TWISTED ACTIONS AND OBSTRUCTIONS IN GROUP COHOMOLOGY

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This article is intended to answer the question "Why do you guys always want to twist everything?" We review the various ways in which twists, twisted actions and twisted crossed products arise, and then discuss some cohomological obstructions to the existence and triviality of twisted actions.

Our review begins with the classical problems of classifying group extensions and irreducible unitary representations, and explains how cocycles and twisted actions arise. We then describe the various kinds of twisted actions occurring in the operator-algebra literature. For completeness, we have included brief descriptions of how cocycles arise in quantum mechanics and deformation theory.

In §2, we discuss three cohomological obstructions. These lie in, respectively, a second cohomology group, a relative cohomology group introduced by Huebschmann and Ratcliffe, and a third cohomology group. All these cohomology groups have descriptions in terms of extensions, and we use these descriptions wherever possible. The groups are related by an eight-term exact sequence which extends the usual five-term restriction-inflation sequence; our main point is that this exact sequence is compatible with our use of cohomology classes as obstructions to problems involving actions on operator algebras (see Theorem 4).

Although we are primarily interested in group actions on C^*-algebras, the underlying ideas are intrinsically algebraic, and we consider here actions of groups (what analysts call "discrete groups") on algebras and $*$-algebras with identities. One relic of our prejudices is our use of $\mathbb{C}G$ to denote the group $*$-algebra, which is universal for unitary representations, rather than the usual group algebra. In a short final section, we briefly discuss what would need to be changed to get a theory directly applicable to actions of locally compact groups on C^*-algebras.

Many of the ideas discussed in this paper are folklore. They are prominent in the classification of group actions on von Neumann algebras, as developed by Jones, Sutherland, Takesaki and Kawahigashi [15, 36, 16]. They have also appeared in the classification of group actions on C^*-algebras, especially in the work of Rosenberg [13, 28, 33], and in mathematical physics [7, 6], but do not seem to be so well-known in these contexts. We have no idea how familiar they will be to algebraists.

1. How twisted actions arise

1.1. **Group extensions.** Twists first arise when we try to form products of two groups N and G: we seek groups E containing N as a normal subgroup with quotient E/N

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isomorphic to G. In other words, we are looking for short exact sequences

$$e \longrightarrow N \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow e, \tag{1}$$

or *extensions E of N by G*. If N is abelian, we can further ask that i has range in the centre of E: that E is a *central extension of N by G*. We can always take E to be the direct product $N \times G$, but are there others?

To answer this, suppose we have a central extension (1). For each $s \in G$, we can choose an element $c(s)$ of E such that $\pi(c(s)) = s$. The map $c : G \to E$ need not be a homomorphism, but if it is, the map $(n, s) \mapsto i(n)c(s)$ is an isomorphism of $N \times G$ onto E; we say that c is a splitting for (1), and we are asserting that a split central extension is trivial. In general, we can decide whether the extension (1) is split (or, strictly speaking, splittable) by comparing $c(s)c(t)$ with $c(st)$: since π is a homomorphism, these two elements of E have the same image st in G, and hence by exactness there is a unique element $\omega(s, t)$ of N such that

$$c(s)c(t) = i(\omega(s, t))c(st). \tag{2}$$

By associativity, $(c(r)c(s))c(t)$ and $c(r)(c(s)c(t))$ are the same multiple of $c(rst)$, and it follows that the function $\omega : G \times G \to N$ satisfies the *cocycle identity*

$$\omega(r, s)\omega(rs, t) = \omega(s, t)\omega(r, st) \quad \text{for } r, s, t \in G; \tag{3}$$

it is handy to assume that we chose $c(e) = e$, and then ω satisfies $\omega(s, e) = \omega(e, s) = e$. One way to obtain functions ω satisfying (3) is to start with a function $\rho : G \to N$ such that $\rho(e) = e$, and take $\omega(s, t) = \rho(s)\rho(st)^{-1}\rho(t)$; we write $\omega = \delta\rho$ and call ω a *coboundary*. If the cocycle ω arising from our extension is a coboundary $\delta\rho$, then $s \mapsto i(\rho(s))^{-1}c(s)$ is a splitting for (1).

We often call the cocycle ω a *twist* because we can use it to define a new "twisted" multiplication $\cdot_\omega$ on $N \times G$ by $(m, s) \cdot_\omega (n, t) := (mn\omega(s, t), st)$; the cocycle identity implies that $\cdot_\omega$ is an associative multiplication. The resulting group $E_\omega := (N \times G, \cdot_\omega)$ is an extension of N by G: take $i(n) := (n, e)$ and $\pi(n, s) = s$. Indeed, carrying out the construction of the preceding paragraph with $E = E_\omega$ and $c(s) := (e, s)$ gives the cocycle ω we started with.

The cocycles form an abelian group $Z^2(G, N)$ under pointwise multiplication, and the coboundaries a subgroup $B^2(G, N)$. The quotient $H^2(G, N) := Z^2(G, N)/B^2(G, N)$ is called the *second cohomology group of G with coefficients in N*. One can show without great difficulty that, given a central extension (1), the class $[\omega]$ of the cocycle ω defined by (2) vanishes in $H^2(G, N)$ precisely when the extension is split; we say that $[\omega]$ is a *complete obstruction* to the splitting of (1). The twisting construction of the previous paragraph shows that every class in $H^2(G, N)$ arises, and one can alternatively view $H^2(G, N)$ as the set of central extensions modulo a natural equivalence relation — indeed, one can even realise the product in H^2 in terms of extensions [21, Chapter IV.4].

As the name suggests, the group $H^2(G, N)$ is just the second in a family of cohomology groups $H^n(G, N)$ parametrised by $\{n : n \geq 0\}$. These are groups of cocycles (functions $f : G^n \to N$ satisfying a multivariable version of (3)) modulo a subgroup of coboundaries, and are interesting because they often have interpretations like the one we have given in terms of extensions. Having the entire theory, though, makes a great deal of machinery available to help with computations. Here we shall be emphasising the interpretations: we are interested in group cohomology because it provides a natural and computable home for obstructions.

1.2. **Representation theory: the Mackey machine.** One way to find the irreducible unitary representations of a group G is to choose a normal subgroup N whose representation theory is known, and apply the following procedure, which is called the Mackey machine.

