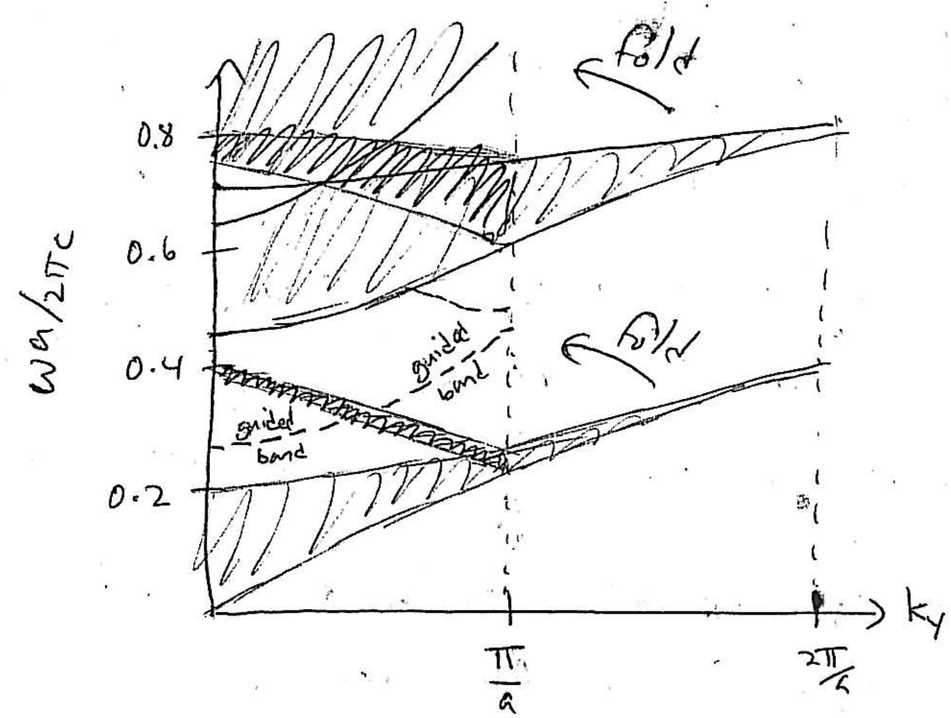
18.369 Midterm 2009 Solutions

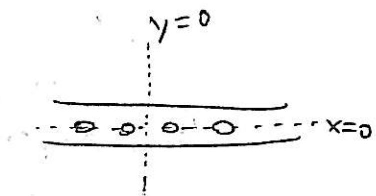
Problem 1

(a) First, we should plot the projected band diagram *without* a defect mode. The periodicity Λ in y means that the original projected bands will "fold" at $k_y = \frac{\pi}{\Lambda} = \frac{\pi}{a}$ (shown by dark shading below):



The air holes should then push a band *up* into the gap, resulting in a guided band that should look something like the dashed line above. (Note $\frac{d\omega}{dk_y} = 0$ at $k_y = 0, \pi/a$!)

At $k_y = 0$, we have $x=0$ and $y=0$ mirror planes:

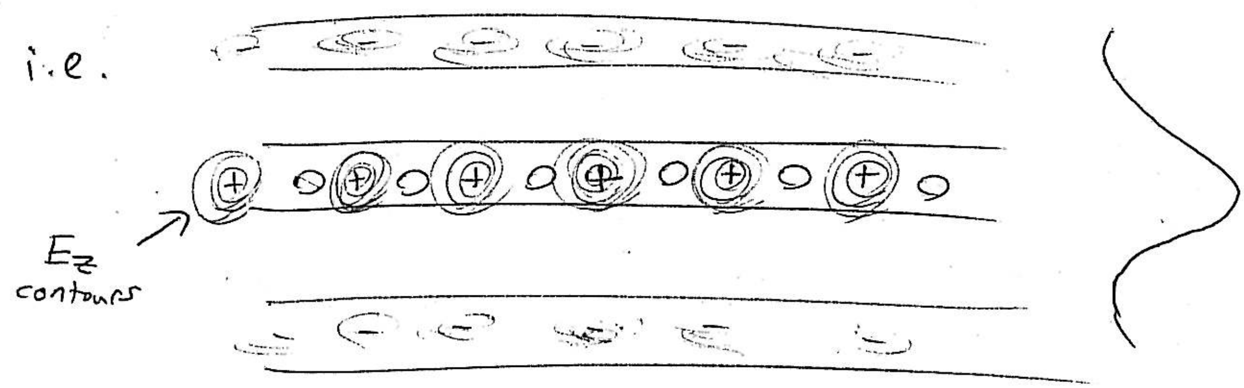


The solution should be even or odd around these.

