THE GAUSSIAN INTEGRAL

KEITH CONRAD

Let

$$I = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x, \quad J = \int_0^{\infty} e^{-x^2}\,\mathrm{d}x, \quad \text{and } K = \int_{-\infty}^{\infty} e^{-\pi x^2}\,\mathrm{d}x.$$

These positive numbers are related: $J = I/(2\sqrt{2})$ and $K = I/\sqrt{2\pi}$.

Theorem. *With notation as above, $I = \sqrt{2\pi}$, or equivalently $J = \sqrt{\pi}/2$, or equivalently $K = 1$.*

We will give multiple proofs of this. (Other lists of proofs are in [5] and [10].) It is subtle since $e^{-\frac{1}{2}x^2}$ has no simple antiderivative. For comparison, $\int_0^\infty xe^{-\frac{1}{2}x^2}\,\mathrm{d}x$ can be computed with the antiderivative $-e^{-\frac{1}{2}x^2}$ and equals 1. In the last section, the Gaussian integral's history is presented.

1. First Proof: Polar coordinates

The most widely known proof, due to Poisson [10, p. 3], expresses J^2 as a double integral and then uses polar coordinates. To start, write J^2 as an iterated integral using single-variable calculus:

$$J^2 = J\int_0^\infty e^{-y^2}\,\mathrm{d}y = \int_0^\infty Je^{-y^2}\,\mathrm{d}y = \int_0^\infty \left(\int_0^\infty e^{-x^2}\,\mathrm{d}x\right) e^{-y^2}\,\mathrm{d}y = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y.$$

View this as a double integral over the first quadrant. To compute it with polar coordinates, the first quadrant is $\{(r,\theta) : r \geq 0 \text{ and } 0 \leq \theta \leq \pi/2\}$. Writing x^2+y^2 as r^2 and $\mathrm{d}x\,\mathrm{d}y$ as $r\,\mathrm{d}r\,\mathrm{d}\theta$,

$$\begin{aligned} J^2 &= \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^\infty re^{-r^2}\,\mathrm{d}r \cdot \int_0^{\pi/2}\mathrm{d}\theta \\ &= -\frac{1}{2}e^{-r^2}\Big|_0^\infty \cdot \frac{\pi}{2} \\ &= \frac{1}{2}\cdot\frac{\pi}{2} \\ &= \frac{\pi}{4}. \end{aligned}$$

Since $J > 0$, $J = \sqrt{\pi}/2$.[1] It is argued in [2] that this method can't be used on any other integral.

2. Second Proof: Another change of variables

Our next proof uses another change of variables to compute J^2. As before,

$$J^2 = \int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)}\,\mathrm{d}x\right)\mathrm{d}y.$$

[1]For a visualization of this calculation as a volume, in terms of $\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x$ instead of J, see `https://www.youtube.com/watch?v=cy8r7WSuT1I`. We'll do a volume calculation for I^2 in Section 5.

Instead of using polar coordinates, set $x = yt$ in the inner integral (y is fixed). Then $\mathrm{d}x = y\,\mathrm{d}t$ and

$$J^2 = \int_0^\infty \left(\int_0^\infty e^{-y^2(t^2+1)} y\,\mathrm{d}t\right) \mathrm{d}y = \int_0^\infty \left(\int_0^\infty ye^{-y^2(t^2+1)}\,\mathrm{d}y\right) \mathrm{d}t, \tag{2.1}$$

where the interchange of integrals is justified by Fubini's theorem for improper Riemann integrals. (The appendix gives an approach using Fubini's theorem for Riemann integrals on rectangles.) Since $\displaystyle\int_0^\infty ye^{-ay^2}\,\mathrm{d}y = \frac{1}{2a}$ for $a > 0$, we have

$$J^2 = \int_0^\infty \frac{\mathrm{d}t}{2(t^2+1)} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4},$$

so $J = \sqrt{\pi}/2$. This proof is due to Laplace [8, pp. 94–96] and historically precedes the widely used technique of the previous proof. We will see in Section 9 what Laplace's first proof was.

3. Third Proof: Differentiating under the integral sign

For $t > 0$, set

$$A(t) = \left(\int_0^t e^{-x^2}\,\mathrm{d}x\right)^2.$$

The integral we want to calculate is $A(\infty) = J^2$ and then take a square root.