The action of G by conjugation on N lifts to an action on the space $\widehat{N}$ of equivalence classes of irreducible unitary representations of N: $(s \cdot U)_n := U_{sns^{-1}}$. The isotropy or stabiliser subgroup G_U consists of the elements $s \in G$ such that $s \cdot U$ is equivalent to U; thus for each $s \in G_U$, there is a unitary V_s on the Hilbert space H_U of U such that $U_{sns^{-1}} = V_s U_n V_s^*$. Since $U_{stn(st)^{-1}} = U_{s(tnt^{-1})s^{-1}}$, the operator $V_{st}^* V_s V_t$ commutes with all the U_n, and hence by irreducibility of U must be a multiple $\omega(s,t)1$ of the identity operator. The associativity of multiplication in $G_U \subset G$ implies that $\omega : G \times G \to \mathbb{T}$ satisfies the cocycle identity (3); we then call V an ω-representation of G_U, or a *projective representation with cocycle* ω. Because we can choose $V_n = U_n$ for $n \in N$, we can assume that the cocycle ω is identically 1 on $N \times N$; because $V_s V_n V_s^* = V_{sns^{-1}}$, ω satisfies

$$\omega(s,n) = \omega(sns^{-1}, s) \ \text{ for } s \in G_U,\ n \in N.$$

One can deduce from this (see, for example, [27, Proposition A2][1]) that ω differs by a coboundary from a cocycle which has been *inflated* from a cocycle σ on G_U/N: $\omega = \delta\rho \cdot \sigma \circ (q \times q)$, where $q : G_U \to G_U/N$ is the quotient map and $\rho : G_U \to \mathbb{T}$; adjusting our choices for V_s by the scalars $\rho(s)^{-1}$ shows that we may as well assume $\omega = \inf \sigma := \sigma \circ (q \times q)$. The cohomology class $[\sigma]$ in $H^2(G_U/N, \mathbb{T})$, which vanishes precisely when U extends to a unitary representation of G_U, is called the *Mackey obstruction* at $U \in \widehat{N}$.

To get unitary representations of G, we take a unitary $\overline{\sigma}$-representation W of G_U/N, tensor it with V to get a unitary representation $V \otimes (W \circ q)$ of G_U, and induce to get a unitary representation $\operatorname{Ind}_{G_U}^G V \otimes (W \circ q)$ of G. Mackey's theory says that if $\widehat{N}$ and the action of G on $\widehat{N}$ are nice enough, then all irreducible representations of G arise this way for some irreducible $\overline{\sigma}$-representation W of the "little group" G_U/N [20]; up to equivalence, $\operatorname{Ind} V \otimes (W \circ q)$ depends on the orbit of U and the equivalence class of W, but not on V or any choices we made in getting from ω to σ.

The message is that, even if we are only interested in ordinary unitary representations, we might well be forced to consider projective representations, albeit of smaller groups. The good news is that things don't get any worse if we try to iterate this process: Mackey showed in [20] that his method also works for projective representations.

1.3. **Covariant representations of dynamical systems.** Ten years after Mackey's paper appeared, Takesaki showed that the same ideas apply to a dynamical system (A, G, α) consisting of an action of a group G by automorphisms α_s of a $*$-algebra A. We are interested in *covariant representations* (π, U) consisting of a nondegenerate $*$-representation $\pi : A \to B(H)$ and a unitary representation $U : G \to U(H)$ satisfying

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^* \tag{4}$$

[1]This Proposition is basically exactness at $H^2(G_U, \mathbb{T})$ of the extended restriction-inflation sequence (15) below.

for $a \in A$, $s \in G$. The group G acts on the set $\widehat{A}$ of equivalence classes of irreducible representations of A via $s \cdot \pi := \pi \circ \alpha_s^{-1}$. The arguments of §1.2 show that for every irreducible representation π of A, there is a projective representation U of the stabiliser G_π which satisfies (4) for $s \in G_\pi$. As before, we can kill the cocycle by tensoring U with a projective representation W of opposite cocycle; then $(\pi \otimes 1, U \otimes W)$ is a covariant representation of (A, G_π, α), and we can induce to get a covariant representation of (A, G, α). Takesaki showed that, under stringent conditions on (A, G, α) (including that A be a C^*-algebra), one gets all the irreducible covariant representations of (A, G, α) this way [37]. His results were substantially improved by Green [11].

1.4. **Twisted actions I: Busby-Smith twists.** Takesaki's insight was motivated by the observation that one can recover the group algebra $\mathbb{C}(N \rtimes G)$ of a semidirect product as a crossed product $(\mathbb{C}N) \rtimes G$. To make sense of this, we begin with an action of G by automorphisms $n \mapsto s \cdot n$ of a group N. The *semidirect product* $N \rtimes G$, which is the set $N \times G$ with the product $(m, s)(n, t) = (m(s \cdot n), st)$, is an extension of N by G, but is not central: conjugation implements the given action of G on N, in the sense that $(e, s)(n, e)(e, s)^{-1} = (s \cdot n, e)$. Unitary representations of $N \rtimes G$ on H are given by pairs of representations $W : N \to U(H)$, $U : G \to U(H)$ satisfying

$$U_s W_n U_s^* = W_{s \cdot n}; \tag{5}$$

it is easy to check, for example, that $(n, s) \mapsto W_n U_s$ is a unitary representation of $N \rtimes G$ when (W, U) satisfy (5).

The *crossed product* $A \rtimes_\alpha G$ of a dynamical system (A, G, α) is the $*$-algebra $k(G, A)$ of functions $f : G \to A$ of finite support, with multiplication and involution given by

$$(f * g)(s) = \sum_{r \in G} f(r)\alpha_r(g(r^{-1}s)) \quad \text{and} \quad f^*(s) = \alpha_s(f(s^{-1})^*).$$

For example, when $A = \mathbb{C}$, we recover the usual group $*$-algebra $\mathbb{C}G$ as $k(G)$. The point of the construction is that there is a bijection between the covariant representations of (A, G, α) and the nondegenerate $*$-representations of $A \rtimes_\alpha G$, which takes (π, U) to $\pi \times U : f \mapsto \sum_s \pi(f(s))U_s$.

When $N \rtimes G$ is the semidirect product associated to an action of G by automorphisms of N, linearising converts the action to an action β of G by automorphisms of the group algebra $\mathbb{C}N$: in terms of canonical generators $\{\delta_n : n \in N\}$ for $\mathbb{C}N$, we have $\beta_s(\delta_n) = \delta_{s \cdot n}$, and in terms of functions $f \in k(N)$, we have $\beta_s(f)(n) = f(s^{-1} \cdot n)$. Since linearisation converts unitary representations W of N to $*$-representations π_W of $\mathbb{C}N$, the crossed product $\mathbb{C}N \rtimes_\beta G$ is universal for pairs of unitary representations (W, U) satisfying (5), and hence is canonically isomorphic to the group algebra $\mathbb{C}(N \rtimes G)$. In terms of functions, the isomorphism takes $f \in k(G)$ to the function $s \mapsto (n \mapsto f(n, s))$ in $k(G, k(N))$.

For a general group extension

$$e \longrightarrow N \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow e,$$

$\mathbb{C}E$ is not a crossed product $\mathbb{C}N \rtimes G$ in the usual sense: though $\mathbb{C}E$ and $k(G, \mathbb{C}N)$ are still isomorphic as vector spaces, it takes some work to describe the multiplication. We begin by fixing a section $c : G \to E$ such that $c(e) = e$, and defining $\omega : G \times G \to N$ by

$c(s)c(t) = \omega(s,t)c(st)$; as before, associativity implies that ω satisfies a cocycle identity of the form

$$\omega(r,s)\omega(rs,t) = c(r)\omega(s,t)c(r)^{-1}\omega(r,st). \tag{6}$$

The presence of the section c is unavoidable: if N is nonabelian, the action of E on N by conjugation need not descend to an action of $G = E/N$. However, it does still linearise to give an action β of E on $\mathbb{C}N$, and composing with the section c gives a map $\alpha := \beta \circ c$ of G into $\operatorname{Aut}\mathbb{C}N$. If we view ω as a map u with values in $N \subset U(\mathbb{C}N)$ (strictly speaking, $u(s,t) = \delta_{\omega(s,t)} \in \mathbb{C}N$), then we have

$$\alpha_s \circ \alpha_t = \operatorname{Ad} u(s,t) \circ \alpha_{st} \tag{7}$$

$$u(r,s)u(rs,t) = \alpha_r(u(s,t))u(r,st) \tag{8}$$

and, because $c(e) = e$, we also have

$$\alpha_e = \mathrm{id} \quad \text{and} \quad u(s,e) = u(e,s) = 1. \tag{9}$$

A pair $\alpha : G \to \operatorname{Aut} A$ and $u : G \times G \to U(A)$ satisfying (7), (8) and (9) is called a *Busby-Smith twisted action* of G on A, after [4].

