\sqrt{2} = \left(\frac{\sqrt{2}^4}{4!} - \frac{\sqrt{2}^6}{6!}\right) + (\dots) \dots \quad > 0$$

$$\cos x = \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!}}_{\text{less than } 0} - \left(\frac{x^6}{6!} - \frac{x^8}{8!}\right) \cancel{+}$$

less than 0, and brackets are all less than 0 $\square$

Corollary 4.11 $\sin\frac{\omega}{2} = 1$

Proof $\cos^2\frac{\omega}{2} + \sin^2\frac{\omega}{2} = 1$

But $\cos^2\frac{\omega}{2} = 0$ and $\sin\frac{\omega}{2} > 0$, so $\sin\frac{\omega}{2} = 1$ $\square$

Define $\pi = \omega$

Theorem 4.12

1. $\sin(z + \frac{\pi}{2}) = \cos z$, $\cos(z + \frac{\pi}{2}) = -\sin z$
2. $\sin(z + \pi) = -\sin z$, $\cos(z + \pi) = -\cos z$
3. $\sin(z + 2\pi) = \sin(z)$, $\cos(z + 2\pi) = \cos(z)$

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Analysis (17)

Proof: To prove 1, use addition formulas and $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$

2 and 3 reduce to 1.

Note: $e^{iz+2\pi i} = \cos(z+2\pi) + i\sin(z+2\pi) = \cos(z) + i\sin z = e^{iz}$

$\Rightarrow e^{z}$ is periodic with period $2\pi i$

Side

In $\mathbb{R}^2$, $\underline{x} = (x_1, x_2)$, $\underline{y} = (y_1, y_2)$

$\underline{x}\cdot\underline{y} = x_1 y_1 + x_2 y_2$, $|\underline{x}\cdot\underline{y}| \le |\underline{x}||\underline{y}|$

If $x \neq 0$, $y \neq 0$ then $-1 \le \frac{\underline{x}\cdot\underline{y}}{|\underline{x}||\underline{y}|} \le 1$

$\exists\, \theta \in [0, \pi]$ such that $\cos\theta = \frac{\underline{x}\cdot\underline{y}}{|\underline{x}||\underline{y}|}$

Hyperbolic Functions

Define $\cosh z = \frac{1}{2}(e^{z} + e^{-z})$, $\sinh z = \frac{1}{2}(e^{z} - e^{z})$

$\Rightarrow \cosh z = \cos(iz)$, $i\sinh(z) = \sin(iz)$

$(\cosh z)' = \sinh z$, $(\sinh z)' = \cosh z$

$\cosh^2 z - \sinh^2 z = 1$

The other trigonometric functions ($\tan$, $\cot$ etc) are defined in the usual way.

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Analysis (18)

5. Integration

$f : [a, b] \to \mathbb{R}$, $f \mapsto \int_a^b f$

f is <u>bounded</u> i.e. $|f(x)| \leq k \ \forall x \in [a, b]$

(For us, unbounded functions will not be integrable)

<u>Definition</u> A dissection, or partition, $\mathcal{D}$ of $[a, b]$ is a finite subset of $[a, b]$ containing the end points a and b.

We write $\mathcal{D} = \{x_0, x_1, \ldots, x_n\}$ where $a = x_0 < x_1 < \ldots < x_n =$

<u>Definition</u>

We define the upper sum and lower sum associated with a partition $\mathcal{D}$ as:

$$S(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x) \qquad \text{(upper sum)}$$

$$s(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x) \qquad \text{(lower sum)}$$

<u>Lemma 5.1</u>

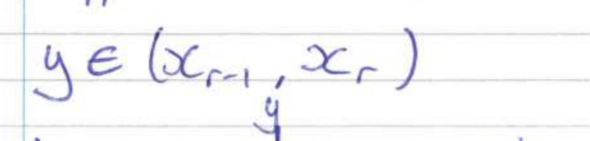
If $\mathcal{D}$ and $\mathcal{D}'$ are two partitions with $\mathcal{D}' \supseteq \mathcal{D}$, then

$$S(f, \mathcal{D}) \geq S(f, \mathcal{D}') \geq s(f, \mathcal{D}') \geq s(f, \mathcal{D})$$

<u>Proof</u> $S(f, \mathcal{D}') \geq s(f, \mathcal{D}')$ is obvious

Let's prove $S(f, \mathcal{D}) \geq S(f, \mathcal{D}')$.

Suppose $\mathcal{D}'$ contains one more point than $\mathcal{D}$, for example $y \in (x_{r-1}, x_r)$



$$\sup_{x \in [x_{r-1}, y]} f(x), \ \sup_{x \in [y, x_r]} f(x) \leq \sup_{[x_{r-1}, x_r]} f(x)$$

This implies $(y-x_{r-1})\sup\limits_{[x_{r-1},y]} f(x) + (x_r - y)\sup\limits_{[y,x_r]} f(x) \leq (y - x_{r-1} + x_r - y) \sup\limits_{x \in [x_{r-1}} f(x$

$\Rightarrow S(f, \mathcal{D}') \leq S(f, \mathcal{D})$

By induction, $S(f, \mathcal{D}') \leq S(f, \mathcal{D})$ for any $\mathcal{D}' \supseteq \mathcal{D}$. A similar argument shows that $s(f, \mathcal{D}') \geq s(f, \mathcal{D})$ $\square$

