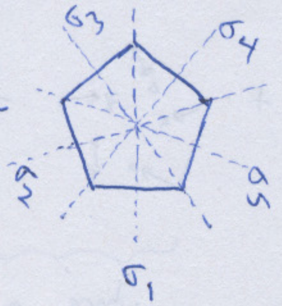
Problem #1

a



$\curvearrowright C_5$

$\mathcal{A} = E, C_5, C_5^2, C_5^3 = C_5^{-2}, C_5^4 = C_5^{-1}, 5\sigma$

(omitting $\vec{0}$ translations)

classes: E, $\{C_5, C_5^{-1}\}$, $\{C_5^2, C_5^{-2}\}$, 5σ

$\{C_5, C_5^{-1}\}$: equivalent under any σ; $\{C_5^2, C_5^{-2}\}$: equivalent under any σ; 5σ: equivalent under C_5 rotations

$\Rightarrow$ character table:

	E	$2C_5$	$2C_5^2$	5σ
Γ_1	1	1	1	1
Γ_2	1	1	1	-1
Γ_3	2	$-\frac{\sqrt5+1}{2}$	$\frac{\sqrt5-1}{2}$	0
Γ_4	2	$\frac{\sqrt5-1}{2}$	$-\frac{\sqrt5+1}{2}$	0

Γ_1, Γ_2: by orthogonality of rows; Γ_3, Γ_4: see below

4 classes $\Rightarrow$ 4 rep's

$\sum d^2 = 1^2+1^2+2^2+2^2 = |\mathcal{A}| = 10$

Γ_4 row:

let Γ_4 = 2×2 coord. transformations (given to be an irreducible rep.)

$$\Rightarrow D^{(4)}(C_5) = \begin{pmatrix} \cos\frac{2\pi}{5} & -\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{pmatrix} \Rightarrow \chi^{(4)}(C_5) = 2\cos\frac{2\pi}{5} = \frac{\sqrt5-1}{2}$$

similarly, $\chi^{(4)}(C_5^2) = 2\cos\frac{4\pi}{5} = -\frac{\sqrt5+1}{2}$

$D^{(4)}(\sigma_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ by inspection $\Rightarrow \chi^{(4)}(\sigma_1) = \chi^{(4)}(\sigma) = 0$

(we could also get this by orthogonality of rows in the char. table)

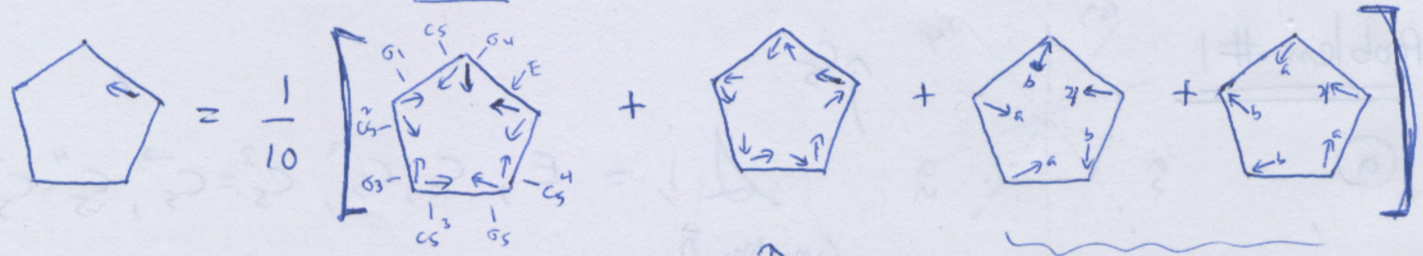
(Note: Γ_4 row orthogonal to Γ_1, Γ_2 rows since $2 + 2\frac{\sqrt5-1}{2} - 2\frac{\sqrt5+1}{2} = 0$)

Γ_3 row: by column orthogonality: $1 + 1 + 2\chi^{(3)}(C_5) + 2\left(\frac{\sqrt5-1}{2}\right) = 0$

$\Rightarrow \chi^{(3)}(C_5) = \chi^{(3)}(C_5^{-1}) = -\left(\frac{\sqrt5+1}{2}\right)$; similarly for $\chi^{(3)}(C_5^2)$, $\chi^{(3)}(\sigma)$

$\underline{\Gamma_1}$ $\underline{\Gamma_2}$ $\underline{\Gamma_3}$ $\underline{\Gamma_4}$

(b)