Differentiating $A(t)$ with respect to t and using the Fundamental Theorem of Calculus,

$$A'(t) = 2\int_0^t e^{-x^2}\,\mathrm{d}x \cdot e^{-t^2} = 2e^{-t^2}\int_0^t e^{-x^2}\,\mathrm{d}x.$$

Let $x = ty$, so

$$A'(t) = 2e^{-t^2}\int_0^1 te^{-t^2y^2}\,\mathrm{d}y = \int_0^1 2te^{-(1+y^2)t^2}\,\mathrm{d}y.$$

The function under the integral sign is easily antidifferentiated *with respect to* t:

$$A'(t) = \int_0^1 -\frac{\partial}{\partial t}\frac{e^{-(1+y^2)t^2}}{1+y^2}\,\mathrm{d}y = -\frac{d}{dt}\int_0^1 \frac{e^{-(1+y^2)t^2}}{1+y^2}\,\mathrm{d}y.$$

Letting

$$B(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x,$$

we have $A'(t) = -B'(t)$ for all $t > 0$, so there is a constant C such that

$$A(t) = -B(t) + C \tag{3.1}$$

for all $t > 0$. To find C, we let $t \to 0^+$ in (3.1). The left side tends to $\displaystyle\left(\int_0^0 e^{-x^2}\,\mathrm{d}x\right)^2 = 0$ while the right side tends to $\displaystyle -\int_0^1 \mathrm{d}x/(1+x^2) + C = -\pi/4 + C$. Thus $C = \pi/4$, so (3.1) becomes

$$\left(\int_0^t e^{-x^2}\,\mathrm{d}x\right)^2 = \frac{\pi}{4} - \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x.$$

Letting $t \to \infty$ in this equation, we obtain $J^2 = \pi/4$, so $J = \sqrt{\pi}/2$.

A comparison of this proof with the first proof is in [22].

4. Fourth Proof: Another differentiation under the integral sign

Here is a second approach to finding J by differentiation under the integral sign. I heard about it from Michael Rozman [15], who modified an idea on math.stackexchange [24], and in a slightly less elegant form it appeared much earlier in [20].

For $t \in \mathbf{R}$, set

$$F(t) = \int_0^\infty \frac{e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x.$$

Then $F(0) = \int_0^\infty dx/(1+x^2) = \pi/2$ and $F(\infty) = 0$. Differentiating under the integral sign,

$$F'(t) = \int_0^\infty -2te^{-t^2(1+x^2)}\,\mathrm{d}x = -2te^{-t^2}\int_0^\infty e^{-(tx)^2}\,\mathrm{d}x.$$

Make the substitution $y = tx$, with $\mathrm{d}y = t\,\mathrm{d}x$, so

$$F'(t) = -2e^{-t^2}\int_0^\infty e^{-y^2}\,\mathrm{d}y = -2Je^{-t^2}.$$

For $b > 0$, integrate both sides from 0 to b and use the Fundamental Theorem of Calculus:

$$\int_0^b F'(t)\,\mathrm{d}t = -2J\int_0^b e^{-t^2}\,\mathrm{d}t \Longrightarrow F(b) - F(0) = -2J\int_0^b e^{-t^2}\,\mathrm{d}t.$$

Letting $b \to \infty$ in the last equation,

$$0 - \frac{\pi}{2} = -2J^2 \Longrightarrow J^2 = \frac{\pi}{4} \Longrightarrow J = \frac{\sqrt{\pi}}{2}.$$

5. Fifth Proof: A volume integral

Our next proof is due to T. P. Jameson [6] and was rediscovered by A. L. Delgado [4]. Revolve $z = e^{-\frac{1}{2}x^2}$ in the xz-plane around the z-axis to produce the "bell surface" $z = e^{-\frac{1}{2}(x^2+y^2)}$. See Figure 1, where the z-axis is vertical and passes through the top point, the x-axis lies just under the surface through the point 0 in front, and the y-axis lies just under the surface through the point 0 on the left. We will compute the volume V below the surface and above the xy-plane in two ways.

First we compute V by *horizontal slices*, which are discs: $V = \int_0^1 A(z)\,\mathrm{d}z$ where $A(z)$ is the area of the disc formed by slicing the surface at height z. Writing the radius of the disc at height z as $r(z)$, $A(z) = \pi r(z)^2$. To compute $r(z)$, the surface cuts the xz-plane at a pair of points $(x, e^{-\frac{1}{2}x^2})$ where the height is z, so $e^{-\frac{1}{2}x^2} = z$. Thus $x^2 = -2\ln z$. Since x is the distance of these points from the z-axis, $r(z)^2 = x^2 = -2\ln z$, so $A(z) = \pi r(z)^2 = -2\pi \ln z$. Therefore

$$V = \int_0^1 -2\pi\ln z\,\mathrm{d}z = -2\pi\,(z\ln z - z)\Big|_0^1 = -2\pi(-1 - \lim_{z\to 0^+} z\ln z).$$

By L'Hospital's rule, $\lim_{z\to 0^+} z\ln z = 0$, so $V = 2\pi$. (A calculation of V by shells is in [12].)