<u>Lemma 5.2</u>

If $\mathcal{D}_1$, $\mathcal{D}_2$ are two arbitrary partitions, then

$S(f, \mathcal{D}_1) \geq S(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq s(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq s(f, \mathcal{D}_1$

<u>Proof</u>

Let $\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \mathcal{D}_1, \mathcal{D}_2$. By the previous lemma, the statement follows. $\square$

$$\boxed{S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2) \text{ for any } \mathcal{D}_1, \mathcal{D}_2}$$

<u>Remark:</u>

Since f is bounded, $-k \leq f(x) \leq k \ \forall x \in [a,b]$

$S(f, \mathcal{D}) \geq -k(b-a)$ so $\inf\limits_{\mathcal{D}} S(f, \mathcal{D})$ exists

$s(f, \mathcal{D}) \leq k(b-a)$ so $\sup\limits_{\mathcal{D}} s(f, \mathcal{D})$ exists

<u>Definition</u>

The <u>upper integral</u> of f is $I^*(f) = \inf\limits_{\mathcal{D}} S(f, \mathcal{D})$

The <u>lower integral</u> of f is $I_*(f) = \sup\limits_{\mathcal{D}} s(f, \mathcal{D})$

$S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$

$\inf\limits_{\mathcal{D}_1} S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$

$I^*(f) \geq s(f, \mathcal{D}_2)$

$\Rightarrow I^*(f) \geq \sup\limits_{\mathcal{D}_2} s(f, \mathcal{D}_2) = I_*(f)$

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Analysis (18)

<u>Definition</u> We say that f is Riemann Integrable (or just integrable in this course) if $I_*(f) = I^*(f)$. In this case we write $I^*(f) = I_*(f) = \int_a^b f(x)\,dx$, $\int_a^b f$

<u>Examples</u>

1. $f(x) = 1$, $x \in [0,1]$

$$S(f,\mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \overset{1}{\sup_{[x_{j-1},x_j]} f} = b-a$$

$s(f, \mathcal{D}) = b-a$ similarly

$$\int_a^b 1 = b - a$$

2. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \notin \mathbb{Q} \cap [0,1] \end{cases}$

$$S(f,\mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \overset{1}{\sup_{[x_{j-1},x_j]} f} = 1$$

$$s(f,\mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1},x_j]} f = 0$$

(every interval contains both rational and irrational numbers)

So this function is not Riemann Integrable as $I^*(f) = 1 \neq 0 = I_*(f)$

<u>Question's to resolve</u>

1. We need to provide good classes of integrable functions.
2. We need to establish some properties of integrals.

We now prove the following very useful criterion for integrability:

<u>Theorem 5·3</u> (Riemann)

A bounded function $f[a,b] \to \mathbb{R}$ is Riemann Integrable if and only if given $\varepsilon > 0$, $\exists\, \mathcal{D}$ such that

$$S(f,\mathcal{D}) - s(f,\mathcal{D}) < \varepsilon$$

Proof

Assume that given $\varepsilon > 0$, $\exists \mathcal{D}$ such that $S(f, \mathcal{D}) - s(f, \mathcal{D}) < \varepsilon$. Let's prove that f is integrable. $0 \leq I^*(f) - I_*(f) \leq S(f, \mathcal{D}) - s(f, \mathcal{D})$ for any $\mathcal{D}$.

Therefore, given $\varepsilon > 0$, $0 \leq I^*(f) - I_*(f) < \varepsilon$

$$\Rightarrow I^*(f) = I_*(f)$$

03/03/11 Analysis (19)

Theorem 5.3 (Riemann Criterion)

A bounded function $f:[a,b]\to\mathbb{R}$ is integrable if and only if given $\varepsilon>0$, $\exists\ \mathcal{D}$, a partition, such that $\underbrace{S(f,\mathcal{D})-s(f,\mathcal{D})<\varepsilon}_{(*)}$

Last time we proved that $(*) \Rightarrow$ integrability.

Assume now that f is Riemann Integrable, i.e. $I^*(f)=I_*(f)$.

By definition of supremum and infimum, given $\varepsilon>0$, $\exists\ \mathcal{D}_1,\mathcal{D}_2$ such that $S(f,\mathcal{D}_1)<I^*(f)+\varepsilon$, $s(f,\mathcal{D}_2)>I_*(f)-\varepsilon$

Take $\mathcal{D}=\mathcal{D}_1\cup\mathcal{D}_2$

$$S(f,\mathcal{D})-s(f,\mathcal{D}) \overset{\text{Lemma 5.2}}{\le} S(f,\mathcal{D}_1)-s(f,\mathcal{D}_2)$$

$$S(f,\mathcal{D})-s(f,\mathcal{D}) < I^*(f)+\varepsilon-I_*(f)+\varepsilon = 2\varepsilon \quad \square$$

Goal:

1. Monotonic functions
2. Continuous

} integrable

Note: Monotonic and continuous functions on $[a,b]$ are bounded (Theorem 2.6)

Theorem 5.4

If $f:[a,b]\to\mathbb{R}$ is monotone, f is integrable

Proof: We prove this for f increasing. The proof ~~of~~ for f decreasing is very similar.