(over →)

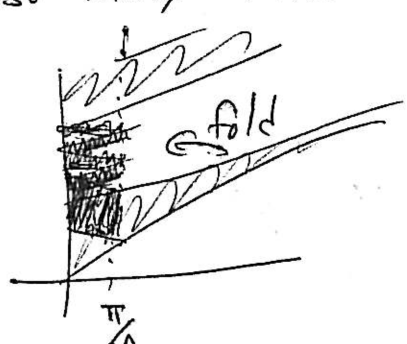
Since the solution comes from the lower band, it should be concentrated in $\varepsilon = 12$ — even with both x and y (for the mirror planes shown).

i.e.



Note that the sign is the same in adjacent periods along y ($k_y = 0$), but the field should be <u>oscillating</u> + exponentially decaying in the x direction $\sim e^{i\frac{\pi}{a}x - \kappa|x|}$ as discussed in class (from analytic continuation)

(b) If Λ is too big, the band will fold so many times it will fill the gap:



clearly, as Λ gets bigger & the folding becomes more dense, at some point the gap is filled.

Air holes (which push <u>up</u> a state) can only guide light in a band gap $\Rightarrow$ for some $\Lambda > \Lambda_0$ they cannot guide.

(*) Analysis of the upper gaps is more tricky & less obvious
I only asked you to look at the first gap.

Problem 2

(a) $\hat{\Theta} = \nabla \times \frac{1}{\varepsilon(\omega)} \nabla \times$ is still Hermitian for any given ω

$\Rightarrow$ if $\hat{\Theta}\vec{H} = \frac{\omega^2}{c^2}\mu\vec{H}$,

then $\langle \hat{H}, \hat{\Theta}\hat{H} \rangle = \frac{\omega^2}{c^2}\langle H, \mu H \rangle$

$= \langle \hat{\Theta} H, H \rangle = \frac{\omega^2}{c^2}^* \langle \mu H, H \rangle$

$= \frac{\omega^2}{c^2}^* \langle H, \mu H \rangle$ (μ is real $\Rightarrow$ Hermitian)

$\Rightarrow \omega^2 = \omega^{2*}$ since $\langle H, \mu H \rangle > 0$ $(\mu(\omega) > 0)$

$\Rightarrow \omega^2$ is real

also, $\frac{\omega^2}{c^2} = \frac{\langle H, \hat{\Theta} H \rangle}{\langle H, \mu H \rangle} = \frac{\int \frac{1}{\varepsilon(\omega)} |\nabla \times H|^2}{\int \mu(\omega) |H|^2} \geq 0$

since $\varepsilon, \mu > 0$

$\Rightarrow \omega$ is real

(b) Let ω_0, $\vec{H}_0$ be the unperturbed solution

$\Rightarrow$ $\nabla \times \frac{1}{\varepsilon(\omega+\Delta\omega) + \Delta\varepsilon(\omega+\Delta\omega)} \nabla \times (\vec{H} + \Delta\vec{H}) = \frac{(\omega+\Delta\omega)^2}{c^2}[\mu(\omega+\Delta\omega) + \Delta\mu(\omega+\Delta\omega)] \cdot (H + \Delta H)$

must be solved to 1st order.

(over →)

since $\varepsilon, \mu, \Delta\varepsilon, \Delta\mu$ are given to be smooth functions of ω, we can write (to 1st order):

$$\varepsilon(\omega+\Delta\omega) \approx \varepsilon(\omega) + \frac{d\varepsilon}{d\omega}\Delta\omega$$

$$\mu(\omega+\Delta\omega) \approx \mu(\omega) + \frac{d\mu}{d\omega}\Delta\omega$$

$$\Delta\varepsilon(\omega+\Delta\omega) \approx \Delta\varepsilon(\omega) + \cancel{\frac{d\Delta\varepsilon}{d\omega}\Delta\omega} \xrightarrow{} \text{2nd order}$$

$$\Delta\mu(\omega+\Delta\omega) \approx \Delta\mu(\omega) + \cancel{\frac{d\Delta\mu}{d\omega}\Delta\omega} \xrightarrow{} \text{2nd order}$$

$\Rightarrow$ we can neglect ω-dependence of $\Delta\varepsilon, \Delta\mu$!