Given a Busby-Smith twisted action (α, u) of G on a $*$-algebra A, we can form the *twisted crossed product* $A \rtimes_{\alpha,u} G$ by putting a new multiplication and involution on $k(G, A)$:

$$(f * g)(s) = \sum_{r \in G} f(r)\alpha_r(g(r^{-1}s))u(r, r^{-1}s) \quad \text{and} \quad f^*(s) = u(s, s^{-1})^*\alpha_s(f(s^{-1})^*).$$

This $*$-algebra is universal for pairs consisting of a representation $\pi : A \to B(H)$ and a map $U : G \to U(H)$ satisfying

$$\pi(\alpha_s(a)) = U_s\pi(a)U_s^* \quad \text{and} \quad U_sU_t = \pi(u(s,t))U_{st}.$$

Some messy calculations show that if (α, u) is the twisted action of G on $\mathbb{C}N = k(N)$ arising from the extension (1) and the section $c : G \to E$, then $\phi(f)(s)(n) := f(nc(s))$ defines an isomorphism ϕ of $\mathbb{C}E$ onto $\mathbb{C}N \rtimes_{\alpha,u} G$. More generally, if we start with a twisted action (α, u) of E on A, then there is a twisted action (β, v) of E/N on A such that $A \rtimes_{\alpha,u} E \cong (A \rtimes_{\alpha,u} N) \rtimes_{\beta,v} E/N$ [25, Theorem 4.1]. (Formally, the twisted action (β, v) depends on the choice of a section $c : E/N \to E$, as it did in our construction involving $\mathbb{C}N$ above, but this can be handled by introducing a notion of exterior equivalence of twisted actions [25, §3].)

One generally expects these twisted crossed products to behave like ordinary ones. Indeed, the *stabilisation trick* of [25] says that, after tensoring with a large matrix algebra $\mathcal{K}$ (the finite rank operators on $\ell^2(G)$, or the compact operators if we are working with C^*-algebras), every twisted crossed product $(A \rtimes_{\alpha,u} G) \otimes \mathcal{K}$ is isomorphic to an ordinary crossed product of the form $(A \otimes \mathcal{K}) \rtimes G$. So when dealing with invariants such as K-theory, which are not affected by stabilisation, twisted crossed products are no harder than ordinary ones [26]. Representation theory, though, is changed by stabilising, and for technical reasons there is as yet no Mackey machine for Busby-Smith twisted crossed products. Nevertheless, a good deal is known; for example, see Packer's survey of twisted transformation group algebras (twisted crossed products of commutative C^*-algebras) in [24].

1.5. **Twisted actions II: Green twisting maps.** Green [11] introduced an alternative notion of twisted crossed products which has some technical advantages. We consider again a general group extension (1). Green observed that the action of E by conjugation on $\mathbb{C}N$ is implemented on N by the canonical copy $\{\delta_n : n \in N\}$ of N inside $U(\mathbb{C}N)$. More generally, we say that a homomorphism $\tau : N \to \operatorname{Aut} A$ is a *Green twisting map* for an action $\alpha : E \to \operatorname{Aut} A$ if

$$\alpha_n = \operatorname{Ad} \tau(n) \quad \text{and} \quad \alpha_s(\tau(n)) = \tau(sns^{-1}) \quad \text{for } n \in N,\, s \in E, \tag{10}$$

and that the pair (α, τ) is a *Green twisted action* of (E, N). Green's twisted crossed product $A \rtimes_{\alpha,\tau} (E, N)$ is then by definition the quotient of $A \rtimes_\alpha E$ by the ideal $I(\tau)$ generated by $\{\delta_e \tau(n) - \delta_n 1_A\}$; equivalently, the representations of $A \rtimes_{\alpha,\tau} (E, N)$ are given by the covariant representations (π, U) of (A, E, α) satisfying $\pi \circ \tau = U|_N$.

The theory of Green's twisted crossed products of C^*-algebras is much better developed than that of Busby-Smith twisted crossed products. There is a decomposition theorem like that of [25, Theorem 4.1]: if M is a normal subgroup of E containing N, then there is an isomorphism of the form $A \rtimes_{\alpha,\tau} (E, N) \cong (A \rtimes_{\alpha,\tau} (M, N)) \rtimes (E, M)$ [11, Proposition 1]. Echterhoff has proved a version of the stabilisation trick which uses Morita equivalence instead of stabilisation [10]. And, most importantly, Green has shown that there is a Morita-equivalence-based version of the Mackey machine for his twisted crossed products [11, 12]. The main technical advantage is that, because $A \rtimes_{\alpha,\tau} (E, N)$ is a quotient of $A \rtimes_\alpha E$, one can extend theorems from ordinary to twisted crossed products by passing to quotients.

In the algebraic setting, Green twisted actions are pretty much equivalent to Busby-Smith twisted actions. Given a Green twisted action (β, τ) of (E, N) and a section $c : G = E/N \to E$ such that $c(e) = e$, there is a Busby-Smith twisted action of G on B given by

$$\alpha_s = \beta_{c(s)}, \quad u(s, t) = \tau(c(s)c(t)c(st)^{-1});$$

then the systems $(A, (E, N), (\beta, \tau))$ and $(A, G, (\alpha, u))$ have the same representation theory and crossed products [25, Proposition 5.1].

On the other hand, given a Busby-Smith twisted action (α, u) of G on A, we can construct a Green twisted action of the group-theoretic twisted crossed product $E := U(A) \rtimes_{\alpha,u} G$, which is the set $U(A) \times G$ with the product

$$(u, s)(v, t) = (u\alpha_s(v)u(s, t), st)$$

(see Proposition 6 below). The formula $\beta_{(u,s)} := \operatorname{Ad} u \circ \alpha_s$ defines an action of E on A for which $\tau(v) := (v, e)$ is a Green twisting map (verifying this requires the formulas in the proof of Proposition 6). Applying the construction of the previous paragraph with $c(s) := (1, s)$ gives $u(s, t) = \tau(c(s)c(t)c(st)^{-1})$, so it again follows from [25, Proposition 5.1] that the two systems have the same representation theory and crossed products.