$$S(f,\mathcal{D})=\sum_{j=1}^{n}(x_j-x_{j-1})f(x_j) \qquad (f \text{ is increasing})$$

$$s(f,\mathcal{D})=\sum_{j=1}^{n}(x_j-x_{j-1})f(x_{j-1})$$

$$S(f,\mathcal{D})-s(f,\mathcal{D})=\sum_{j=1}^{n}(x_j-x_{j-1})[f(x_j)-f(x_{j-1})]$$

Take $\mathcal{D}_n=\{a, a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}, \dots, b\}$, $x_j=a+\frac{j(b-a)}{n}$

For this special $\mathcal{D}_n$:

$$S(f, \mathcal{D}_n) - s(f, \mathcal{D}_n) = \frac{b-a}{n} \sum_{j=1}^{n} \left[f(x_j) - f(x_{j-1}) \right]$$

$$= \frac{b-a}{n} \left[f(b) - f(a) \right]$$

If we take sufficiently large n, $\frac{b-a}{n}\left[f(b) - f(a)\right] < \varepsilon$

Now by the Riemann Criterion, f is integrable. $\square$

Now we look at continuous functions. First, an auxiliary Lemma.

Lemma 5·5

Let $f: [a,b] \to \mathbb{R}$ be continuous. Then, given $\varepsilon > 0$, $\exists \delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

This is called Uniform Continuity (the definition of continuity is for f continuous at x, if given $\varepsilon > 0, \ldots$, rather than $f: [a,b] \to \mathbb{R}$ be continuous) as we can choose a δ which works for $\forall x \in [a,b]$

Proof

Suppose this statement is not true. Then

$\exists \varepsilon > 0$ such that $\forall \delta > 0$, $\exists x, y \in [a,b]$

with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$

Choose $\delta = \frac{1}{n}$, then we have $x_n, y_n \in [a,b]$ with $|x_n - y_n| < \frac{1}{n}$

and $|f(x_n) - f(y_n)| \geq \varepsilon$

By the Bolzano Weierstrass Theorem, x_n has a convergent subsequence

$x_{n_k} \to c \in [a,b]$

$|y_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| \to 0$ as $k \to \infty$

so $y_{n_k} \to c$

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$|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$

Let $k \to \infty$. By continuity $|f(c) - f(c)| \geq \varepsilon$, $0 \geq \varepsilon$ which is absurd. $\square$

__Theorem 5·6__

If f is Riemann Integrable on $[a,b]$ with $m \leq f \leq M$ and ϕ is a continuous function on $[m, M]$, then $\phi \circ f$ is Riemann Integrable on $[a,b]$. In particular, continuous functions are Riemann Integrable (since $f(x) = x$ is an increasing function and therefore integrable).

__Proof__

Let $\varepsilon > 0$ be given. By Lemma 5·5, $\exists\, \delta > 0$ such that if $s, t \in [m, M]$ and $|s-t| < \delta$, $|\phi(s) - \phi(t)| < \epsilon$.

It will be convenient to choose $\delta < \varepsilon$ for later. Since f is integrable by Riemann's Criterion, $\exists\, \mathcal{D}$ of $[a,b]$ such that

$S(f, \mathcal{D}) - s(f, \mathcal{D}) < \delta^2$

M_j and m_j are the supremum and infemum of f in $[x_j, x_{j-1}]$

M_j^* and m_j^* are the supremum and infemum of $\phi \circ f$ in $[x_j, x_{j-1}]$

$\Delta x_j = x_j - x_{j-1}$

$A = \{ j : M_j - m_j < \delta \}$ $(1 \leq j \leq n)$

$k = \sup |\phi|$

3

$$S(\phi \circ f, \mathcal{D}) - s(\phi \circ f, \mathcal{D}) = \sum_{j=1}^{n} (M_j^* - m_j^*) \Delta x_j$$
$$= \underbrace{\sum_{j \in A} (M_j^* - m_j^*) \Delta x_j}_{(\mathrm{I})} + \underbrace{\sum_{j \notin A} (M_j^* - m_j^*) \Delta x_j}_{(\mathrm{II})}$$

We ~~bound~~ estimate I and II

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$$S(\phi\circ f, \mathcal{D}) - s(\phi\circ f, \mathcal{D}) = \underbrace{\sum_{j\in A} (M_j^* - m_j^*)\Delta x_j}_{\mathrm{I}} + \underbrace{\sum_{j\notin A} (M_j^* - m_j^*)\Delta x_j}_{\mathrm{II}}$$

M_j and m_j are the sup and inf of f on $[x_{j-1}, x_j]$

M_j^* and m_j^* are the sup and inf of $\phi\circ f$ on $[x_{j-1}, x_j]$

$k = \sup|\phi| \qquad A = \{ j : M_j - m_j < \delta \}$

Look at I

Take $x, y \in [x_{j-1}, x_j]$, $f(x), f(y) \in [m_j, M_j]$

For $j \in A$, $M_j - m_j < \delta \Rightarrow |f(x) - f(y)| < \delta$

$\Rightarrow |\phi(f(x)) - \phi(f(y))| < \varepsilon$

$\Rightarrow M_j^* - m_j^* \leq \varepsilon$

$\Rightarrow \mathrm{I} \leq \varepsilon(b-a)$

Now, looking at II.