Next we compute V by *vertical slices* in planes $x =$ constant. Vertical slices are scaled bell curves: look at the black contour lines in Figure 1. The equation of the bell curve along the top of the vertical slice with x-coordinate x is $z = e^{-\frac{1}{2}(x^2+y^2)}$, where y varies and x is fixed. Then

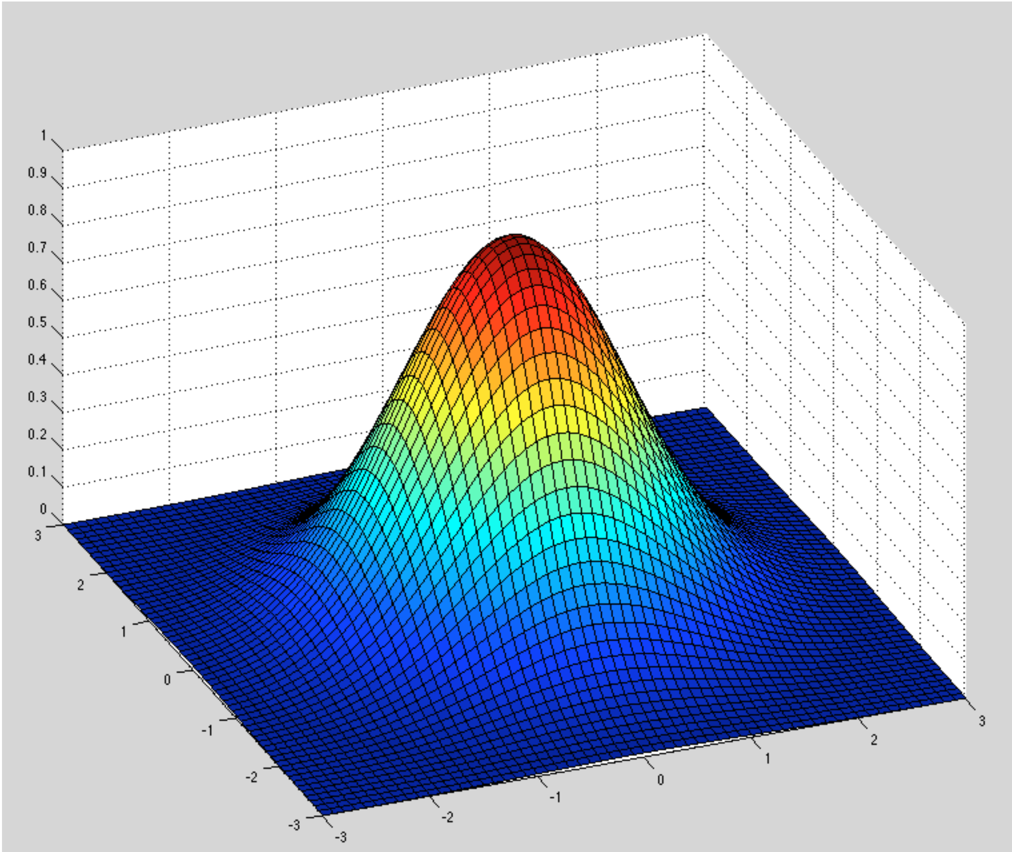

FIGURE 1. The bell surface $z = e^{-\frac{1}{2}(x^2+y^2)}$

$V = \int_{-\infty}^{\infty} A(x)\,\mathrm{d}x$, where $A(x)$ is the area of the x-slice:

$$A(x) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,\mathrm{d}y = e^{-\frac{1}{2}x^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,\mathrm{d}y = e^{-\frac{1}{2}x^2} I.$$

Thus $V = \int_{-\infty}^{\infty} A(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} I\,\mathrm{d}x = I\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x = I^2$.

Comparing the two formulas for V, we have $2\pi = I^2$, so $I = \sqrt{2\pi}$.

6. Sixth Proof: The Γ-function

For any integer $n \geq 0$, we have $n! = \int_0^{\infty} t^n e^{-t}\,\mathrm{d}t$. For $x > 0$ we define

$$\Gamma(x) = \int_0^{\infty} t^x e^{-t}\,\frac{\mathrm{d}t}{t},$$

so $\Gamma(n) = (n-1)!$ when $n \geq 1$. Using integration by parts, $\Gamma(x+1) = x\Gamma(x)$. One of the basic properties of the Γ-function [16, pp. 193–194] is

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t. \tag{6.1}$$

Set $x = y = 1/2$:

$$\Gamma\left(\frac{1}{2}\right)^2 = \int_0^1 \frac{\mathrm{d}t}{\sqrt{t(1-t)}}.$$

Note

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \sqrt{t}e^{-t}\frac{\mathrm{d}t}{t} = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,\mathrm{d}t = \int_0^\infty \frac{e^{-x^2}}{x}2x\,\mathrm{d}x = 2\int_0^\infty e^{-x^2}\,\mathrm{d}x = 2J,$$

so $4J^2 = \int_0^1 \mathrm{d}t/\sqrt{t(1-t)}$. With the substitution $t = \sin^2\theta$,

$$4J^2 = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\,\mathrm{d}\theta}{\sin\theta\cos\theta} = 2\frac{\pi}{2} = \pi,$$

so $J = \sqrt{\pi}/2$. Equivalently, $\Gamma(1/2) = \sqrt{\pi}$. Any method that proves $\Gamma(1/2) = \sqrt{\pi}$ is also a method that calculates $\int_0^\infty e^{-x^2}\,\mathrm{d}x$.