Thus Busby-Smith twisted actions of G are equivalent to Green twisted actions of some pair (E, N) with $E/N = G$. Nevertheless, it is reasonable to fix an extension (E, N) with $E/N = G$, and ask whether a given Busby-Smith twisted action is equivalent to a Green twisted action of (E, N). We will see in the next section that questions like this have interesting interpretations in group cohomology. This question is particularly interesting for actions of locally compact groups on C^*-algebras: the group $U(A) \rtimes_{\alpha,u} G$

need not be locally compact even if G is, so a given Busby-Smith twisted action may not be realisable as a Green twisted action of a locally compact group E, and Green's theory is formally less general. However, Green's theory does seem to suffice for most applications, and his deep analysis in [11, 12] makes his theory a powerful tool.

1.6. **Quantum mechanics.** The states of a quantum mechanical system are represented by unit vectors h in a Hilbert space H — or, more precisely, by the unit rays $[h] := \{zh : z \in \mathbb{T}\}$. Symmetries of the system are bijections of the set $S(H)$ of unit rays which preserve the transition probabilities $P_{[h],[k]} := |(h \,|\, k)|$. A theorem of Wigner asserts that every symmetry is represented by a unitary or antiunitary operator on H, and two operators U, V implement the same symmetry precisely when there is a scalar $z \in \mathbb{T}$ such that $U = zV$ (see [5] for a recent discussion).

The symmetry of the underlying physical system is realised by a homomorphism of the appropriate symmetry group G into the group $\operatorname{Symm} H$ of symmetries. If G is a connected Lie group and the action of G on $\operatorname{Symm} H$ is suitably continuous, it follows from Wigner's theorem that there are unitary operators U_s on H such that $s\cdot[h] = [U_s h]$. Since $(st)\cdot[h] = s\cdot(t\cdot[h])$, U_{st} and U_sU_t implement the same symmetry, and there exists $\omega(s,t) \in \mathbb{T}$ such that $U_sU_t = \omega(s,t)U_{st}$. Comparing $U_{r(st)}$ with $U_{(rs)t}$ shows that ω satisfies the cocycle identity (3), and this cocycle is a coboundary precisely when we can choose U to be a homomorphism.

In general, though, the action of G lifts to a projective unitary representation U with cocycle ω, as in §1.2, and non-trivial cocycles ω can arise. For example, when the system consists of just one particle moving freely in $\mathbb{R}^3$, the full symmetry group $E(3,\mathbb{R})$ of $\mathbb{R}^3$ should act as symmetries. Since $E(3,\mathbb{R})$ contains the group $SO(3,\mathbb{R})$ of rotations, and this group is a connected Lie group, there is an action $SO(3,\mathbb{R}) \to \operatorname{Symm} H$ which lifts to a projective representation $U : SO(3,\mathbb{R}) \to U(H)$. When G is a connected compact Lie group, like $SO(3,\mathbb{R})$, the cohomology group $H^2(G,\mathbb{T})$ is isomorphic to the fundamental group $\pi_1(G)$ (see, for example, [22, Proposition 2.1]); since $\pi_1(SO(3,\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$ [34, page 150], it is definitely possible that the projective representation U has non-trivial cocycle. (One way to deal with this is to inflate the representation to the simply-connected covering group $\operatorname{Spin}(3,\mathbb{R})$ of $SO(3,\mathbb{R})$, where every cocycle is a coboundary. This circle of ideas leads to the concept of spin of a particle. See [34, §VII.7] for a discussion of $\operatorname{Spin}(3,\mathbb{R})$, and [38, Chapter IX] for a mathematical treatment of spin.)

More recently, twisted crossed products of C^*-algebras have been used in several different ways in the mathematical physics literature. Particularly interesting is their appearance in Bellissard's explanation of the quantum Hall effect: the underlying algebra of observables in his model is a Busby-Smith twisted crossed product (see [2, 8]).

1.7. **Deformations.** In one approach to quantum mechanics, one takes the commutative algebra A of observables modelling a classical system — that is, an algebra of functions on the state space — and constructs a family of noncommutative algebras $A_\hbar$ which have the same underlying vector space and multiplication depending on a real parameter $\hbar$, and which converge in some sense to A as $\hbar \to 0$ (see, for example, [32]). In key examples, the state space is the compact dual of an abelian group G, and A is the Fourier transform of the group $*$-algebra $\mathbb{C}G$; $\mathbb{C}G$ is spanned by a canonical unitary representation $\delta : G \to U(\mathbb{C}G) = \{u \in \mathbb{C}G : u^*u = uu^* = 1\}$, with multiplication given on generators by $\delta_s\delta_t = \delta_{s+t}$.

To deform $\mathbb{C}G$, we choose a cocycle $\omega : G \times G \to \mathbb{T}$, and define a new multiplication by $\delta_s *_\omega \delta_t = \omega(s,t)\delta_{s+t}$, to obtain a $*$-algebra $\mathbb{C}(G,\omega)$ which is universal for the projective representations with cocycle ω; $\mathbb{C}(G,\omega)$ is called a *twisted group $*$-algebra.* (Alternatively, one can define $\mathbb{C}(G,\omega)$ to be the Busby-Smith twisted crossed product $\mathbb{C} \rtimes_{\mathrm{id},\omega} G$.) For the groups $\mathbb{Z}^2$ and $\mathbb{R}^2$, every cocycle is equivalent to one of the form

$$\omega_\theta((x_1,y_1),(x_2,y_2)) := \exp(2\pi i\theta y_1 x_2),$$

where θ is a real number, and the twisted group $*$-algebras $\mathbb{C}(G,\omega_{\hbar\theta})$ (or their enveloping C^*-algebras) form a quantum deformation of $\mathbb{C}G$ [32]. These examples are more pervasive than one might expect: many apparently quite different C^*-algebraic quantum deformations can be obtained from deformations of this sort by inducing and taking generalised fixed-point algebras [17, 18].

2. Obstructions to the existence of twisted actions

In this section we consider various questions associated with actions of groups on algebras. These questions are all of the general flavour "Is the situation nicer than it seems at first sight?" We show that each of these questions has a complete obstruction in an appropriate cohomology group. These cohomology groups are related by an exact sequence extending the usual inflation-restriction sequence; we show that this exact sequence respects our interpretation of cohomology classes as obstructions.

We consider an algebra B and a fixed group $U(B)$ of units (invertible elements) in B. For each $u \in U(B)$, there is an inner automorphism $\operatorname{Ad} u : b \mapsto ubu^{-1}$; we denote by $\operatorname{Inn} B$ the group of inner automorphisms of B. Notice that we have an extension

$$1 \longrightarrow UZ(B) \longrightarrow U(B) \xrightarrow{\operatorname{Ad}} \operatorname{Inn} B \longrightarrow \mathrm{id},$$

where $UZ(B)$ is the intersection of $U(B)$ with the centre $Z(B)$ of B, and id denotes the identity automorphism of B.