$\displaystyle\sum_{j\notin A} (M_j^* - m_j^*)\Delta x_j \leq 2k \sum_{j\notin A} \Delta x_j$ $\qquad \boxed{\leq 2k\delta}$

$\displaystyle\sum_{j=1}^{n} (M_j - m_j)\Delta x_j < \delta^2 \Rightarrow \sum_{j\notin A} (M_j - m_j)\Delta x_j < \delta^2$

By definition of A, $\displaystyle\delta \sum_{j\notin A} \Delta x_j \leq \sum_{j\notin A} (M_j - m_j)\Delta x_j < \delta^2$

$\displaystyle\Rightarrow \sum_{j\notin A} \Delta x_j < \delta$

$\Rightarrow \mathrm{I} + \mathrm{II} < \varepsilon(b-a) + 2k\delta < \varepsilon[b-a+2k]$ $\quad\square$

<u>Examples</u>

1. $f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in its lowest possible form} \\ 0 & \text{otherwise} \end{cases}$ for $x \in [0,1]$

$\displaystyle s(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f = 0 \quad \forall \mathcal{D}$

We will show that given $\varepsilon > 0$, $\exists \mathcal{D}$ such that $S(f, \mathcal{D}) < \varepsilon$. This implies f is Riemann Integrable and that

$\int^1 f = 0$ because $I_*(f) = 0$

Choose a positive integer N such that $\frac{1}{N} < \frac{\varepsilon}{2}$

Look at $\{x \in [0,1] : f(x) \geqslant \frac{1}{N}\}$

$= \{\frac{p}{q} : 1 \leq q \leq N, 0 \leq p \leq q\}$, a finite set.

We call points in this set $0 = t_1 < t_2 < \dots < t_r = 1$

Consider a partition $\mathcal{D}$ of $[0,1]$ such that

1. each t_k for $1 < k < R$ is in some (x_{j-1}, x_j)
2. $\forall k$, the unique interval containing t_k has length at most $\varepsilon/2R$

$$S(f,\mathcal{D}) = \underbrace{\sum_{\text{intervals containing } t_k}}_{(f \leq 1) \quad < \frac{\varepsilon}{2}} + \underbrace{\sum_{\text{other intervals}}}_{\frac{1}{N} < \frac{\varepsilon}{2}} \; (f \leq \tfrac{1}{N}) < \varepsilon$$

Example

$g(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$ on $[0,1]$ is Riemann Integrable

Now consider $g \circ f(x) = \begin{cases} 0 & x \in \mathbb{Q}^c \\ 1 & x \in \mathbb{Q} \end{cases}$ which is NOT Riemann Integrable.

This shows that

a) Compositions of integrable functions are not necessarily integrable

b) Theorem 5·6 is very sharp, because, in our example, g fails to be continuous at only a single point.

Elementary Properties of Integrals

$f, g : [a,b] \to \mathbb{R}$, bounded and integrable

1. If $f(x) \leq g(x)$ $\forall x \in (a,b)$, then $\int_a^b f \leq \int_a^b g$

Proof $\int_a^b f = I^*(f) \leq S(f,\mathcal{D}) \leq S(g,\mathcal{D}) \quad \forall \mathcal{D}$

hence $\int_a^b f \leq I^*(g) = \int_a^b g$

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2. $f+g$ is integrable, and $\int_a^b f+g = \int_a^b f + \int_a^b g$

Proof

$$f(x) + g(x) \leq \sup_{[x_{j-1}, x_j]} f(x) + \sup_{[x_{j-1}, x_j]} g(x) \qquad \forall x \in [x_{j-1}, x_j]$$

$$\Rightarrow \sup_{[x_{j-1}, x_j]} (f+g) \leq \sup_{[x_{j-1}, x_j]} f + \sup_{[x_{j-1}, x_j]} g$$

$$S(f+g, \mathcal{D}) \leq S(f, \mathcal{D}) + S(g, \mathcal{D}) \quad \forall \mathcal{D}$$

Now take any two partitions $\mathcal{D}_1$, $\mathcal{D}_2$:

$$I^*(f+g) \leq S(f+g, \mathcal{D}_1 \cup \mathcal{D}_2) \leq S(f, \mathcal{D}_1 \cup \mathcal{D}_2) + S(g, \mathcal{D}_1 \cup \mathcal{D}$$

$$\leq S(f, \mathcal{D}_1) + S(g, \mathcal{D}_2)$$

Take infemums over all $\mathcal{D}_1$ to get

$I^*(f+g) \leq I^*(f) + S(g, \mathcal{D}_2)$, then over all $\mathcal{D}_2$ to get

$$I^*(f+g) \leq I^*(f) + I^*(g) = \int_a^b f + \int_a^b g$$

Similarly $I_*(f+g) \geq I_*(f) + I_*(g) = \int_a^b f + \int_a^b g$

Since $I^*(f+g) \geq I_*(f+g)$ then

$$I^*(f+g) = I_*(f+g) = \int_a^b f + \int_a^b g$$

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3. For any constant k, kf is integrable and $\int_a^b kf = k\int_a^b f$
4. $|f|$ is integrable and $\left|\int_a^b f\right| \leq \int_a^b |f|$

<u>Proof</u> $\phi(x) = |x|$ is continuous. By theorem 5·6, $\phi \circ f$ is integrable so $|f|$ is integrable. Note, $-|f| \leq f < |f|$

By 1, $-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f| \Rightarrow \left|\int_a^b f\right| \leq \int_a^b |f|$

5. The product function $f(x)\,g(x)$ is integrable.

<u>Proof</u>

$\phi(x) = x^2$ is continuous, so f^2 is Riemann integrable as $\phi\circ f$ is integrable by Theorem 5·6.