$\Rightarrow$ collecting 1st order terms,

(noting that $\frac{1}{\varepsilon+\Delta\varepsilon} \approx -\frac{\Delta\varepsilon}{\varepsilon^2} + \frac{1}{\varepsilon}$), we

obtain:

$$-\nabla\times\left(\frac{\Delta\varepsilon + \frac{d\varepsilon}{d\omega}\Delta\omega}{\varepsilon^2}\right)\nabla\times H + \nabla\times\frac{1}{\varepsilon}\nabla\times(\Delta H) = 2\frac{\omega\Delta\omega}{c^2}\mu H + \frac{\omega^2}{c^2}\left(\Delta\mu + \frac{d\mu}{d\omega}\Delta\omega\right)H + \frac{\omega^2}{c^2}\mu(\Delta H)$$

take the inner product of both sides with $\langle H, \ldots\rangle$:

$$\langle H, \nabla\times\tfrac{1}{\varepsilon}\nabla\times\Delta H\rangle = \langle\hat{\Theta}H, \Delta H\rangle = \frac{\omega^2}{c^2}\langle H, \mu\Delta H\rangle$$

which cancels the term $\langle H, \frac{\omega^2}{c^2}\mu\Delta H\rangle$ on the other side

$\Rightarrow$ collecting $\Delta\omega$ terms on the right hand side ...

(over $\rightarrow$)

$$-\left\langle H, \nabla\times \frac{\Delta\varepsilon}{\varepsilon^2}\nabla\times H\right\rangle - \left\langle H, \frac{\omega^2}{c^2}\Delta\mu\, H\right\rangle = \Delta\omega\left(\frac{2\omega}{c^2}\langle H, \mu H\rangle + \left\langle H, \nabla\times\frac{d\varepsilon/d\omega}{\varepsilon^2}\nabla\times H\right\rangle + \frac{\omega^2}{c^2}\left\langle H, \frac{d\mu}{d\omega} H\right\rangle\right)$$

$$= -\left\langle \nabla\times H, \frac{\Delta\varepsilon}{\varepsilon^2}\overbrace{\nabla\times H}^{=-i\frac{\omega}{c}\varepsilon E}\right\rangle - \left\langle H, \frac{\omega^2}{c^2}\Delta\mu\, H\right\rangle = \Delta\omega\left(\frac{2\omega}{c^2}\int\mu|H|^2 + \frac{\omega^2}{c^2}\int\frac{d\varepsilon}{d\omega}|E|^2 + \frac{\omega^2}{c^2}\int\frac{d\mu}{d\omega}|H|^2\right)$$

$$= -\frac{\omega^2}{c^2}\int\Delta\varepsilon|E|^2 + \Delta\mu|H|^2$$

$$\Rightarrow \Delta\omega = \frac{-\frac{\omega^2}{\cancel{c^2}}\int\Delta\varepsilon|E|^2+\Delta\mu|H|^2}{\underbrace{\frac{2\omega}{\cancel{c^2}}\int\mu|H|^2}+\frac{\omega^2}{\cancel{c^2}}\int\frac{d\varepsilon}{d\omega}|E|^2+\frac{\omega^2}{\cancel{c^2}}\int\frac{d\mu}{d\omega}|H|^2}$$

simplify

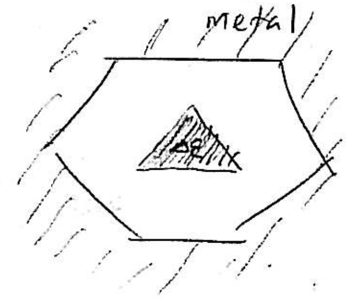
$$= \frac{\omega}{c^2}\int\mu|H|^2+\varepsilon|E|^2$$

since $\int\mu|H|^2 = \int\varepsilon|E|^2$

note: $\omega\varepsilon + \omega^2\frac{d\varepsilon}{d\omega} = \omega\frac{d}{d\omega}(\omega\varepsilon)$

$$= \boxed{\frac{-\omega\int\Delta\varepsilon|E|^2+\Delta\mu|H|^2}{\int\frac{d}{d\omega}(\omega\varepsilon)|E|^2+\frac{d}{d\omega}(\omega\mu)|H|^2}}$$

Problem 3



(a) if we put in the perturbation we lose the $3\sigma'$, $2C_6$, and C_2 symmetries from C_{6v}

– looking at the remaining columns, we set:

	E	$2C_3$	3σ		
Γ_1	1	1	1	=	R_1
Γ_2	1	1	−1	=	R_2
Γ_3	1	1	1	=	R_1
Γ_4	1	1	−1	=	R_2
Γ_5	2	−1	0	=	R_3
Γ_6	2	−1	0	=	R_3

!

$\Longrightarrow$ the original eigenfunctions, which were partner functions of $\Gamma_1 \cdots \Gamma_6$, are also partner functions of $R_1 \cdots R_3$ as shown.

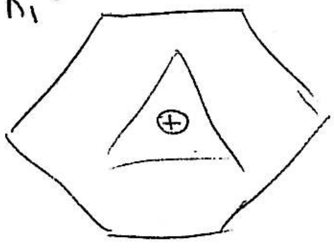
[A consequence of this is that ΔE does <u>not</u> break non-accidental degeneracies.]