2.1. Actions by inner automorphisms.

Suppose we have an action $\alpha : G \to \operatorname{Inn} B$ of a group G by inner automorphisms of B. Each automorphism α_s has the form $\operatorname{Ad} u_s$ for some $u_s \in U(B)$; because $\operatorname{Ad} u_s u_t = \operatorname{Ad} u_{st}$, there exists $\mu(s,t) \in UZ(B)$ such that $u_s u_t = \mu(s,t)u_{st}$, and (as usual) comparing $u_{r(st)}$ with $u_{(rs)t}$ shows that μ satisfies the cocycle identity (3). The class $c(\alpha) := [\mu]$ of the cocycle in $H^2(G, UZ(B))$ vanishes if and only if α is implemented by by a homomorphism $u : G \to U(B)$. If so, the system (B, G, α) is in some sense trivial, because the interaction between G and B can be untangled. More formally, the map $(\pi, U) \mapsto (\pi, \pi(u^{-1})U)$ is a bijection between the covariant representations of the system and pairs of commuting representations of B and G, and the crossed product $B \rtimes_\alpha G$ which is universal for covariant representations decomposes as a tensor product $B \otimes \mathbb{C}G$.

If one prefers to think of $H^2(G, UZ(B))$ as equivalences classes of extensions, one can construct the obstruction $c(\alpha)$ directly:

Lemma 1. *Suppose $\alpha : G \to \operatorname{Inn} B$ is an action of G by inner automorphisms. Let*

$$E = \{(u,s) \in U(B) \times G : \operatorname{Ad} u = \alpha_s\},$$

with multiplication given by $(u,s)(v,t) = (uv, st)$*, and define* $i(u) = (u,e)$ *and* $q(u,s) = s$*. Then*

$$1 \longrightarrow UZ(B) \xrightarrow{i} E \xrightarrow{q} G \longrightarrow e$$

is an extension of $UZ(B)$ *by* G *whose class* $c(\alpha)$ *in* $H^2(G, UZ(B))$ *vanishes if and only if there is a homomorphism* $u : G \to U(B)$ *such that* $\alpha = \operatorname{Ad} u$*.*

Proof. It is easy to verify that E is an extension as claimed. The class $c(\alpha)$ is trivial if and only if the extension is split. Now any map c of G into E for which $q \circ c$ is the identity must have the form $c(s) = (u_s, s)$, and it is straightforward to check that c is a homomorphism if and only if u is. □

We think of the class $c(\alpha)$ as the obstruction to triviality of the action $\alpha : G \to \operatorname{Inn} B$. Our goal here is to describe similar obstructions to the existence of Green and Busby-Smith twisted actions, and the relationships between them.

2.2. **The obstruction to existence of a Green twisting map.** Consider an action $\alpha : G \to \operatorname{Aut} B$ of a group on an algebra B and a fixed normal subgroup N of G such that $\alpha(N) \subset \operatorname{Inn} B$. We ask whether there is a Green twisting map $\tau : N \to U(B)$ for α. Proceeding in the familiar way, we choose $u_n \in U(B)$ such that $\alpha_n = \operatorname{Ad} u_n$ for $n \in N$, and find that there is a cocycle $\mu : N \times N \to UZ(B)$ such that $u_m u_n = \mu(m,n) u_{mn}$ (so μ represents $c(\alpha|_N) \in H^2(N, UZ(B))$); there is also a map $\lambda : G \times N \to UZ(B)$ such that $\alpha_s(u_{s^{-1}ns}) = \lambda(s,n) u_n$. The pair (λ, μ) satisfies a complicated set of cocycle identities and compatibility conditions (see [29, Lemma 5.1]), and such pairs are the cocycles for a relative cohomology group $\Lambda(G, N; UZ(B))$; the class $\lambda(\alpha) := [\lambda, \mu]$ vanishes if and only if we can choose u to be a Green twisting map for α on N [29, Proposition 5.4].

To make contact with the algebra literature, we shall use the following description of $\Lambda(G, N; A)$ from [14] and [31]. Let N be a normal subgroup of G and A an (abelian) G/N-module. (In the application above, the action of G/N on $UZ(B)$ comes from the action α, which passes to an action of G/N because $\operatorname{Inn} B$ acts trivially on $Z(B)$.) A G*-crossed extension* of A by N is a group extension

$$e \longrightarrow A \xrightarrow{i} C \xrightarrow{\partial} N \longrightarrow e, \tag{11}$$

together with an action of G on C such that i and ∂ are G-homomorphisms and

$$d^{-1} c d = \partial(d) \cdot c \tag{12}$$

for all $c, d \in C$. This last condition says that $\partial : C \to N$ is a G*-crossed module.* This extension is congruent to another C' if there is a G-isomorphism $\phi : C \to C'$ such that

$$\begin{tikzcd}
 & & C \arrow[dd, "\phi"] \arrow[dr, "\partial"] & & \\
e \arrow[r] & A \arrow[ur] \arrow[dr] & & N \arrow[r] & e \\
 & & C' \arrow[ur, "\partial'"'] & &
\end{tikzcd}$$

commutes. Let $\operatorname{Xext}(G, N, A)$ denote the set of congruence classes. Ratcliffe [31, §3] proves that $\operatorname{Xext}(G, N, A)$ is a group with identity represented by

$$e \longrightarrow A \longrightarrow A \times N \longrightarrow N \longrightarrow e,$$

with the diagonal action of G on $A \times N$, and that a crossed extension C is trivial if and only if $\partial : C \to N$ is split as a G-homomorphism [31, Proposition 2.1]. The proof that Xext is a group involves characterising crossed extensions in terms of a pair (λ, μ) satisfying the conditions of [29, Lemma 5.1]: μ measures the obstruction to triviality of the extension as a group extension, and λ whether the resulting splitting is a G-homomorphism. So almost by definition, Xext $= \Lambda$.

Lemma 2. *Suppose $A : G \to \operatorname{Aut} B$ and N is a normal subgroup of G such that $\alpha(N) \subset \operatorname{Inn} B$. Let*

$$C = \{(u, n) \in U(B) \times N : \operatorname{Ad} u = \alpha_n\},$$

with multiplication $(u, m)(v, n) = (uv, mn)$. Define actions of G on $U(B)$ and C by $s \cdot u = \alpha_s(u)$ and $s \cdot (u, n) = (\alpha_s(u), sns^{-1})$, and define homomorphisms $i : UZ(B) \to C$, $\partial : C \to N$ by $i(u) = (u, e)$, $\partial(u, n) = n$. Then

$$1 \longrightarrow UZ(B) \xrightarrow{\ i\ } C \xrightarrow{\ \partial\ } N \longrightarrow e,$$

is a G-crossed extension of $UZ(B)$ by N, whose class $\lambda(\alpha)$ in $\operatorname{Xext}(G, N, UZ(B))$ is trivial if and only if there is a Green twisting map for (A, G, α) on N.