$(f+g)^2 = f^2 + g^2 + 2fg$, $2fg = (f+g)^2 - f^2 - g^2$

Using our previous results, it follows that fg is integrable.

6. Take $a < c < b$, then f is integrable in $[a,c]$ and $[c,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$

<u>Convention:</u> If $a > b$, $\int_a^b f = -\int_b^a f$.

If $a = b$ we agree that the integral is zero.

<u>Proof</u> Follows from 4, 1 and above.

If $a < b$, $\left|\int_a^b f\right| \leq \int_a^b |f| \leq \int_a^b k = k(b-a)$

So if $|f| \leq k$ for every point in the interval,

$$\left|\int_a^b f\right| \leq k|b-a|$$

Fundamental Theorem of Calculus

$f : [a,b] \to \mathbb{R}$, bounded and integrable

$$F(x) = \int_a^x f(t)\,dt \quad , \; x \in [a,b]$$

Theorem 5.7

F is continuous.

Proof $F(x+h) - F(x) = \int_a^{x+h} f - \int_a^x f = \int_x^{x+h} f$

$$|F(x+h) - F(x)| = \left|\int_x^{x+h} f\right| \leq k|h|$$

If $h \to 0$, $F(x+h) - F(x) \to 0$ $\square$

Theorem 5.8 (Fundamental Theorem of Calculus)

Assume $f : [a,b] \to \mathbb{R}$ is continuous

Then F is differentiable on $[a,b]$ and $F'(x) = f(x)$ $\forall x \in [a,b]$

Proof

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| = \frac{1}{|h|}\left|F(x+h) - F(x) - hf(x)\right|$$

$$= \frac{1}{|h|}\left|\int_x^{x+h} f(t)\,dt - hf(x)\right| = \frac{1}{|h|}\left|\int_x^{x+h} f(t) - f(x)\,dt\right|$$

$$\leq \frac{1}{|h|} \max_{\theta \in [0,1]} |f(x+\theta h) - f(x)|\,|h|$$

$$= |f(x+\theta(h)\,h) - f(x)| \to 0 \text{ as } h \to 0 \text{ since } f \text{ is continuous } \square$$

Example $f(x) = \begin{cases} 1 & [0,1] \\ -1 & [-1,0) \end{cases}$

not continuous at 0 but integrable in $[-1,1]$

$$F(x) = \int_{-1}^x f(t)\,dt = \begin{cases} -1-x & x \in [-1,0) \\ -1+x & x \in [0,1] \end{cases}$$

$$= -1 + |x|$$

F is continuous but not differentiable at 0

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<u>Corollary 5·9</u> (integration is inverse of differentiation)

If $f = g'$ is continuous on $[a,b]$ then
$$\int_a^x f(t)\,dt = g(x) - g(a) \quad \forall x \in [a,b]$$

<u>Proof</u>

From 5·8 $\quad (F-g)' = F' - g' = f - f = 0$

$\Rightarrow F - g$ is constant.

$$F(x) - g(x) = F(a) - g(a)$$
$$\int_a^x f(t)\,dt - g(x) = 0 - g(a) \qquad \square$$

Any continuous function f has an anti derivative and moreover, two anti derivatives differ by a constant. Anti derivatives are denoted $\int f(x)\,dx$

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Analysis (22)

Fundamental Theorem of Calculus:

$$F(x) = \int_a^x f(t)\,dt \qquad \text{and } F'(x) = f(x)$$

Any continuous function has an anti-derivative

Consider the differential equation:

$$\begin{cases} y'(x) = f(x) & \text{on } [a,b],\ f \text{ continuous} \\ y(a) = y_0 \end{cases}$$

This has a unique solution $y(x) = y_0 + \int_a^x f(t)\,dt$

<u>Corollary 5·10</u> (Integration by parts)

Suppose f' and g' exist and are continuous on $[a,b]$. Then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$$

<u>Proof</u> Remember the product rule for differentiation:

$$(fg)' = f'g + fg'$$

By 5·9, $\int_a^b (f'g + fg') = f(b)g(b) - f(a)g(a)$

$$\int_a^b f'g + \int_a^b fg' = f(b)g(b) - f(a)g(a) \qquad \square$$

<u>Corollary 5·1</u> (Integration by substitution)

$g: [\alpha, \beta] \to [a,b]$ with $g(\alpha) = a$, $g(\beta) = b$

g' exists and is continuous on $[\alpha, \beta]$

$f: [a,b] \to \mathbb{R}$ is continuous. Then $\int_a^b f(x)\,dx = \int_\alpha^\beta f[g(t)]g'(t)\,dt$

<u>Proof</u>

Set $F(x) = \int_a^x f(t)\,dt$

Let $h(t) = F[g(t)]$. $h'(t) = F'[g(t)]g'(t) = f[g(t)]g'(t)$

Therefore $\int_\alpha^\beta f[g(t)]g'(t)\,dt \overset{(5\cdot 9)}{=} h(\beta) - h(\alpha) = F[g(\beta)] - F[g(\alpha)]$

$$= F(b) - F(a) = \int_a^b f(x)\,dx$$

We want to use the integral to give an expression of the remainder in Taylor's Theorem.

