Fibred Fibration Categories

Taichi Uemura

June 26, 2016

Fibred type-theoretic fibration categories

A "good" notion of fibred category between categorical models of dependent type theory.

- Construct total type-theoretic structure from fiberwise one
- Change of base

Application

Categorical description of "logical relations" [Hermida, 1993] on HoTT.

Theorem

$t : \Pi_{A:\mathcal{U}} \Pi_{x:A} x = x \to x = x$ *is homotopic to* $\lambda(p : x = x).p^n$ *for some* $n \in \mathbb{Z}$.

Type-theoretic fibration categories

Type-theoretic fibration categories [Shulman, 2015] are sound and complete categorical models of Martin-Löf's intentional theory.

- A category $\mathbb{C}$ equipped with specific morphisms called *fibrations* corresponding to type families.
- Path induction is modeled by the *lifting property*.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathsf{refl}"'] & X \arrow[d, two heads] \\ \mathsf{Id}_A \arrow[r, equal] \arrow[ur, dotted, "J"] & \mathsf{Id}_A \end{tikzcd}$$

  - A morphism like refl is called an *acyclic cofibration*.

Fibred type-theoretic fibration categories

A fibred type-theoretic fibration category is a fibred category $p : \mathbb{E} \to \mathbb{B}$ such that:

1. $\mathbb{E}$ and $\mathbb{B}$ are type-theoretic fibration categories and p preserves all structures of type-theoretic fibration category.
2. A fibration in $\mathbb{E}$ factors as a *vertical* fibration followed by a *horizontal* fibration.

$$
\begin{tikzcd}
 & X \arrow[d, dotted, two heads] \arrow[dr, two heads] & \\
\mathbb{E} \arrow[d, "p"'] & u^*Y \arrow[r, dotted, two heads] & Y \\
\mathbb{B} & \alpha \arrow[r, two heads, "u"'] & \beta
\end{tikzcd}
$$

3. Every Cartesian morphism above an acyclic cofibration is an acyclic cofibration.

Proposition

A fibred category $p : \mathbb{E} \to \mathbb{B}$ is a fibred type-theoretic fibration category if and only if:

1. $\mathbb{B}$ and all fibers $\mathbb{E}_\alpha$ are type-theoretic fibration categories.
2. For any morphism $u : \alpha \to \beta$ in $\mathbb{B}$, $u^* : \mathbb{E}_\beta \to \mathbb{E}_\alpha$ is a strong fibration functor.
3. For every acyclic cofibration $u : \alpha \xrightarrow{\sim} \beta$ and fibration $f : Y \twoheadrightarrow X$ in $\mathbb{E}_\beta$, $u^* : \mathbb{E}_\beta/X(X, Y) \to \mathbb{E}_\alpha/u^*X(u^*X, u^*Y)$ is surjective.
4. For every fibration $u : \alpha \twoheadrightarrow \beta$ in $\mathbb{B}$, $u^* : \mathbb{E}_\beta \to \mathbb{E}_\alpha$ has a right adjoint u_* satisfying the "Beck-Chevalley condition".

Proof of the Proposition, syntactically

A fully categorical proof is in arXiv:1602.08206. I give a syntactic description.
A fibred category $p : \mathbb{E} \to \mathbb{B}$ models a type theory with two sorts Kind and Type.

Category theory	Type theory
$\alpha \in \mathbb{B}$	α : Kind
$X \in \mathbb{E}_\alpha$	$X : \alpha \to$ Type

The Proposition says the pairs of
$(\alpha : \text{Kind}, X : \alpha \to \text{Type})$ form a new type theory.

Proof of the Proposition, syntactically

Concepts	Definition
type	$(\alpha : \mathrm{Kind},\ X : \alpha \to \mathrm{Type})$
element	$(a : \alpha,\ x : X(a))$
family	$(\beta : \alpha \to \mathrm{Kind},$
	$Y : \Pi_{a:\alpha}\beta(a) \to X(a) \to \mathrm{Type})$
section	$(u : \Pi_{a:\alpha}\beta(a),$
	$f : \Pi_{a:\alpha}\Pi_{x:X(a)}Y(a, u(a), x))$
pair	$((a, b) : \Sigma_{a:\alpha}\beta(a),$
	$(x, y) : \Sigma_{x:X(a)}Y(a, b, x))$
identity type	?

Proof of the Proposition, syntactically

We need the path induction on an identity kind w.r.t. any type family over the identity kind.

$$\frac{\begin{array}{l}\alpha : \mathrm{Kind} \\ X : \Pi_{a,a':\alpha} a = a' \to \mathrm{Type} \\ x : \Pi_{a:\alpha} X(a, a, \mathrm{refl}_a)\end{array}}{\begin{array}{l}\mathrm{ind}_{=_\alpha}(X, x) : \Pi_{a,a':\alpha}\Pi_{p:a=a'} X(a, a', p) \\ \mathrm{ind}_{=_\alpha}(X, x, a, a, \mathrm{refl}_a) \equiv x\end{array}}$$

In particular, for α : Kind, $X : \alpha \to \mathrm{Type}$ and $p : a =_\alpha a'$, we have the *transport along* p $p_* : X(a) \to X(a')$.