Proof. It is easy to check that C is a G-crossed extension. Any map $\phi : N \to C$ which splits ∂ has the form $\phi(n) = (u_n, n)$; note that p is a homomorphism if and only if u is, and G-equivariant if and only if $\alpha_s(u_n) = u_{sns^{-1}}$, which gives the result. $\square$

2.3. **The obstruction to existence of a Busby-Smith twisted action.** Now we consider an action of G by outer automorphisms of B: a homomorphism γ of G into the outer-automorphism group $\operatorname{Out} B := \operatorname{Aut} B / \operatorname{Inn} B$. For each $s \in G$ we can choose an automorphism α_s which represents the coset γ_s, and the usual considerations show that $\alpha_s\alpha_t$ differs from α_{st} by an inner automorphism $\operatorname{Ad} u(s, t)$. Associativity implies that $u : G \times G \to U(B)$ satisfies

$$\operatorname{Ad}\big(u(r, s)u(rs, t)\big) = \operatorname{Ad}\big(\alpha_r(u(s, t))u(r, st)\big),$$

and hence there exists $\nu : G \times G \times G \to UZ(B)$ such that

$$\nu(r, s, t)u(r, s)u(rs, t) = \alpha_r(u(s, t))u(r, st).$$

A computation (see [9, pages 172–3]) shows that ν satisfies the cocycle identity

$$\alpha_p(\nu(r, s, t))\nu(p, r, s)\nu(p, rs, t) = \nu(pr, s, t)\nu(p, r, st),$$

and hence represents a class $d(\gamma)$ in the cohomology group $H^3(G, UZ(B))$. The class $d(\gamma)$ vanishes precisely when there is a Busby-Smith twisted action (α, u) of G such that α_s represents γ_s [9, Lemma 4.6]. As in §2.2, the module $UZ(B)$ carries the possibly nontrivial action of G coming from γ and the natural action of $\operatorname{Aut} B / \operatorname{Inn} B$ on $Z(B)$, which is given in practice by any lifting α as above; for the definition of cohomology with coefficients in nontrivial modules, see [30, §7.4].

To make contact with the construction of [14] and [31], suppose A is a G-module. A *crossed sequence* from A to G is an exact sequence

$$e \longrightarrow A \xrightarrow{\ i\ } C \xrightarrow{\ \partial\ } H \xrightarrow{\ \pi\ } G \longrightarrow e$$

together with an action of H on C such that ∂ is an H-homomorphism, $dcd^{-1} = \partial(d) \cdot c$, and i is a G-homomorphism. (Note that elements of $\ker \partial$ are central in C, and hence $G = H/\operatorname{im}\partial$ acts naturally on $\ker\partial$.) If there is a commutative diagram

$$\begin{tikzcd}
& & C \arrow[r] \arrow[dd, "\phi"] & H \arrow[dr] \arrow[dd, "\psi"] & & \\
e \arrow[r] & A \arrow[ur] \arrow[dr] & & & N \arrow[r] & e \\
& & C' \arrow[r] & H' \arrow[ur] & &
\end{tikzcd}$$

such that $\phi(h \cdot c) = \psi(h) \cdot \phi(c)$, then the top and bottom sequences are said to be simply congruent; this generates (but is not itself) an equivalence relation on the set of crossed sequences. We denote by $\operatorname{Xseq}(G, A)$ the set of equivalence classes. Ratcliffe proves [31, §9] that there is a natural isomorphism of $\operatorname{Xseq}(G, A)$ onto $H^3(G, A)$, constructed as follows. Given such a sequence S, choose a section $c : G \to H$ for π. Then choose elements $\mu(s, t)$ of $\mathbb{C}$ such that

$$\partial(\mu(s, t))c(st) = c(s)c(t),$$

and define $\lambda : G \times G \times G \to A$ by

$$i(\lambda(r, s, t)) = c(r) \cdot (\mu(s, t))\mu(r, st)\mu(rs, t)^{-1}\mu(r, s)^{-1}.$$

It turns out that λ is a cocycle whose class in $H^3(G, A)$ is independent of any of the choices made, and is called the Mac Lane-Whitehead obstruction of the sequence. The map $[S] \mapsto [\lambda]$ is the required isomorphism.

The algebraic version of [9, Lemma 4.6] is:

Lemma 3. *Suppose γ is a homomorphism of G into the outer automorphism group* $\operatorname{Out} B$. *Then we have a crossed sequence*

$$1 \longrightarrow UZ(B) \xrightarrow{i} U(B) \xrightarrow{\partial} H \xrightarrow{\pi} G \longrightarrow e$$

in which

$$H = \{(\alpha, s) \in \operatorname{Aut} B \times G : [\alpha] = \gamma_s \text{ in } \operatorname{Out} B\},$$

H acts on $U(B)$ by $(\alpha, s) \cdot u = \alpha(u)$, ∂ is defined by $\partial(u) = (\operatorname{Ad} u, e)$, and i is inclusion. The Mac Lane-Whitehead obstruction of this sequence is the obstruction $d(\gamma)$ constructed in [9, Lemma 4.6], *and vanishes if and only if there is a twisted action (α, u) of G on B such that $\gamma_s = [\alpha_s]$ in* $\operatorname{Out} B$ *for each $s \in G$.*

Proof. It seems clear that the constructions of the Mac Lane-Whitehead obstruction and $d(\gamma)$ can be done with the same data, and yield the same cocycle, so this follows from [9, Lemma 4.6]. □

2.4. **The exact sequence.** Suppose N is a normal subgroup of G and A is a G/N-module. Ratcliffe [31] defines homomorphisms

$$\gamma : H^2(G, A) \to \operatorname{Xext}(G, N, A) \quad \text{and} \quad \delta : \operatorname{Xext}(G, N, A) \to H^3(G/N, A)$$

as follows. If

$$E : \; e \longrightarrow A \xrightarrow{\zeta} K \xrightarrow{\eta} G \longrightarrow e \tag{13}$$

is an extension of A by G compatible with the given action of G on A, then $\gamma(E)$ is the crossed extension

$$1 \longrightarrow A \xrightarrow{\zeta} \eta^{-1}(N) \xrightarrow{\partial} N \longrightarrow 1$$

in which ∂ is the restriction of η and the action of G on $\eta^{-1}(N)$ is induced by conjugation in K; since N acts trivially on A, A is central in $\eta^{-1}(N)$ and the action of K by conjugation on $\eta^{-1}(N)$ factors through η (see [31, §5]). If

$$E:\ e \longrightarrow A \longrightarrow C \xrightarrow{\partial} N \longrightarrow e \tag{14}$$

is a crossed extension, then $\delta(E)$ is the Mac Lane-Whitehead class of the crossed sequence

$$e \longrightarrow A \longrightarrow C \xrightarrow{\partial} G \longrightarrow G/N \longrightarrow e$$

(see [31, page 85]). Ratcliffe proves that these maps are well-defined homomorphisms on the appropriate sets of equivalence classes, and that the sequence

$$H^2(G/N, A) \xrightarrow{\text{inf}} H^2(G, A) \xrightarrow{\gamma} \operatorname{Xext}(G, N, A) \xrightarrow{\delta} H^3(G/N, A) \xrightarrow{\text{inf}} H^3(G, A) \tag{15}$$

is exact [31, Theorem 8.1] (and continues the usual restriction-inflation sequence).