7. Seventh Proof: Asymptotic estimates

We will show $J = \sqrt{\pi}/2$ by a technique whose steps are based on [17, p. 371].

For $x \geq 0$, power series expansions show $1 + x \leq e^x \leq 1/(1-x)$. Reciprocating and replacing x with x^2, we get

$$1 - x^2 \leq e^{-x^2} \leq \frac{1}{1+x^2}. \tag{7.1}$$

for all $x \in \mathbf{R}$.

For any positive integer n, raise the terms in (7.1) to the nth power and integrate from 0 to 1:

$$\int_0^1 (1-x^2)^n\,\mathrm{d}x \leq \int_0^1 e^{-nx^2}\,\mathrm{d}x \leq \int_0^1 \frac{\mathrm{d}x}{(1+x^2)^n}.$$

Using the changes of variables $x = \sin\theta$ on the left, $x = y/\sqrt{n}$ in the middle, and $x = \tan\theta$ on the right,

$$\int_0^{\pi/2} (\cos\theta)^{2n+1}\,\mathrm{d}\theta \leq \frac{1}{\sqrt{n}}\int_0^{\sqrt{n}} e^{-y^2}\,\mathrm{d}y \leq \int_0^{\pi/4} (\cos\theta)^{2n-2}\,\mathrm{d}\theta < \int_0^{\pi/2} (\cos\theta)^{2n-2}\,\mathrm{d}\theta. \tag{7.2}$$

Set $I_k = \int_0^{\pi/2}(\cos\theta)^k\,\mathrm{d}\theta$, so $I_0 = \pi/2$, $I_1 = 1$, and (7.2) implies

$$\sqrt{n}I_{2n+1} \leq \int_0^{\sqrt{n}} e^{-y^2}\,\mathrm{d}y < \sqrt{n}I_{2n-2}. \tag{7.3}$$

We will show that as $k \to \infty$, $kI_k^2 \to \pi/2$. Then

$$\sqrt{n}I_{2n+1} = \frac{\sqrt{n}}{\sqrt{2n+1}}\sqrt{2n+1}I_{2n+1} \to \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2}$$

and

$$\sqrt{n}I_{2n-2} = \frac{\sqrt{n}}{\sqrt{2n-2}}\sqrt{2n-2}I_{2n-2} \to \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2},$$

so by (7.3), $\int_0^{\sqrt{n}} e^{-y^2}\,\mathrm{d}y \to \sqrt{\pi}/2$. Thus $J = \sqrt{\pi}/2$.

To show $kI_k^2 \to \pi/2$, first we compute several values of I_k explicitly by a recursion. Using integration by parts,

$$I_k = \int_0^{\pi/2} (\cos\theta)^k \, d\theta = \int_0^{\pi/2} (\cos\theta)^{k-1} \cos\theta \, d\theta = (k-1)(I_{k-2} - I_k),$$

so

$$I_k = \frac{k-1}{k} I_{k-2}. \tag{7.4}$$

Using (7.4) and the initial values $I_0 = \pi/2$ and $I_1 = 1$, the first few values of I_k are computed and listed in Table 1.

k	I_k	k	I_k
0	$\pi/2$	1	1
2	$(1/2)(\pi/2)$	3	$2/3$
4	$(3/8)(\pi/2)$	5	$8/15$
6	$(15/48)(\pi/2)$	7	$48/105$

Table 1.

From Table 1 we see that

$$I_{2n} I_{2n+1} = \frac{1}{2n+1} \frac{\pi}{2} \tag{7.5}$$

for $0 \le n \le 3$, and this can be proved for all n by induction using (7.4). Since $0 \le \cos\theta \le 1$ for $\theta \in [0, \pi/2]$, we have $I_k \le I_{k-1} \le I_{k-2} = \frac{k}{k-1} I_k$ by (7.4), so $I_{k-1} \sim I_k$ as $k \to \infty$. Therefore (7.5) implies

$$I_{2n}^2 \sim \frac{1}{2n} \frac{\pi}{2} \Longrightarrow (2n) I_{2n}^2 \to \frac{\pi}{2}$$

as $n \to \infty$. Then

$$(2n+1) I_{2n+1}^2 \sim (2n) I_{2n}^2 \to \frac{\pi}{2}$$

as $n \to \infty$, so $kI_k^2 \to \pi/2$ as $k \to \infty$. This completes our proof that $J = \sqrt{\pi}/2$.

Remark 7.1. This proof is closely related to the fifth proof using the Γ-function. Indeed, by (6.1)

$$\frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{k+1}{2} + \frac{1}{2})} = \int_0^1 t^{(k+1)/2+1} (1-t)^{1/2-1} \, dt,$$

and with the change of variables $t = (\cos\theta)^2$ for $0 \le \theta \le \pi/2$, the integral on the right is equal to $2\int_0^{\pi/2} (\cos\theta)^k \, d\theta = 2I_k$, so (7.5) is the same as

$$\begin{aligned} I_{2n} I_{2n+1} &= \frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{2n+2}{2})} \frac{\Gamma(\frac{2n+2}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{2n+3}{2})} \\ &= \frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})^2}{4\Gamma(\frac{2n+1}{2} + 1)} \\ &= \frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})^2}{4\frac{2n+1}{2}\Gamma(\frac{2n+1}{2})} \\ &= \frac{\Gamma(\frac{1}{2})^2}{2(2n+1)}. \end{aligned}$$

By (7.5), $\pi = \Gamma(1/2)^2$. We saw in the fifth proof that $\Gamma(1/2) = \sqrt{\pi}$ if and only if $J = \sqrt{\pi}/2$.