Proof of the Proposition, syntactically

The identity type of $(\alpha : \mathsf{Kind}, X : \alpha \to \mathsf{Type})$ is the pair of

- $=: \alpha \to \alpha \to \mathsf{Kind}$ and
- $\lambda aa'pxx'.p_* x = x' : \Pi_{a,a':\alpha}\Pi_{p:a=a'} X(a) \to X(a') \to \mathsf{Type}$.

It is the type of "path over path" but in different sorts.

Universes in a fibred setting

Let $\mathcal{U}$: Kind and $\mathcal{V}$: Type be universes of kinds and types. Then $(\mathcal{U}, \lambda(\alpha : \mathcal{U}).\alpha \to \mathcal{V})$ is a universe in the new type theory if, for any $\alpha : \mathcal{U}$ and $X : \alpha \to \mathcal{V}$, $\Pi_{a:\alpha}.X(a) : \mathcal{V}$. Its elements are $(\alpha : \mathcal{U}, X : \alpha \to \mathcal{V})$.

Equivalences in a fibred setting

For $(u : \alpha \to \beta, f : \Pi_{a:\alpha} X(a) \to Y(u(a)))$, An element of is-equiv$(u, f)$ is (v : homotopy inverse of u, g : homotopy inverse of f above v).

Lemma

Suppose the function extensionality holds. Then

$$\text{is-equiv}(u, f) \simeq (\text{is-equiv}(u), \lambda\_.\Pi_{a:\alpha}\text{is-equiv}(f_a))$$

for all $(u : \alpha \to \beta, f : \Pi_{a:\alpha} X(a) \to Y(u(a)))$ in the new type theory.

Univalence in a fibred setting

A universe U in a type theory is *univalent* if the canonical map $\lambda(A : U).(A, A, \mathrm{id}_A) : U \to \Sigma_{A,A':U} A \simeq A'$ is an equivalence.
The new universe $(\mathcal{U}, \lambda(\alpha : \mathcal{U}).\alpha \to \mathcal{V})$ is univalent if $\lambda(\alpha : \mathcal{U}).(\alpha, \alpha, \mathrm{id}_\alpha)$ is an equivalence and for all $\alpha : \mathcal{U}$, $\lambda X.(X, X, \lambda(a : \alpha).\mathrm{id}_{X(a)}) : (\alpha \to \mathcal{V}) \to \Sigma_{X,Y:\alpha\to\mathcal{V}} \Pi_{a:\alpha} X(a) \simeq Y(a)$ is an equivalence. This holds if $\mathcal{U}$ and $\mathcal{V}$ are univalent and the function extensionality holds.

Outline

Introduction

Foundations

Examples

Appendix

Arrow categories

Let $\mathbb{C}$ be a type-theoretic fibration category and write $(\mathbb{C}^{\to})_{\mathsf{f}} \subset \mathbb{C}^{\to}$ for the full subcategory of all the fibrations. Then $\mathbf{cod} : (\mathbb{C}^{\to})_{\mathsf{f}} \to \mathbb{C}$ is a fibred type-theoretic fibration category. In this case $\mathsf{Type} \equiv \mathsf{Kind}$.
If $\mathbb{C}$ has a univalent universe, so does $(\mathbb{C}^{\to})_f$.
Originally, Shulman proved $(\mathbb{C}^{\to})_f$ is a type-theoretic fibration category [Shulman, 2015], and I give a fibred categorical description.

Change of base

Let $p : \mathbb{E} \to \mathbb{B}$ be a fibred type-theoretic fibration category and $F : \mathbb{A} \to \mathbb{B}$ be a functor preserving fibrations, pullbacks of fibrations and acyclic cofibrations. Then the change of base or pullback

$$\begin{tikzcd}
F^*\mathbb{E} \arrow[r] \arrow[d] & \mathbb{E} \arrow[d, "p"] \\
\mathbb{A} \arrow[r, "F"'] & \mathbb{B}
\end{tikzcd}$$

is a fibred type-theoretic fibration category.

Relational model

Let $p : \mathbb{E} \to \mathbb{B}$ be a fibred type-theoretic fibration category. We have the change of base

$$
\begin{tikzcd}
\mathcal{R}el(p) \arrow[r] \arrow[d] & \mathbb{E} \arrow[d, "p"] \\
\mathbb{B} \arrow[r, "\lambda\alpha.\alpha\times\alpha"'] & \mathbb{B}.
\end{tikzcd}
$$

$\mathcal{R}el(p)$ is the category of binary families $(\alpha : \mathrm{Kind}, R : \alpha \to \alpha \to \mathrm{Type})$.