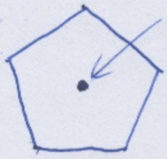
(we have labelled each $\vec{J}$ copy by the symmetry operation that gives it)

call σ $\vec{J}$'s are flipped)

where $a = \sqrt{5} - 1$

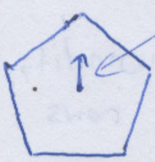
$b = \sqrt{5} + 1$

(c)



$\vec{J}_1 = \delta(\vec{x})\hat{z}$

excites Γ_1 only : $\vec{J}_1$ is fully symmetric , so $\hat{P}^{(1)}\vec{J}_1 = \vec{J}_1$

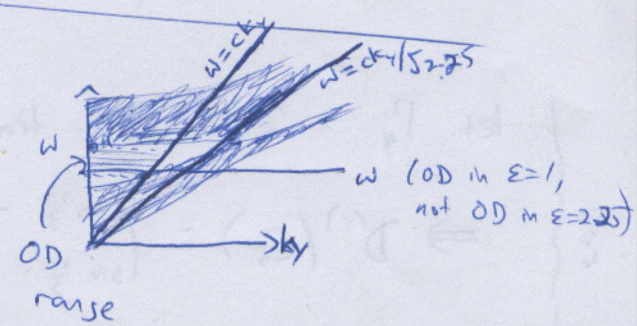


$\vec{J}_4 = \delta(\vec{x})\,\hat{y}$ excites Γ_4 only : $\vec{J}_4$ manifestly transforms as $(\hat{x}, \hat{y}) \Rightarrow \hat{P}^{(4)}\vec{J}_4 = \vec{J}_4$

(or any combination of $\hat{x}, \hat{y}$)

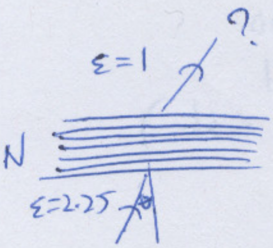
Problem 2 :

(a) mirror is OD from $\varepsilon = 1$ $\Rightarrow$ band diagram looks like



mirror is $\underline{\text{not}}$ OD from $\varepsilon = 2.25$ at ω

$\Rightarrow$ at ω, $\omega = ck_y/\sqrt{2.25}$ light line is $\underline{\text{inside}}$ (or below) mirror continuum as shown above.



$\rightarrow k_y$ is conserved

~~Three~~ Two $\underline{\text{cases}}$:

(1) incident light has k_y $\underline{\text{above}}$ $\omega \geq ck_y > ck_y/\sqrt{2.25}$

$\Rightarrow$ in this range, there are no modes in the mirror

$\Rightarrow$ $\underline{\text{exponentially decaying transmission}}$ with N (due to tails that penetrate mirror)

(3)

(2) incident light has k_y <u>above</u> $\varepsilon = 2.25$ light cone but <u>below</u> $\varepsilon = 1$ light cone: $ck_y > \omega \geq ck_y / \sqrt{2.25}$

$\Longrightarrow$ cannot couple to propagating modes in $\varepsilon = 1 \Rightarrow$ <u>0 transmission</u>

— we <u>may</u> couple to modes propagating in the mirrors, since this portion of the $\varepsilon = 2.25$ light cone overlaps the mirror modes, but these mirror modes are still reflected at the $\varepsilon = 1$ interface $\Longrightarrow$ reflection is still 1

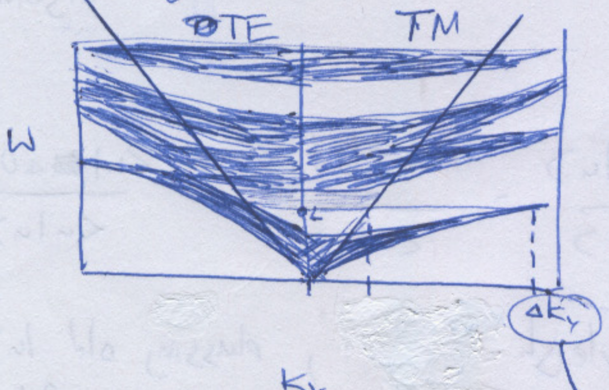
Case (2) corresponds to $\theta > \theta_c = \sin^{-1}\left(\frac{1}{\sqrt{2.25}}\right)$.