Theorem 5·12 (Taylor's Theorem with the remainder as an integral)

Let $f^{(n)}(x)$ be continuous for $x \in [0,h]$. Then

$$f(h) = \sum_{j=0}^{n-1} \frac{h^j f^{(j)}(0)}{j!} + R_n$$

where $R_n = \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(th)\,dt$

Observation: We are assuming that $f^{(n)}$ is continuous, whereas before in Chapter 3, we only assume the existence of $f^{(n)}$.

Proof First we do a substitution in the formula for R_n.

Set $u = th$, $du = h\,dt$

$$R_n = \frac{h^n}{(n-1)!}\int_0^h \left(1-\frac{u}{h}\right)^{n-1} f^{(n)}(u)\,\frac{du}{h} = \frac{1}{(n-1)!}\int_0^h \overbrace{(h-u)^{n-1}}^{u}\,\overbrace{f^{(n)}(u)\,du}^{dv}$$

We integrate by parts to get:

$$R_n = \frac{n-1}{(n-1)}\int_0^h (h-u)^{n-2} f^{(n-1)}(u)\,du - \frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!}$$

$$R_n = \frac{1}{(n-2)!}\int_0^h (h-u)^{n-2} f^{(n-1)}(u)\,du - \frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!}$$

$$R_n = R_{n-1} - \frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!}$$

If we integrate by parts $n-1$ times, we arrive at:

$$R_n = -\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) - \dots - hf'(0) + \int_0^h f'(u)\,du \text{ or } f(h) - f(0) \quad \square$$

$$\boxed{R_n = \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(th)\,dt}$$

Now we show that we can obtain both the Cauchy and Lagrange forms of remainder from this

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Analysis (22)

First we show:

For $f, g : [a,b] \to \mathbb{R}$ continuous with $g(x) \neq 0 \ \forall x \in (a,b)$

(*) Then $\exists c \in (a,b)$ such that $\int_a^b f(x) g(x)\, dx = f(c) \int_a^b g(x)\, dx$

This is like a mean value theorem for integrals.

If we take $g(x) = 1$ for example, $\int_a^b f(x)\, dx = f(c)(b-a)$

Proof [of (*)]

$$G(x) = \int_a^x g(t)\, dt \qquad F(x) = \int_a^x f(t)\, g(t)\, dt$$

The Cauchy Mean Value Theorem applied to F and G means

$$\exists c \in (a,b) \ \ G'(c)[F(b) - F(a)] = F'(c)[G(b) - G(a)]$$

$$\left(\int_a^b f(x) g(x)\, dx - 0 \right) g(c) = f(c)\, g(c) \left(\int_a^b g(x)\, dx \right) = 0$$

We can cancel as $g(c) \neq 0$ $\square$

Let's apply (*) to R_n. First we choose (*) for the case $g = 1$.

"f" $= (1-t)^{n-1} f^{(n)}(th)$ $\qquad \exists \theta \in (0,1)$:

(*) $\Rightarrow R_n = \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta h)$, Cauchy's Form of Remainder.

Finally, use (*) with:

"f" $= f^{(n)}(th)$ $\qquad$ "g" $= (1-t)^{n-1}$ $\qquad$ (>0 on $(0,1)$)

$\Rightarrow \exists \theta \in (0,1)$

$$R_n = \frac{h^n}{(n-1)!} f^{(n)}(h\theta) \underbrace{\int_0^1 (1-t)^{n-1}\, dt}_{\left[-\frac{(1-t)^n}{n}\right]_0^1} = \frac{h^n}{n!} f^{(n)}(\theta h)$$

Lagrange's Form of Remainder

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Analysis (23)

We would like to make sense of $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^{\infty} f(x)\,dx$

<u>Improper Integrals</u>

<u>Definition</u>

Suppose $f\,[0,\infty) \to \mathbb{R}$ is integrable on every interval $[a,R]$, and that $\lim_{R\to\infty} \int_a^R f(x)\,dx$ exists and equals L. Then we say that $\int_0^\infty f(x)\,dx$ <u>exists</u> and <u>converges</u> and that its value is L. If $\lim_{R\to\infty}\int_a^R f(x)\,dx$ does not exist, then we say that $\int_a^\infty f(x)\,dx$ <u>diverges</u>.

<u>Example</u>

$\int_1^\infty \frac{dx}{x^k}$: We must compute $\int_1^R \frac{dx}{x^k}$.

$$\int_1^\infty \frac{dx}{x^k} = \left[\frac{x^{k-1}}{1-k}\right]_1^R \quad \text{for } k \neq 1$$
$$= \frac{R^{1-k}-1}{1-k}$$

Let $R \to \infty$. We conclude that the limit exists if and only if $k > 1$ (with limit $-\frac{1}{1-k}$)

If $k=1$, $\int_1^R \frac{dx}{x} = [\log x]_1^R = \log R$, $\to \infty$ as $R \to \infty$

So $\int_1^\infty \frac{dx}{x^k}$ converges if and only if $k > 1$.