Application

Let $t : \Pi_{A:\mathcal{U}} \Pi_{x:A} x = x \to x = x$.

Theorem

- t is "natural": for any $f : A \to B$ in $\mathcal{U}$, $\mathsf{ap}_f \circ t \sim t \circ \mathsf{ap}_f$.

$$\begin{tikzcd} x = x \arrow[r, "t"] \arrow[d, "\mathsf{ap}_f"'] & x = x \arrow[d, "\mathsf{ap}_f"] \\ fx = fx \arrow[r, "t"'] & fx = fx \end{tikzcd}$$

- If the type theory has $\mathbb{S}^1$, then for some $n \in \mathbb{Z}$, $tp = p^n$ for all $p : x = x$.

Application

Let $\mathbb{C}$ be the syntactic category of Martin-Löf type theory with a univalent universe $\mathcal{U}$. Then $(\mathcal{U}, \lambda(A, B : \mathcal{U}).A \to B \to \mathcal{U})$ is a univalent universe in $\mathcal{R}el(\mathbf{cod})$. We have a strong fibration functor

$$\begin{tikzcd} & \mathcal{R}el(\mathbf{cod}) \arrow[d] \\ \mathbb{C} \arrow[ur, dotted, "R"] \arrow[r, equal] & \mathbb{C}. \end{tikzcd}$$

In particular, for every closed term $t : A$, we have $R_A : A \to A \to \text{Type}$ and $R_t : R_A(t, t)$.

Application

Let $t : \Pi_{A:\mathcal{U}}\Pi_{x:A} x = x \to x = x$. Then $R_t :$ $\Pi_{A,B:\mathcal{U},W:A\to B\to\mathcal{U}}\Pi_{x:A,y:B,v:W(x,y)}\Pi_{p:x=x,q:y=y}(p,q)_* v = v \to (tp, tq)_* v = v$.

$$\begin{tikzcd} x \arrow[r, equal, "p"] \arrow[d, dash, "v"'] \arrow[dr, phantom, "="] & x \arrow[d, dash, "v"] \\ y \arrow[r, equal, "q"'] & y \end{tikzcd} \to \begin{tikzcd} x \arrow[r, equal, "tp"] \arrow[d, dash, "v"'] \arrow[dr, phantom, "="] & x \arrow[d, dash, "v"] \\ y \arrow[r, equal, "tq"'] & y \end{tikzcd}$$

Application

Given $f : A \to B$ in $\mathcal{U}$, let $W(x, y) \equiv fx = y$. Then R_t looks like

$$\begin{array}{ccc} fx & \overset{\mathsf{ap}_f\, p}{=\!=\!=} & fx \\ v \Big\| & = & \Big\| v \\ y & \underset{q}{=\!=\!=} & y \end{array} \quad \to \quad \begin{array}{ccc} fx & \overset{\mathsf{ap}_f(tp)}{=\!=\!=} & fx \\ v \Big\| & = & \Big\| v \\ y & \underset{tq}{=\!=\!=} & y. \end{array}$$

Let $y \equiv fx$, $q \equiv \mathsf{ap}_f\, p$ and $v \equiv \mathsf{refl}_{fx}$ and apply the function to $\mathsf{refl}_{\mathsf{ap}_f\, p}$. Then $\mathsf{ap}_f(tp) = t(\mathsf{ap}_f\, p)$ for all $x : A$ and $p : x = x$.

Application

Suppose the type theory has $\mathbb{S}^1 : \mathcal{U}$ with $b : \mathbb{S}^1$ and $l : b = b$. Then $(\mathbb{S}^1, =_{\mathbb{S}^1})$ is a unit circle in $\mathcal{R}el(\mathbf{cod})$ and we still have the functor $R : \mathbb{C} \to \mathcal{R}el(\mathbf{cod})$. Let $y : B$ and $q : y = y$ which corresponds to $f : \mathbb{S}^1 \to B$. $t(l) = l^n$ for some $n \in \mathbb{Z}$ and

$$\begin{aligned} tq &= t(\mathrm{ap}_f(l)) \\ &= \mathrm{ap}_f(t(l)) \\ &= \mathrm{ap}_f(l^n) \\ &= (\mathrm{ap}_f(l))^n \\ &= q^n \end{aligned}$$

References I

Hermida, C. (1993).

Fibrations, Logical Predicates and Indeterminates.

PhD thesis, University of Edinburgh.

Shulman, M. (2015).

Univalence for inverse diagrams and homotopy canonicity.

Mathematical Structures in Computer Science, 25(05):1203–1277.