In short: $\theta > \theta_c \Rightarrow$ 0 transmission, 100% reflection

$\theta < \theta_c \Rightarrow$ transmission T exponentially decreasing with N, reflection $= 1 - T$

— this phenomenon does <u>not</u> depend on polarization, but polarization will affect transmission T in the $\theta < \theta_c$ regime by some constant factor

(b)

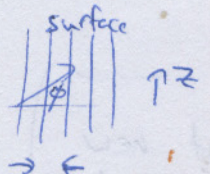


(i) Λ couples k_y to $k_y + \frac{2\pi}{\Lambda}$

$\Rightarrow$ if Λ is small enough, we cannot couple light cone to mirror states

$$\Longrightarrow \Lambda < \underset{\substack{\text{max } \Delta k_y \\ \text{that couples} \\ \text{light cone} \\ \text{to mirror}}}{\frac{2\pi}{}} < \underset{\substack{\text{marked} \\ \text{on} \\ \text{graph} \\ \text{(limited by} \\ \text{TM modes)} \\ \text{at lowest } \omega}}{\frac{2\pi}{\Delta k}}$$

(ii) No. If we look out of plane:



period at an angle ϕ $= \Lambda \sec\phi$

$\rightarrow \infty$ for $\phi \rightarrow \frac{\pi}{2}$

$\Longrightarrow$ there is always some out-of-plane angle ϕ so that $\Lambda \sec\phi > 2\pi / \Delta k_y$

Problem 3

(a) O is Hermitian $\Longrightarrow$ use 1^{st}-order perturbation theory

$\Longrightarrow$ for <u>non-degenerate</u> case, Δeigenvalue $= \dfrac{\langle u|\Delta O|u\rangle}{\langle u|u\rangle} + O(\Delta O^2)$

$\Longrightarrow$ for $\begin{pmatrix}1\\0\\0\\\pm1\end{pmatrix}$: Δeigenvalue $\cong \dfrac{\emptyset}{2} = \emptyset$ zero!

for $\begin{pmatrix}0\\1\\\pm1\\0\end{pmatrix}$ we must use <u>degenerate</u> perturbation theory

$\Longrightarrow$ find superposition that diagonalizes ΔO

by inspection: $\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\right)$

$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\right)$ works

$\Longrightarrow$ $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ has Δeigenvalue $= \dfrac{\delta}{1} = \delta$

$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ has Δeigenvalue $= 0$

degeneracy is <u>split</u> and these are the new eigenvectors

(b) 1^{st}-order approx for new eigenvalue

$= \text{old eigenvalue} + \dfrac{\langle u|\Delta O|u\rangle}{\langle u|u\rangle} = \dfrac{\langle u|O|u\rangle}{\langle u|u\rangle} + \dfrac{\langle u|\Delta O|u\rangle}{\langle u|u\rangle}$

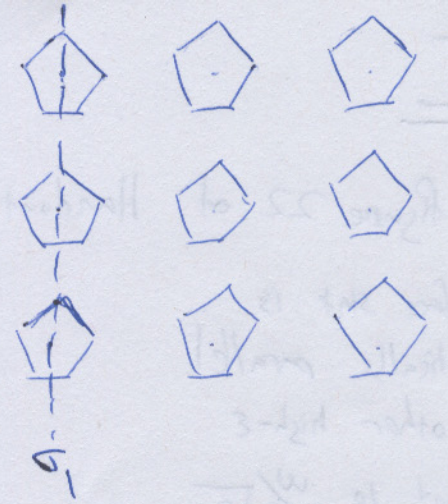
$= \dfrac{\langle u|O+\Delta O|u\rangle}{\langle u|u\rangle}$ = Rayleigh quotient, plugging old $|u\rangle$ into new $O+\Delta O$

O Hermitian, ΔO Hermitian $\Longrightarrow$ $O+\Delta O$ Hermitian

$\Longrightarrow$ Variational thm. holds

$\Longrightarrow$ $\boxed{\text{smallest new eigenvalue} \leq 1^{st}\text{ order approximation} \leq \text{largest new eigenvalue}}$

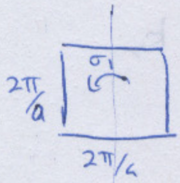
Problem 4



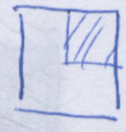
ⓐ same lattice $\Rightarrow$ same B.Z.

point-group symmetries: E, σ_1

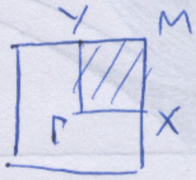
+ time-reversal only!