Similarly, we say that $\int_{-\infty}^a f(x)\,dx$ exists if $\lim_{R\to-\infty}\int_R^a f(x)\,dx$ exists.

Finally, to make sense of $\int_{-\infty}^\infty f(x)\,dx$, we consider:

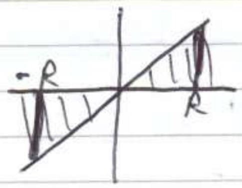
$\lim_{R\to\infty}\int_a^R f(x)\,dx = L_1 \qquad \lim_{R\to-\infty}\int_{-\infty}^a f(x)\,dx = L_2$

If L_1 and L_2 exist, $\int_{-\infty}^\infty f(x)\,dx$ ~~exists~~ converges, and is equal to $L_1 + L_2$. This is independent of a.

Warning $f(x) = x$

$\int_0^R x\,dx \to \infty$ as $R \to \infty$ $\qquad \int_R^0 x\,dx \to -\infty$ as $R \to -\infty$

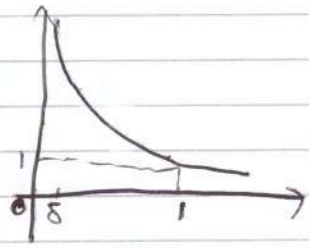
However, $\lim_{R\to\infty} \int_{-R}^R x\,dx = 0$



This does not fit our definition.

Remarks

1. There are other types of improper integrals.



$f(x) = \frac{1}{\sqrt{x}}$, $(0, 1]$, which is unbounded.

$\int_\delta^1 \frac{dx}{\sqrt{x}} = 2 - 2\sqrt{\delta} \to 2$ as $\delta \to 0$ $\quad$ (Improper Integrals of the second kind !!)

So it makes sense to say $\int_0^1 \frac{dx}{\sqrt{x}} = 2$

$f(x) = \frac{1}{x}$, $(0, 1]$

$\int_\delta^1 \frac{dx}{dx} = [\log x]_\delta^1 \Rightarrow \lim_{\delta\to 0} \int_\delta^1 \frac{dx}{x}$ does not exist.

2. Suppose we have 2 functions, f, g on $[a, \infty)$, such that $\exists k > 0$ satisfying $0 \leq f(x) \leq k\,g(x)$ $\forall x \in [a, \infty)$

Then if $\int_a^\infty g(x)\,dx$ converges, so does $\int_a^\infty f(x)\,dx$, with

$\int_a^\infty f(x)\,dx \leq k \int_a^\infty g(x)\,dx$ (comparison test)

Proof

Note that since both functions are ≥ 0, $R \to \int_a^R f$, $R \to \int_a^R g$ are increasing. $\int_a^R g(x)\,dx \leq \int_a^\infty g(x)\,dx$

$f(x) \leq kg(x) \Rightarrow \int_a^R f(x)\,dx \leq k \int_a^R g(x)\,dx$ by the properties of integrals

$\Rightarrow \sup_{R \geq a} \int_a^R f(x)\,dx$ exists. Let this supremum be L.

Now we conclude $\lim_{R\to\infty} \int_a^\infty f(x)\,dx = L$

~~By definition of supremum~~,

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Analysis (23)

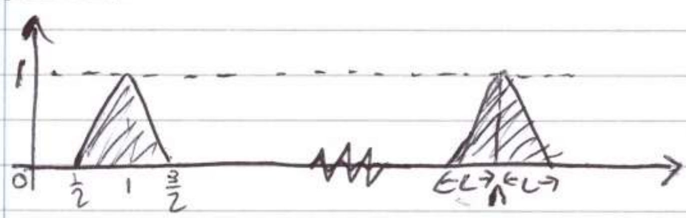
By definition of supremum, given $\varepsilon > 0$, $\exists R_0$ such that $\int_a^{R_0} f(x)\,dx > L - \varepsilon$

If $R \geq R_0$, $\int_a^R f(x)\,dx \geq \int_a^{R_0} f(x)\,dx > L - \varepsilon$

i.e. $0 \leq L - \int_a^R f(x)\,dx$ $\square$

3. Another warning : for series we saw that $\sum a_n$ converges $\Rightarrow a_n \to 0$.
This is not quite the same for improper integrals.

__Example__



where $L = \frac{2}{(n+1)^2}$

$\int_0^\infty f(x)\,dx$ exists because $\sum \frac{2}{(n+1)^2}$ converges.

$f(n) = 1 \quad \forall n \in \mathbb{N}$ so $f(x) \not\to 0$ as $x \to \infty$

__Theorem 5.13__ (The integral test)

Let $f(x)$ be, for $x \geq 1$, a positive, __decreasing__ function of x. Then

1. The integral $\int_1^\infty f(x)\,dx$ and the series $\sum_{n=1}^{\infty} f(n)$ both converge or diverge.
2. As $n \to \infty$, $\sum_{r=1}^{n} f(r) - \int_1^n f(x)\,dx$ tends to a limit L, such that $0 \leq L \leq f(1)$

__Example__

1. $\sum_{n=1}^{\infty} \frac{1}{n^k}$. Consider $f(x) = \frac{1}{x^k}$

We saw that $\int_1^\infty f(x)\,dx$ converges if and only if $k > 1$, so by the integral test, the same is true for the __series__.