Theorem 4. (1) *Suppose $\alpha : G \to \operatorname{Inn} B$ is an action of a group G on an algebra B by inner automorphisms, and N is a normal subgroup of G. Then the homomorphism γ carries the obstruction $c(\alpha)$ to implementing α by a homomorphism $u : G \to U(B)$ (described in Lemma 1) into the obstruction $\lambda(\alpha)$ to implementing α by a Green twisting map on N (described in Lemma 2).*

(2) *Suppose $\alpha : G \to \operatorname{Aut} B$ and N is a normal subgroup of G such that $\alpha(N) \subset \operatorname{Inn} B$. Then the homomorphism δ carries the obstruction $\lambda(\alpha)$ to implementing α by a Green twisting map on N into the obstruction $\delta(\tilde{\alpha})$ to implementing the induced map $\tilde{\alpha} : G/N \to \operatorname{Aut} B/\operatorname{Inn} B$ by a Busby-Smith twisted action (described in Lemma 3).*

Proof. The restriction of the extension of Lemma 1, namely

$$1 \to UZ(B) \to q^{-1}(N) = \{(u, n) : \operatorname{Ad} u = \alpha_n\} \to N \to e$$

is exactly the same as the underlying extension of Lemma 2. Therefore to prove (1), all we have to do is check that the two G-actions on $q^{-1}(N)$ coincide. But this is easy: if $s \in G$ and $\operatorname{Ad} v = \alpha_s$, then

$$(v, s)(u, n)(v, s)^{-1} = (v, s)(u, n)(v^{-1}, s^{-1}) = (vuv^{-1}, sns^{-1}) = (\alpha_s(u), sns^{-1}),$$

as required.

The class $\delta(\lambda(\alpha))$ is represented by the crossed sequence

$$1 \to UZ(B) \to C = \{(u, n) : \operatorname{Ad} u = \alpha_n\} \xrightarrow{\partial_1} G \to G/N \to e, \tag{16}$$

where $\partial_1(u, n) = n$, and $\delta(\tilde{\alpha})$ by the crossed sequence

$$1 \to UZ(B) \to U(B) \xrightarrow{\partial} H = \{(\alpha, sN) : [\alpha] = [\alpha_s]\} \xrightarrow{\pi} G/N \to e. \tag{17}$$

If we define $\phi(u, n) = u$ and $\psi(s) = (\alpha_s, sN)$, then we have a commutative diagram

$$\begin{tikzcd}
 & & C \arrow[r] \arrow[dd, "\phi"] & G \arrow[dr] \arrow[dd, "\psi"] & & \\
1 \arrow[r] & UZ(B) \arrow[ur] \arrow[dr] & & & G/N \arrow[r] & \mathrm{e} \\
 & & U(B) \arrow[r] & H \arrow[ur] & &
\end{tikzcd}$$

and we can easily verify that $\phi(s \cdot (u,n)) = \alpha_s(u) = \psi(s) \cdot \phi(u,n)$. Thus the crossed sequence (16) is simply congruent to (17), and they therefore determine the same class in $H^3(G/N, UZ(B))$. $\square$

This theorem says that if we have an action $\alpha : G \to \operatorname{Aut} B$ with $\alpha(N) \subset \operatorname{Inn} B$, then it is easier for it to give rise to a Busby-Smith twisted action of G/N than a Green twisted action of (G,N): we could have $\lambda(\alpha) \neq 0$ but $\delta(\tilde{\alpha}) = 0$. As we saw in §1.5, this is not true if we are allowed to replace (G,N) by other pairs (E,M) with $E/M = G/N$. In a similar vein, every crossed sequence

$$e \longrightarrow A \longrightarrow C \longrightarrow G \xrightarrow{\pi} Q \longrightarrow e$$

comes from an element

$$e \longrightarrow A \longrightarrow C \longrightarrow \ker\pi \longrightarrow e$$

of $\operatorname{Xext}(G, \ker\pi, A)$ for some pair $(G, \ker\pi)$ with quotient Q. Ratcliffe uses this to deduce facts about $H^3(Q,A)$ from properties of $\operatorname{Xext}(G,N,A)$ as G, N vary.

2.5. **Obstructions to extensions.** Classes in H^3 have more traditionally arisen as the obstructions to the existence of extensions. If we start with an extension

$$e \longrightarrow N \xrightarrow{i} E \longrightarrow G \longrightarrow e, \tag{18}$$

then E acts by conjugation on N; of course N acts by inner automorphisms, so this action induces a homomorphism of $G = E/N$ into the outer automorphism group $\operatorname{Out} N$, which is called the *conjugation class* of the extension. If we start with an action $\gamma : G \to \operatorname{Out} N$ by outer automorphisms of a group N, we can ask whether there is an extension (18) with γ as its conjugation class. Proceeding as in §2.3, we choose representatives $\ell_s \in \operatorname{Aut} N$ for γ_s, choose $f(s,t) \in N$ such that $\ell_s \circ \ell_t = \operatorname{Ad} f(s,t) \circ \ell_{st}$, and associativity implies that there is a cocycle $\nu : G \times G \times G \to Z(N)$ such that

$$\nu(r,s,t) f(r,s) f(rs,t) = \ell_r(f(s,t)) f(r,st).$$

The action of G on $Z(N)$ and the class $\theta(N,G,\gamma)$ of the cocycle in $H^3(G,Z(N))$ do not not depend on any of the choices we made, and $\theta(N,G,\gamma)$ vanishes if and only if there is an extension (18) with conjugation class γ (see [1, Chapter 1] or [21, §IV.8]). Every class in $H^3(G,A)$ arises this way for some group N with centre A [21, §IV.9].

When we start with an action $\gamma : G \to \operatorname{Out} B$ on an algebra B, we can try to apply this interpretation by viewing γ_s as an outer automorphism of $U(B)$. To make this work, we need two assumptions. First, we need to know that γ is unit-preserving, in the sense that the individual outer automorphisms γ_s restrict to automorphisms $\operatorname{res}\gamma_s$ of $U(B)$. Second, since the construction of the preceding paragraph talks about $ZU(B)$ rather than $UZ(B) := U(B) \cap Z(B)$, we need to assume that $U(B)$ spans B, which ensures that $ZU(B) = UZ(B)$. Fortunately, both these assumptions are automatically satisfied in the main examples of interest to us:

Examples 5. (1) In any algebra, every outer automorphism preserves the full group of units. In any Banach algebra B, $b - \lambda 1$ is a unit whenever $|\lambda|$ is bigger than the spectral radius of b, and in particular whenever $|\lambda| > \|b\|$; thus we can write any element $b = \big((b - 2\|b\|1) + (b + 2\|b\|1)/2$ as a sum of two units.

(2) When B is a C^*-algebra, we would usually restrict attention to $*$-automorphisms, and these (and the corresponding outer automorphisms) always leave the group $U(B)$ of unitary elements invariant. The Russo-Dye Theorem says that $U(B)$ spans B.

(3) In the algebra $\mathbb{C}[x]$ of complex polynomials, the only invertible elements are the constants, which certainly do not span. However, if we complete $\mathbb{C}[x]$ in the norm $\|p(x)\|_\infty := \sup\{|p(x)| : |x| \leq 1\}$, for example, then the completion is a Banach algebra, and we are back in case (1): some polynomials have inverses given by convergent power series. (Basically, chaps, algebra sucks.)