8. Eighth Proof: Stirling's Formula

Besides the integral formula $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}$ that we have been discussing, another place in mathematics where $\sqrt{2\pi}$ appears is in Stirling's formula:

$$n! \sim \frac{n^n}{e^n}\sqrt{2\pi n} \text{ as } n \to \infty.$$

In 1730 De Moivre proved $n! \sim C(n^n/e^n)\sqrt{n}$ for some positive number C without being able to determine C. Stirling soon thereafter showed $C = \sqrt{2\pi}$ and wound up having the whole formula named after him. We will show that determining that the constant C in Stirling's formula is $\sqrt{2\pi}$ is equivalent to showing that $J = \sqrt{\pi}/2$ (or, equivalently, that $I = \sqrt{2\pi}$).

Applying (7.4) repeatedly,

$$\begin{aligned} I_{2n} &= \frac{2n-1}{2n} I_{2n-2} \\ &= \frac{(2n-1)(2n-3)}{(2n)(2n-2)} I_{2n-4} \\ &\vdots \\ &= \frac{(2n-1)(2n-3)(2n-5)\cdots(5)(3)(1)}{(2n)(2n-2)(2n-4)\cdots(6)(4)(2)} I_0. \end{aligned}$$

Inserting $(2n-2)(2n-4)(2n-6)\cdots(6)(4)(2)$ in the top and bottom,

$$I_{2n} = \frac{(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\cdots(6)(5)(4)(3)(2)(1)}{(2n)((2n-2)(2n-4)\cdots(6)(4)(2))^2}\frac{\pi}{2} = \frac{(2n-1)!}{2n(2^{n-1}(n-1)!)^2}\frac{\pi}{2}.$$

Applying De Moivre's asymptotic formula $n! \sim C(n/e)^n\sqrt{n}$, ,

$$I_{2n} \sim \frac{C((2n-1)/e)^{2n-1}\sqrt{2n-1}}{2n(2^{n-1}C((n-1)/e)^{n-1}\sqrt{n-1})^2}\frac{\pi}{2} = \frac{(2n-1)^{2n}\frac{1}{2n-1}\sqrt{2n-1}}{2n\cdot 2^{2(n-1)}Ce(n-1)^{2n}\frac{1}{(n-1)^2}(n-1)}\frac{\pi}{2}$$

as $n \to \infty$. For any $a \in \mathbf{R}$, $(1+a/n)^n \to e^a$ as $n \to \infty$, so $(n+a)^n \sim e^a n^n$. Substituting this into the above formula with $a = -1$ and n replaced by $2n$,

$$I_{2n} \sim \frac{e^{-1}(2n)^{2n}\frac{1}{\sqrt{2n}}}{2n\cdot 2^{2(n-1)}Ce(e^{-1}n^n)^2\frac{1}{n^2}n}\frac{\pi}{2} = \frac{\pi}{C\sqrt{2n}}. \tag{8.1}$$

Since $I_{k-1} \sim I_k$, the outer terms in (7.3) are both asymptotic to $\sqrt{n}I_{2n} \sim \pi/(C\sqrt{2})$ by (8.1). Therefore

$$\int_0^{\sqrt{n}} e^{-y^2}\,dy \to \frac{\pi}{C\sqrt{2}}$$

as $n \to \infty$, so $J = \pi/(C\sqrt{2})$. Therefore $C = \sqrt{2\pi}$ if and only if $J = \sqrt{\pi}/2$.

9. Ninth Proof: The original proof

The original proof that $J = \sqrt{\pi}/2$ is due to Laplace [9] in 1774. (An English translation of Laplace's article is mentioned in the bibliographic citation for [9], with preliminary comments on that article in [19].) He wanted to compute

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{-\log x}}. \tag{9.1}$$

Setting $y = \sqrt{-\log x}$, this integral is $2\int_0^\infty e^{-y^2}\,\mathrm{d}y = 2J$, so we expect (9.1) to be $\sqrt{\pi}$.

