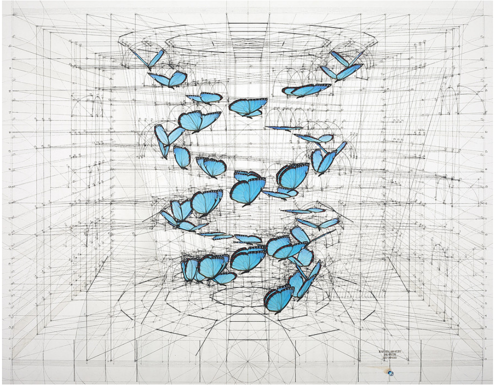
On manifolds and fixed-point theorems



Rafael Araujo: *Blue Morpho, Double Helix*

The birth of manifold theory

In 1895 Poincaré publishes the seminal paper *Analysis Situs* – the first systematic treatment of topology.

The birth of manifold theory

L.E.J. Brouwer (1881 - 1966)

Dutch mathematician interested in the philosophy of the foundations of mathematics (cf. *intuitionism*).

In 1909 meets Poincaré, Hadamard, Borel, and is convinced of the importance of better understanding the topology of Euclidean space.

This led to what we now know as *Brouwer's fixed-point theorem*.

The birth of manifold theory

Brouwer's fixed-point theorem, 1910

Any continuous self-map

$$f : \mathbb{B}^n \to \mathbb{B}^n$$

from the closed unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ has a fixed point.

The birth of manifold theory

Fundamental theorems on the topology of Euclidean space

- Brouwer fixed-point theorem, 1910.
- Jordan-Brouwer separation theorem, 1911.
- Invariance of domain, 1912.
- Invariance of dimension, 1912.
- Hairy ball theorem for S^{2n}, 1912.

- Borsuk-Ulam theorem.
- No-retraction theorem.

The birth of manifold theory

Foundations of manifold theory

Builiding on ideas of Gauss / Riemann / Poincaré, and new ideas from the emerging field of *topology* people like Weyl and Whitney formalized the concept of a manifold.

Whitney embedding theorem, 1936:
Unifies extrinsic and intrinsic approaches – first complete exposition of the concept of a manifold.

- Brouwer's fixed-point theorem proved fundamental in this development.
- Can we extend this theorem to other manifolds?

Lefschetz fixed-point theorem

Solomon Lefschetz (1884 - 1972)

Known for:

- Topology of algebraic varieties.
- The fixed-point theorem.
- Relative homology.
- Duality for manifolds with boundary.
- Editor of the Annals of Mathematics (1928 - 1958).
- Works on dynamical systems.

Lefschetz fixed-point theorem

The problem

Let X be a compact manifold and $f\colon X \to X$ a continuous map. When can we guarantee that f has a fixed point?

The strategy

A fixed point is a point of intersection between the graph of f and the diagonal Δ, i.e.

$$\mathrm{Fix}(f) = \Gamma_f \cap \Delta \subset X \times X.$$

The intersection number $\#(\Gamma_f \cap \Delta)$ depends only on the homology classes of Γ_f and Δ. We should be able to detect whether or not Γ_f and Δ intersect by using homology theory.

Lefschetz fixed-point theorem

The theorem

Define the *Lefschetz number* of f to be

$$L(f) = \sum_k (-1)^k \operatorname{tr}(f_*\colon H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})).$$

Theorem: *If $L(f) \neq 0$ then f has a fixed point.*

Theorem (Lefschetz fixed-point formula, 1926)

Assume X is oriented and f has isolated fixed points only. Then $L(f) \in \mathbb{Z}$ and f has exactly $L(f)$ fixed points (counted with multiplicity).

$$\sum_{x \in \operatorname{Fix}(f)} \iota(f,x) = L(f).$$

Lefschetz fixed-point theorem

Some immediate corollaries

- Brouwer's fixed point theorem.
- Every self-map of a contractible manifold has a fixed point.
- Every self-map of a $\mathbb{Q}$-acyclic manifold has a fixed point.
    - Particular case: $\mathbb{R}\mathrm{P}^{2k}$.

The Lefschetz formula is very powerful because it not only asserts the existence of fixed points but tells us exactly how many there are.

Remark

The Lefschetz formula can also be used to deduce the Poincaré-Hopf index theorem for vector fields.