(over →)

(b) We just need $\vec{J}$'s that are partner functions of $R_1 \ldots R_3$. For example, dipole currents located / oriented as shown:

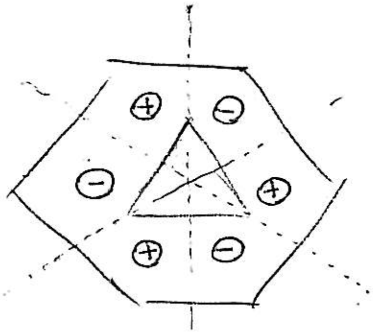
(for TM fields, $\vec{J}$ must be in the $\hat{z}$ direction, with the sign of J_z indicated by $\pm$)

R_1:



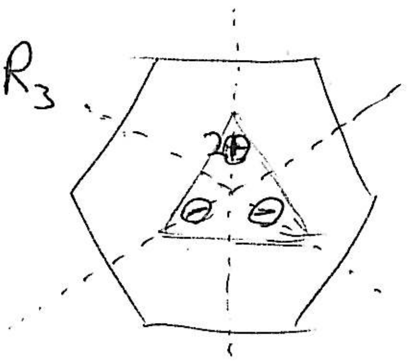
($J_z \sim \delta(x,y)$ at center $\Rightarrow$ fully symmetric)

R_2



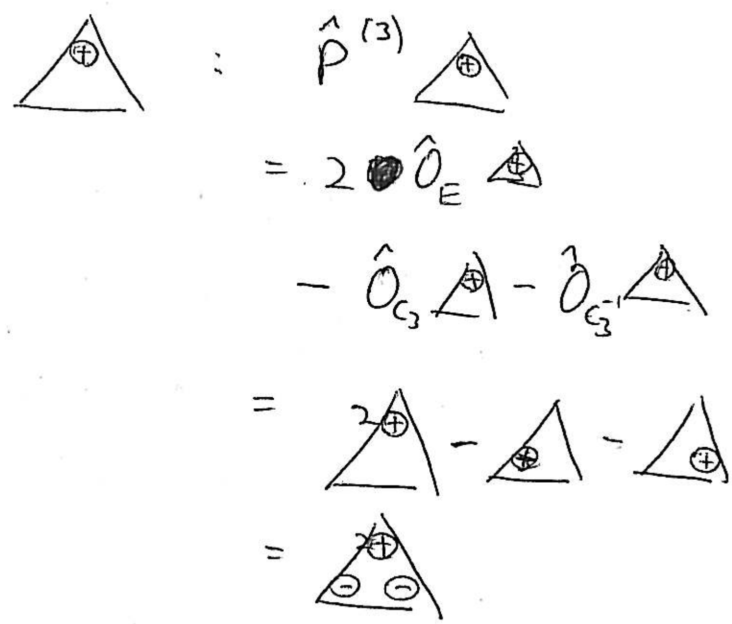
(odd across mirror planes)

R_3



(note that one of the currents has <u>twice</u> the amplitude)

$\leftarrow$ this is the trickiest one, but is easy if we do the projection operator $\hat{P}^{(3)}$ on



(neglecting overall normalization)

(c) given a field ψ, (assumed scalar for simplicity here)

we know that if ψ is a partner function of the irreducible rep. α, then the projection is

$$\hat{P}^{(\beta)}\psi = \begin{cases} \psi & \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

The key thing is that this is true at <u>every</u> point in space: $\hat{P}^{(\beta)}\psi$ is <u>everywhere</u> $= 0$
or <u>everywhere</u> $= \psi$

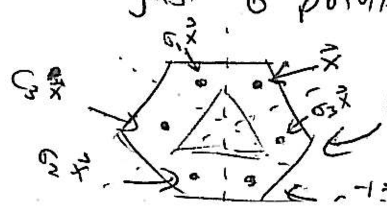
$\Longrightarrow$ we only need to look at a single point $\vec{x}$ where $\psi(\vec{x}) \neq 0$ and $\vec{x}$'s rotations, checking:
(& reflections)

(more generally, if ψ is a vector, we rotate ψ too)

$$\chi^{(\beta)}(E)^* \psi(\vec{x}) + \chi^{(\beta)}(C_3)^* \psi(C_3^{-1}\vec{x}) + \chi^{(\beta)}(C_3^{-1})^* \psi(C_3\vec{x})$$
$$+ \chi^{(\beta)}(\sigma_1)^* \psi(\sigma_1\vec{x}) + \chi^{(\beta)}(\sigma_2)^* \psi(\sigma_2\vec{x}) + \chi^{(\beta)}(\sigma_3)^* \psi(\sigma_3\vec{x})$$

$= \psi(\vec{x})$ for $\beta = \alpha$
0 otherwise

$\Longrightarrow$ evaluate ψ at just 6 points



(the rotations + reflections of $\vec{x}$)

(note that we can't pick $\vec{x}$ on a mirror plane: $\psi = 0$ there for B_2)