Laplace's starting point for evaluating (9.1) was a formula of Euler:

$$\int_0^1 \frac{x^r\,\mathrm{d}x}{\sqrt{1-x^{2s}}}\int_0^1 \frac{x^{s+r}\,\mathrm{d}x}{\sqrt{1-x^{2s}}} = \frac{1}{s(r+1)}\frac{\pi}{2} \tag{9.2}$$

for positive r and s. (Laplace himself said this formula held "whatever be" r or s, but if $s < 0$ then the number under the square root is negative.) Accepting (9.2), let $r \to 0$ in it to get

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^{2s}}}\int_0^1 \frac{x^{s}\,\mathrm{d}x}{\sqrt{1-x^{2s}}} = \frac{1}{s}\frac{\pi}{2}. \tag{9.3}$$

Now let $s \to 0$ in (9.3). Then $1 - x^{2s} \sim -2s\log x$ by L'Hopital's rule, so (9.3) becomes

$$\left(\int_0^1 \frac{\mathrm{d}x}{\sqrt{-\log x}}\right)^2 = \pi.$$

Thus (9.1) is $\sqrt{\pi}$.

Euler's formula (9.2) looks mysterious, but we have met it before. In the formula let $x^s = \cos\theta$ with $0 \le \theta \le \pi/2$. Then $x = (\cos\theta)^{1/s}$, and after some calculations (9.2) turns into

$$\int_0^{\pi/2} (\cos\theta)^{(r+1)/s-1}\,\mathrm{d}\theta \int_0^{\pi/2} (\cos\theta)^{(r+1)/s}\,\mathrm{d}\theta = \frac{1}{(r+1)/s}\frac{\pi}{2}. \tag{9.4}$$

We used the integral $I_k = \int_0^{\pi/2} (\cos\theta)^k\,\mathrm{d}\theta$ before when k is a nonnegative integer. This notation makes sense when k is any positive real number, and then (9.4) assumes the form $I_\alpha I_{\alpha+1} = \frac{1}{\alpha+1}\frac{\pi}{2}$ for $\alpha = (r+1)/s - 1$, which is (7.5) with a possibly nonintegral index. Letting $r = 0$ and $s = 1/(2n+1)$ in (9.4) recovers (7.5). Letting $s \to 0$ in (9.3) corresponds to letting $n \to \infty$ in (7.5), so the proof in Section 7 is in essence a more detailed version of Laplace's 1774 argument.

10. Tenth Proof: Residue theorem

We will calculate $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x$ using contour integrals and the residue theorem. However, we can't just integrate $e^{-z^2/2}$, as this function has no poles. For a long time nobody knew how to handle this integral using contour integration. For instance, in 1914 Watson [21, p. 79] wrote "Cauchy's theorem cannot be employed to evaluate all definite integrals; thus $\displaystyle\int_0^\infty e^{-x^2}\,\mathrm{d}x$ has not been evaluated except by other methods." In the 1940s several contour integral solutions were published using awkward contours such as parallelograms [11], [13, Sect. 5] (see [3, Exer. 9, p. 113] for a recent appearance). Our approach will follow Kneser [7, p. 121] (see also [14, pp. 413–414] or [23]), using a rectangular contour and the function

$$\frac{e^{-z^2/2}}{1 - e^{-\sqrt{\pi}(1+i)z}}.$$

This function comes out of nowhere, so our first task is to motivate the introduction of this function.

We seek a meromorphic function $f(z)$ to integrate around the contour γ_R in Figure 2, with vertices at $-R$, R, $R+ib$, and $-R+ib$, where b will be fixed and we let $R \to \infty$.

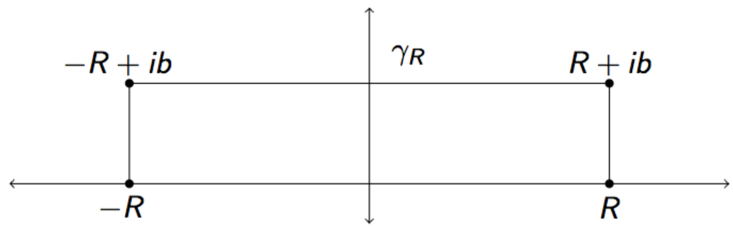

FIGURE 2. Contour to compute Gaussian

Suppose $f(z) \to 0$ along the right and left sides of γ_R uniformly as $R \to \infty$. Then by applying the residue theorem and letting $R \to \infty$, we would obtain (if the integrals converge)

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x + \int_{\infty}^{-\infty} f(x+ib)\,\mathrm{d}x = 2\pi i \sum_a \mathrm{Res}_{z=a} f(z),$$

where the sum is over poles of $f(z)$ with imaginary part between 0 and b. This is equivalent to

$$\int_{-\infty}^{\infty} (f(x) - f(x+ib))\,\mathrm{d}x = 2\pi i \sum_a \mathrm{Res}_{z=a} f(z).$$

Therefore we want $f(z)$ to satisfy

$$f(z) - f(z+ib) = e^{-z^2/2}, \tag{10.1}$$

where $f(z)$ and b need to be determined.