Lefschetz fixed-point theorem

The moral of the story

The global topology of X, and in particular the way f^* acts on $H^\bullet(X;\mathbb{Q})$, constrain the existence and behavior of fixed points.

What does behavior of a fixed point mean anyways?

Lefschetz fixed-point theorem

What about behavior?

Let X be a smooth manifold and $f\colon X \to X$ a smooth map.

Definition

We say that a fixed point $x \in \mathrm{Fix}(f)$ is *non-degenerate* if

$$\det(I - Df(x)) \neq 0.$$

Note that this happens if and only if $\Gamma_f \pitchfork_x \Delta$.

If x is a non-degenerate fixed point then the linear map $I - Df(x)$ determines the first order behavior of f locally around x. This is the actual behavior of f around x.

Lefschetz fixed-point theorem

Question

Does the topology of X constrain this behavior of the fixed points?

Answer

Not really. $H^\bullet(X;\mathbb{Q})$ is too coarse to measure that behavior.

Even $H^\bullet_{dR}(X) \cong H^\bullet(X;\mathbb{R})$ is not enough.

We need extra structure on X and f.

The right extra structure is a complex structure.

Into the complex realm

This slide is more important than you imagine

Let V be a complex vector space and $L\colon V \to \mathbb{C}$ an $\mathbb{R}$-linear map.

- Note that L is in fact $\mathbb{C}$-linear iff $L(iv) = iL(v)$.
- We say that L is $\mathbb{C}$-antilinear iff $L(iv) = -iL(v)$.

Proposition

Any $\mathbb{R}$-linear map $L \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ decomposes uniquely as

$$L = L' + L'',$$

with L' $\mathbb{C}$-linear and L'' $\mathbb{C}$-antilinear. In particular we have

$$\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \oplus \mathrm{Hom}_{\mathbb{C}}(\overline{V}, \mathbb{C}).$$

Into the complex realm

This slide is more important than you imagine

For short, let us rewrite the previous decomposition as

$$V^* = V^{(1,0)} \oplus V^{(0,1)}.$$

In a similar way the exterior powers of V^* decomposes as

$$\wedge^k V^* = \bigoplus_{p+q=k} V^{(p,q)},$$

where $V^{(p,q)} = (\wedge^p V^{(1,0)}) \wedge (\wedge^q V^{(0,1)})$.

Into the complex realm

Ok, now let's talk about complex manifolds

Let X be a *compact complex manifold* of dimension n. Let $\mathcal{A}^k(X)$ denote the space of smooth differential k-forms on X. As before, we have a bigrading:

$$\mathcal{A}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X).$$

The exterior derivative $d\colon \mathcal{A}^k(X) \to \mathcal{A}^{k+1}(X)$ decomposes as

$$d = \partial + \overline{\partial},$$

$$\partial\colon \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q}(X), \qquad \overline{\partial}\colon \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X).$$

Into the complex realm

Ok, now let's talk about complex manifolds

It is immediate that $\overline{\partial}^2 = 0$ and so for every $0 \leq p \leq n$ we have the cochain complex

$$0 \longrightarrow \mathcal{A}^{p,0}(X) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1}(X) \xrightarrow{\overline{\partial}} \quad \dots \quad \xrightarrow{\overline{\partial}} \mathcal{A}^{p,n}(X) \longrightarrow 0.$$

Definition

The *Dolbeault cohomology* of X is defined as

$$H^{p,q}_{\overline{\partial}}(X) = H^q(\mathcal{A}^{p,\bullet}(X), \overline{\partial}).$$

The holomorphic Lefschetz formula

Let X be a compact complex manifold and $f\colon X \to X$ a holomorphic self-map.

Definition

The *holomorphic Lefschetz number* of f is defined to be

$$L(f, \mathcal{O}_X) = \sum_{q=0}^{n} (-1)^q \operatorname{tr}\left(f^*\colon H^{0,q}_{\bar\partial}(X) \to H^{0,q}_{\bar\partial}(X) \right).$$

Theorem (Holomorphic Lefschetz fixed-point theorem)

If f has non-degenerate fixed points only then

$$\sum_{x \in \operatorname{Fix}(f)} \frac{1}{\det(I - Df(x))} = L(f, \mathcal{O}_X).$$

The holomorphic Lefschetz formula
An example on the Riemann sphere

Let $X = \mathbb{CP}^1$ and $f\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ the rotation $f(z) = e^{i\theta}z$.
This map has fixed points at $z_1 = 0$ and $z_2 = \infty$.
The derivative of f at z_1 is $Df(z_1) = e^{i\theta}$.
On the other hand, in coordinates $w = \frac{1}{z}$, the map f is given by

$$w \mapsto \frac{1}{e^{i\theta} \cdot \frac{1}{w}} = e^{-i\theta}w,$$

and so $Df(z_2) = e^{-i\theta}$.