B.Z. = 

$\Rightarrow$ by σ_1 we only need 

by time-reversal, we only need  since  and 

+ σ_1

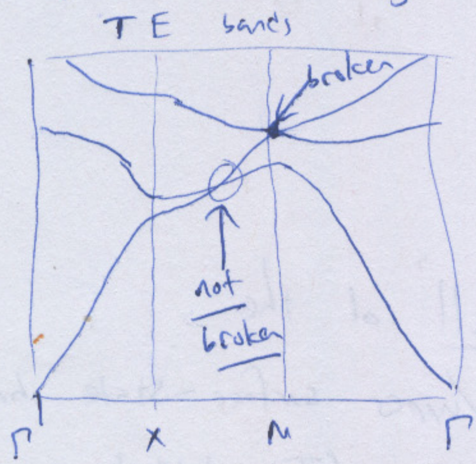
$\Rightarrow$ I.B.Z. is 

ⓑ $\{E, \sigma_1\}$ has no 2-dimensional irreducible representations

$\Longrightarrow$ degeneracies at Γ and M (which came from symmetry) are broken.

But, accidental degeneracies from band crossings remain

e.g.



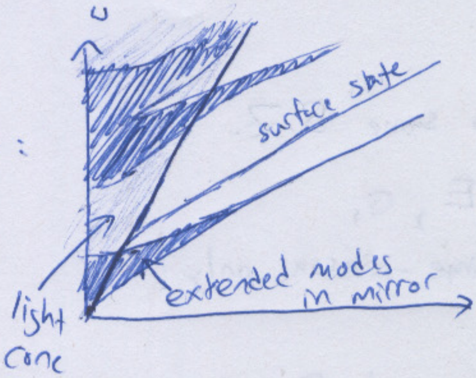
However, these crossing points will shift slightly to a new $\vec{k}$, ω

ⓒ because of mirror σ_1 + time-reversal, ω's just above and below $\Gamma-X$ line are equal $\Rightarrow$ $\Gamma-X$ line is an extremum $\Rightarrow$ $\frac{\partial\omega}{\partial k_y} = 0$ at $\Gamma-X$

$\Rightarrow \underbrace{\vec{v} = \frac{d\omega}{d\vec{k}}}_{=\text{ energy propagation direction}}$ is in $\pm\hat{x}$ direction on $\Gamma-X$. On $\Gamma-M$, there is no such symmetry $\Rightarrow$ $\vec{v}$ can point anywhere!

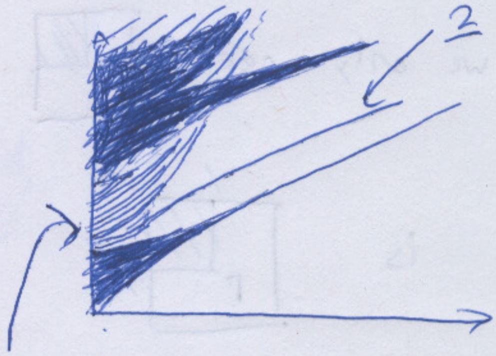
Problem 5

(a) from figure 22 of Handout:

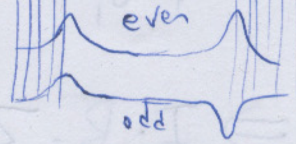


— the surface state is asymptotically parallel to the other high-ε bands and to $\frac{\omega}{c\sqrt{\varepsilon_{hi}}}$ because it is made of the same material

(b) large separation:



2 surface states, nearly degenerate

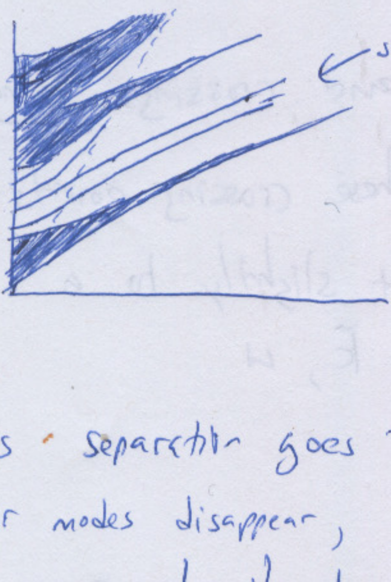


form even and odd superpositions (by mirror symmetry)

— nearly degenerate since only coupled through exponential tails

many discrete modes above light line, guided between two mirrors

smaller separation:



surface states, bigger split between even/odd

fewer modes in air

As separation goes to zero, all of the air modes disappear, and the lower/upper surface-state bands merge with the lower/upper continua. (Two half layers together = 1 whole layer = no defect.)