Let's try $f(z) = e^{-z^2/2}/d(z)$, for an unknown $d(z)$ whose zeros are poles of $f(z)$. We want

$$f(z) - f(z+\tau) = e^{-z^2/2} \tag{10.2}$$

for some τ (which will *not* be purely imaginary, so (10.1) doesn't quite work, but (10.1) is only motivation). Substituting $e^{-z^2/2}/d(z)$ for $f(z)$ in (10.2) gives us

$$e^{-z^2/2}\left(\frac{1}{d(z)} - \frac{e^{-\tau z - \tau^2/2}}{d(z+\tau)}\right) = e^{-z^2/2}. \tag{10.3}$$

Suppose $d(z+\tau) = d(z)$. Then (10.3) implies

$$d(z) = 1 - e^{-\tau z - \tau^2/2},$$

and with this definition of $d(z)$, $e^{-z^2/2}/d(z)$ satisfies (10.2) if and only if $e^{\tau^2} = 1$, or equivalently $\tau^2 \in 2\pi i \mathbf{Z}$. The simplest nonzero solution is $\tau = \sqrt{\pi}(1+i)$. From now on this is the value of τ, so $e^{-\tau^2/2} = e^{-i\pi} = -1$ and $d(z) = 1 + e^{-\tau z}$. Set

$$f(z) = \frac{e^{-z^2/2}}{d(z)} = \frac{e^{-z^2/2}}{1+e^{-\tau z}},$$

which is Kneser's function mentioned earlier. This function satisfies (10.2) and we henceforth ignore the motivation (10.1). Poles of $f(z)$ are at odd integral multiples of $\tau/2$.

We will integrate this $f(z)$ around the contour γ_R in Figure 3, whose height $\sqrt{\pi}$ is $\mathrm{Im}(\tau)$.

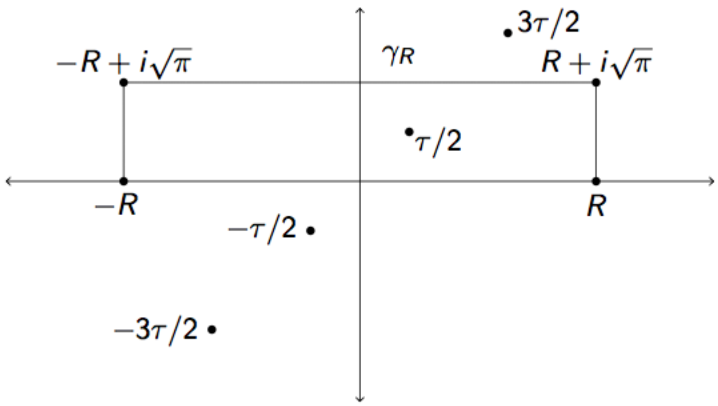

FIGURE 3. The contour γ_R and poles of Kneser's function

The poles of $f(z)$ nearest the origin are plotted in Figure 3; they lie along the line $y = x$. The only pole of $f(z)$ inside γ_R (for $R > \sqrt{\pi}/2$) is at $\tau/2$, so by the residue theorem

$$\int_{\gamma_R} f(z)\,\mathrm{d}z = 2\pi i \mathrm{Res}_{z=\tau/2} f(z) = 2\pi i \frac{e^{-\tau^2/8}}{(-\tau)e^{-\tau^2/2}} = \frac{2\pi i e^{3\tau^2/8}}{-\sqrt{\pi}(1+i)} = \frac{2\pi i e^{3\pi i/4}}{-\sqrt{\pi}(1+i)} = \sqrt{2\pi}.$$

Since the left and right sides of γ_R have the same length, $\sqrt{\pi}$, for all R, to show the integral of f along those sides tends to 0 uniformly as $R \to \infty$, it suffices to show $f(z) \to 0$ uniformly along those sides as $R \to \infty$. Parametrize z along the left and right sides as $-R + it$ and $R + it$ with t running over $[0, \sqrt{\pi}]$ in one direction or the other (which won't matter since we'll be taking absolute values). Then, using the reverse triangle inequality in the denominator, when $R > \sqrt{\pi}$ (so $R > t$)

$$|f(R+it)| = \frac{|e^{-R^2/2 - iRt + t^2/2}|}{|1 + e^{-\tau(R+it)}|} \le \frac{e^{-R^2/2}e^{t^2/2}}{|1 - e^{-\mathrm{Re}(\tau(R+it))}|} \le \frac{e^{-R^2/2}e^{\pi/2}}{1 - e^{-\sqrt{\pi}(R-t)}} < \frac{e^{-R^2/2}e^{\pi/2}}{1 - e^{-\sqrt{\pi}(R-\sqrt{\pi})}},$$

which tends to 0 as $R \to \infty$. Also

$$|f(-R+it)| = \frac{|e^{-R^2/2 + iRt + t^2/2}|}{|1 + e^{-\tau(-R+it)}|} \le \frac{e^{-R^2/2}e^{t^2/2}}{|1 - e^{-\mathrm{Re}(\tau(-R+it))}|} \le \frac{e^{-R^2/2}e^{\pi/2}}{e^{\sqrt{\pi}(R+t)} - 1} < \frac{e^{-R^2/2}e^{\pi/2}}{e^{\sqrt{\pi}R} - 1},$$

which tends to 0 as $R \to \infty$. Thus

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x + \int_{\infty + i\sqrt{\pi}}^{-\infty + i\sqrt{\pi}} f(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} f(x + i\sqrt{\pi})\,\mathrm{d}x.$$