The holomorphic Lefschetz formula now tells us

$$\frac{1}{1-e^{i\theta}} + \frac{1}{1-e^{-i\theta}} = L(f, \mathcal{O}_X) = 1.$$

The holomorphic Lefschetz formula
Some consequences

We can now say:

The global structure of X, as measured by the Dolbeault cohomology, constrains the existence and behavior of fixed points.

Fact

If $X = \mathbb{C}\mathrm{P}^n$ then $L(f, \mathcal{O}_x)$ is always equal to 1.

Corollaries:

- Every holomorphic self-map of $\mathbb{C}\mathrm{P}^n$ has a fixed point.
- Every linear endomorphism of $\mathbb{C}^{n+1}$ has an eigenvector.
- Every non-constant monic polynomial over $\mathbb{C}$ has a root (i.e. the fundamental theorem of algebra).

The holomorphic Lefschetz formula
What's next?

A fundamental result in complex geometry is Dolbeault's theorem, which relates the Dolbeault cohomology to the cohomology of the sheaf $\mathcal{O}_X$ of holomorphic functions on X via the isomorphism

$$H^{0,q}_{\bar{\partial}}(X) \cong H^q(X, \mathcal{O}_X).$$

Conclusion

The holomorphic Lefschetz number $L(f, \mathcal{O}_X)$ measures the action of f on the cohomology of the structure sheaf $\mathcal{O}_X$.

The Woods-Hole trace formula

Let X be a compact complex manifold, $f\colon X \to X$ a holomorphic endomorphism and $\mathscr{F}$ a coherent analytic sheaf.

Question

Does the map f act on the cohomology groups $H^k(X, \mathscr{F})$?

No! There is only a map $f^*\colon H^k(X, \mathscr{F}) \to H^k(X, f^*\mathscr{F})$.

Definition

A *lift* of f to $\mathscr{F}$ is a morphism of $\mathcal{O}_X$-modules $\varphi\colon f^*\mathscr{F} \to \mathscr{F}$.

A lift provides us with linear maps

$$\tilde{\varphi}^k\colon H^k(X, \mathscr{F}) \xrightarrow{f^*} H^k(X, f^*\mathscr{F}) \xrightarrow{\varphi} H^k(X, \mathscr{F}).$$

The Woods-Hole trace formula

Definition

The Lefschetz number of the lift φ is defined to be

$$L(f,\varphi,\mathscr{F}) = \sum_k (-1)^k \operatorname{tr}\left(\tilde{\varphi}^k \colon H^k(X,\mathscr{F}) \to H^k(X,\mathscr{F})\right).$$

On the other hand for any fixed-point $x \in \mathrm{Fix}(f)$ we have maps on the stalks

$$\varphi_x \colon (f^*\mathscr{F})_x \cong \mathscr{F}_x \to \mathscr{F}_x,$$

and linear maps on the fibers

$$\varphi(x) \colon \mathscr{F}(x) \to \mathscr{F}(x).$$

The Woods-Hole trace formula

The Woods-Hole trace formula (Atiyah - Bott, 1965)

If f is a transversal endomorphism then

$$\sum_{x\in \mathrm{Fix}(f)} \frac{\mathrm{tr}(\varphi(x))}{\det(I - Df(x))} = L(f, \varphi, \mathscr{F}).$$

Remarks:

- Woods-Hole contains as particular cases all the formulas discussed today.
- It also implies the *Weyl character formula* from representation theory.
- It was a precursor of the *Atiyah-Singer index theorem* for elliptic complexes.

The Woods-Hole trace formula

Remarks

- M. Atiyah and R. Bott proved this result for *elliptic complexes* on compact manifolds. The previous formula is just a particular case of this.
- J.L. Verdier proved the equivalente formula for smooth projective varieties and étale cohomology.
- The name *Woods-Hole* came from the place where the formula was first proved.