2. $\sum_{2}^{\infty} \frac{1}{n \log n}$ $f(x) = \frac{1}{x \log x}$, a decreasing positive function

$$\int_2^{\infty} \frac{dx}{x \log x}$$

$$\int_2^R \frac{dx}{x \log x} = \int_{\log 2}^{\log R} \frac{du}{u} = \left[\log u\right]_{\log 2}^{\log R} = \log(\log R) - \log(\log 2)$$

$\to \infty$ as $R \to \infty$

so the series diverges by the integral test.

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Theorem 5.13 (The integral test)

Let $f(x)$ be, for $x \geq 1$, a positive decreasing function of x. Then

1. The integral $\int_1^\infty f(x)\,dx$ and the series $\sum_{n=1}^{\infty} f(n)$ both converge or diverge.
2. As $n \to \infty$, $\sum_{r=1}^{n} f(r) - \int_1^n f(x)\,dx$ tends to a limit, L, where $0 \leq L \leq f(1)$

Proof

(Note: f is decreasing, so f is integrable on any interval $[1, R]$ by 5.4)

$n-1 \leq x \leq n$ $\quad$ f decreasing $\Rightarrow f(n-1) \geq f(x) \geq f(n)$

By the properties of integration: $f(n-1) \geq \int_{n-1}^{n} f(x)dx \geq f(n)$ $\quad (*)$

Adding: $\sum_{r=1}^{n-1} f(r) \geq \int_1^n f(x)\,dx \geq \sum_{r=2}^{n} f(r)$ $\quad (**)$

Claim 1 in our Theorem follows right away from $(**)$:

If $\sum_{n=1}^{\infty} f(n)$ converges, then from $(**)$, $\int_1^R f(x)\,dx$ is bounded above, and since $R \mapsto \int_1^R f(x)\,dx$ is increasing, We saw before that $\int_0^\infty f(x)\,dx$ converges, so so does $\int_1^\infty f(x)\,dx$.

Now if $\int_0^\infty f(x)\,dx$ converges, $(**)$ implies $\sum_1^\infty f(r)$ is bounded above and thus $\sum_{n=1}^{\infty} f(n)$ converges.

To prove 2: ~~As n →~~ Let $\phi(n) = \sum_{r=1}^{n} f(r) - \int_1^n f(x)\,dx$.

~~As n →~~ $\phi(n) - \phi(n-1) = \sum_{r=1}^{n} f(r) - \int_1^n f(x)dx - \sum_{r=1}^{n-1} f(r) + \int_1^{n-1} f(x)dx$

$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^{n} f(x)\,dx \leq 0$ by $(*)$

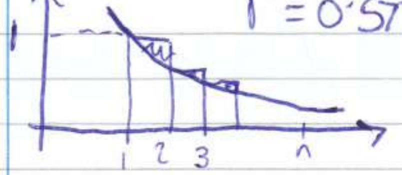
so $\phi(n)$ is decreasing. From $(**)$, $0 \leq \phi(n) \leq f(1)$. By the fundamental axiom $\phi(n)$ converges to a limit, L, and $0 \leq L \leq f(1)$ $\square$

Corollary 5.14 (Euler's Constant)

As $n \to \infty$, $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \to \gamma$, $0 \le \gamma \le 1$

Proof $f(x) = \frac{1}{x}$ is positive decreasing, and apply the previous theorem $\square$

Open problem: Is γ irrational?



$\gamma = 0.577$ We will prove $\frac{1}{2} < \gamma < 1 = f(1)$

Proof $a_n = \int_0^1 \frac{t\,dt}{n(n-t)}$ for $n \ge 2$

$a_n = \frac{1}{n}\int_0^1 \frac{t}{n-t}\,dt$ Use the mean value theorem for integrals.

$= \frac{1}{n}\frac{1}{n-c}\int_0^1 t\,dt$ where $c \in (0,1)$

$= \frac{1}{2n(n-c)} < \frac{1}{2n(n-1)}$

$S_N = \sum_2^N a_n < \sum_2^N \frac{1}{2n(n-1)} = \frac{1}{2}\sum_2^N \left[\frac{1}{n-1} - \frac{1}{n}\right] = \frac{1}{2}\left(1 - \frac{1}{N}\right)$

$S_N \to \frac{1}{2}$ as $N \to \infty$ $\Rightarrow$ $0 < \sum_2^\infty a_n < \frac{1}{2}$

Now we can compute a_n using integration by parts:

$na_n = \int_0^1 \frac{t\,dt}{n-t} = \left[-t\log(n-t)\right]_0^1 + \int_0^1 \log(n-t)\,dt$ $\quad s = n-t$

$= -\log(n-1) + \int_{n-1}^n \log(s)\,ds$

$= -\log(n-1) + \left[s\log s - s\right]_{n-1}^n$

$= n\log\left(\frac{n}{n-1}\right) - 1$

$S_N = \sum_2^N a_n = \sum_2^N \left[\log\left(\frac{n}{n-1}\right) - \frac{1}{n}\right] = \log N - \sum_2^N \frac{1}{n}$

$\to 1 - \gamma$ as $N \to \infty$

$(*) \Rightarrow \frac{1}{2} < \gamma < 1$