Proposition 6. *Suppose that B is an algebra and $U(B)$ is a group of units which spans B. Let $\gamma : G \to \operatorname{Out} B$ be an action by outer automorphisms which preserve $U(B)$.*

(1) *Suppose $\alpha : G \to \operatorname{Aut} B$ is a map such that $[\alpha_s] = \gamma_s$ for all $s \in G$. Then there is a group extension*

$$1 \longrightarrow U(B) \xrightarrow{\ i\ } E \xrightarrow{\ q\ } G \longrightarrow e$$

whose conjugation class is given by the composition $G \xrightarrow{\gamma} \operatorname{Out} B \xrightarrow{\text{res}} \operatorname{Out} U(B)$ if and only if there exists $u : G \times G \to U(B)$ such that (α, u) is a Busby-Smith twisted action of G on B. If so, we can take E to be the group $U(B) \times_{\alpha,u} G$ with underlying set $U(B) \times G$ and multiplication

$$(a, s)(b, t) = (a\alpha_s(b)u(s, t), st).$$

(2) *The obstruction $\theta(U(B), G, \gamma) \in H^3(G, ZU(B))$ is the obstruction $d(\gamma)$ of* [9, Lemma 4.6].

Proof. (1) If (α, u) is a twisted action of G on B such that α lifts γ, then calculations similar to those in [1, Chapter 1, Theorem 5.4] show that $U(B) \times_{\alpha,u} G$ is a group: the identity is $(1, e)$, $(b, s)^{-1} = (\alpha_s^{-1}(b^{-1}u(s, s^{-1})^{-1}), s^{-1})$, and associativity follows from the identity $\operatorname{Ad}(u(s, t)) \circ \alpha_{st} = \alpha_s\alpha_t$. With $i(a) = (a, e)$ and $q((a, s)) = s$,

$$1 \longrightarrow U(B) \xrightarrow{\ i\ } U(B) \times_{\alpha,u} G \xrightarrow{\ q\ } G \longrightarrow e$$

is an extension of $U(B)$ by G with the required conjugation class.

For the converse, suppose we are given a group extension with conjugation class $\text{res} \circ \gamma$. For any lifting $\alpha : G \to \operatorname{Aut} B$ of γ, $\text{res} \circ \alpha : G \to \operatorname{Aut} U(B)$ is a lifting of $\text{res} \circ \gamma$. If $\ell : G \to E$ is any section, then $\operatorname{Ad} \circ \ell : G \to \operatorname{Aut} U(B)$ is also a lifting for $\text{res} \circ \gamma$. Thus each $\operatorname{Ad}(\ell_s)$ differs from $\text{res}(\alpha_s)$ by an inner automorphism of $U(B)$, and we can adjust ℓ by a map $G \to U(B)$ to ensure that $\operatorname{Ad}(\ell_s) = \text{res}\, \alpha_s$.

Now define $u : G \times G \to E$ by $u(s, t) := \ell_s\ell_t\ell_{st}^{-1}$. Routine calculations show that the $(\text{res} \circ \alpha, u)$ satisfy the relations of a twisted action, at least when viewed as automorphisms of $U(B)$. But since $\alpha_s\alpha_t$ and $\operatorname{Ad} u(s, t) \circ \alpha_{st}$ are linear maps which agree on the spanning set $U(B)$, they agree on all of B. Thus (α, u) defines a twisted action of G on B.

(2) Because the class $\theta(U(B), G, \gamma) \in H^3(G, ZU(B))$ is independent of the choices of ℓ and f, we can first choose a lifting α for γ and then use $\ell := \text{res} \circ \alpha$. From there the procedure for defining the cocycle ν is exactly the same in both cases. □

The intrusion of these extra hypotheses suggests that the crossed-sequence description of H^3 is better suited to problems involving twisted actions. However, we should point out that Sutherland has successfully used the "obstruction to extension" approach for group actions on von Neumann algebras [35]. He avoided the problem of getting from $\operatorname{Aut} U(B)$ to $\operatorname{Aut} B$ by working directly in terms of extensions of von Neumann algebras by groups: loosely speaking, his extensions are Busby-Smith twisted crossed products, so he too is looking at the obstructions to existence of twisted actions. Presumably his ideas could be fairly easily adapted to the algebraic setting, but making them work for C^*-algebras might be a bit trickier. A start has been made in [3] for extensions of discrete abelian groups, but the connection with H^3 is not made there.

3. Group actions on C^*-algebras

A C^*-dynamical system consists of an action α of a locally compact group G on a C^*-algebra A such that $s \mapsto \alpha_s(a)$ is continuous for each fixed $a \in A$. Such systems do not slot neatly into the algebraic scheme we have been discussing, for two reasons. First, we need to take account of the topology on G: all the group algebras and crossed products will need to be universal for the strongly continuous unitary representations of G, and will be completions of algebras of continuous or Borel functions. Second, C^*-algebras need not have identities, so $U(A)$ could be empty; even if we assume that the original algebra A has an identity, group algebras and crossed products involving nondiscrete groups will not, so we cannot avoid algebras without identities.

There is a standard way of handling C^*-algebras A without identities: everywhere we have used $U(A)$, use instead the group $UM(A)$ of unitary elements in the multiplier algebra $M(A)$. This comes equipped with a *strict topology*, in which a net $\{m_i\}$ converges to m when $m_i a \to ma$ and $am_i \to am$ for every $a \in A$; with this topology, $UM(A)$ is a topological group. It is not locally compact unless A is finite-dimensional, but if A is separable, $UM(A)$ is Polish: the topology is given by a complete metric. (See [30, page 191]; it will be clear in a minute why this is important.)

Allowing the group G to be nondiscrete causes more serious problems. One's instinct would be to insist that all maps and homomorphisms are continuous, but this quickly becomes untenable: for example, the extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{e} \mathbb{T} \longrightarrow 1,$$

in which $e(r) := \exp(2\pi i r)$, does not admit a continuous section $c : \mathbb{T} \to \mathbb{R}$ (this would give a continuous branch of log on $\mathbb{T} \subset \mathbb{C} \setminus \{0\}$). So we cannot expect to get away with continuous cocycles, even in our first application of cohomology to classifying group extensions. The solution to this problem was known to Mackey: second countable locally compact group extensions always admit Borel sections, and the theory of §1.1 then carries over with H^2 defined in terms of Borel cocycles. The key point is that the model extensions E_ω then have a Borel structure and an invariant measure, and hence automatically acquire a locally compact group topology.

The problem of extending this H^2 to a general cohomology theory based on Borel cocycles was taken up by Moore [22]. He showed that there is a satisfactory theory $H^n(G, M)$ when the group G is second countable and locally compact and when the coefficient module M is Polish [23]. His *Moore cohomology* theory is outlined in [30, §7.4], where examples and further references are given.

That the coefficient module M is not required to be locally compact is crucial for the kind of applications we have been discussing: we can take $M := UZM(A)$, and then the extensions E of Lemma 1 belong to the class of Polish extensions classifiable by Moore's $H^2(G, UZM(A))$. There is a well-developed theory of Busby-Smith twisted actions on C^*-algebras involving Borel cocycles — indeed, all the references we gave earlier allow this. However, because of the asymmetry in the hypotheses on G and M, the interpretations of higher-dimensional cohomology in terms of longer exact sequences are not presently available. So to carry out the program of §2 for actions of locally compact groups on C^*-algebras, one needs to do all the constructions directly in terms of cocycles, and then verify that all these constructions preserve Borel structures. The cocycle-based construction of the exact sequence already exists ([19]; see also [7, §3]), and we believe that all this works. The details, though, are a bit messy.

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