In the second integral, write $i\sqrt{\pi}$ as $\tau - \pi$ and use (real) translation invariance of $\mathrm{d}x$ to obtain

$$\sqrt{2\pi} = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} f(x+\tau)\,\mathrm{d}x = \int_{-\infty}^{\infty} (f(x) - f(x+\tau))\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x \quad \text{by (10.2).}$$

11. Eleventh Proof: Fourier transforms

For a continuous function $f\colon \mathbf{R} \to \mathbf{C}$ that is rapidly decreasing at $\pm\infty$, its Fourier transform is the function $\mathcal{F}f\colon \mathbf{R} \to \mathbf{C}$ defined by

$$(\mathcal{F}f)(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy}\,\mathrm{d}x. \tag{11.1}$$

For example, $(\mathcal{F}f)(0) = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$. Here are three properties of the Fourier transform.

- If f is differentiable, then after using differentiation under the integral sign on the Fourier transform of f we obtain
$$(\mathcal{F}f)'(y) = \int_{-\infty}^{\infty} -ixf(x)e^{-ixy}\,\mathrm{d}x = -i(\mathcal{F}(xf(x)))(y).$$
- Using integration by parts on the Fourier transform of f, with $u = f(x)$ and $\mathrm{d}v = e^{-ixy}\,\mathrm{d}x$, we obtain
$$(\mathcal{F}(f'))(y) = iy(\mathcal{F}f)(y).$$
- If we apply the Fourier transform twice then we recover the original function up to interior and exterior scaling:

$$(\mathcal{F}^2 f)(x) = 2\pi f(-x). \tag{11.2}$$

The 2π is admittedly a nonobvious scaling factor here, and the proof of (11.2) is nontrivial. We'll show the appearance of 2π in (11.2) is equivalent to the evaluation of I as $\sqrt{2\pi}$.

Fixing $a > 0$, set $f(x) = e^{-ax^2}$, so
$$f'(x) = -2axf(x).$$
Applying the Fourier transform to both sides of this equation implies $iy(\mathcal{F}f)(y) = -2a\frac{1}{-i}(\mathcal{F}f)'(y)$, which simplifies to $(\mathcal{F}f)'(y) = -\frac{1}{2a}y(\mathcal{F}f)(y)$. The general solution of $g'(y) = -\frac{1}{2a}yg(y)$ is $g(y) = Ce^{-y^2/(4a)}$, so
$$f(x) = e^{-ax^2} \Longrightarrow (\mathcal{F}f)(y) = Ce^{-y^2/(4a)}$$
for some constant C. We have $1/(4a) = a$ when $a = 1/2$, so set $a = 1/2$: if $f(x) = e^{-x^2/2}$ then

$$(\mathcal{F}f)(y) = Ce^{-y^2/2} = Cf(y). \tag{11.3}$$

Setting $y = 0$ in (11.3), the left side is $(\mathcal{F}f)(0) = \int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x = I$, so $I = Cf(0) = C$.

Applying the Fourier transform to both sides of (11.3) with $C = I$ and using (11.2), we get $2\pi f(-x) = I(\mathcal{F}f)(x) = I^2 f(x)$. At $x = 0$ this becomes $2\pi = I^2$, so $I = \sqrt{2\pi}$ since $I > 0$. That is the Gaussian integral calculation. If we didn't know that the constant on the right side of (11.2) is 2π, whatever its value is would wind up being I^2, so saying 2π appears on the right side of (11.2) is equivalent to saying $I = \sqrt{2\pi}$.

There are other ways to define the Fourier transform besides (11.1), such as
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ixy}\,\mathrm{d}x \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,\mathrm{d}x.$$
These transforms have properties similar to the transform as defined in (11.1), so they can be used in its place to compute the Gaussian integral. Let's see how such a proof looks using the second alternative definition, which we'll write as
$$(\widetilde{\mathcal{F}}f)(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,\mathrm{d}x.$$
For this Fourier transform, the analogue of the three properties above for $\mathcal{F}$ are

- $(\widetilde{\mathcal{F}}f)'(y) = -2\pi i(\widetilde{\mathcal{F}}(xf(x)))(y)$.
- $(\widetilde{\mathcal{F}}(f'))(y) = 2\pi iy(\widetilde{\mathcal{F}}f)(y)$.
- $(\widetilde{\mathcal{F}}^2 f)(x) = f(-x)$.

The last property for $\widetilde{\mathcal{F}}$ looks nicer than that for $\mathcal{F}$, since there is no overall 2π-factor on the right side (it has been hidden in the definition of $\widetilde{\mathcal{F}}$). On the other hand, the first two properties for $\widetilde{\mathcal{F}}$ have overall factors of 2π on the right side while the first two properties of $\mathcal{F}$ do not. You can't escape a role for π or 2π somewhere in every possible definition of a Fourier transform.

Now let's run through the proof again with $\widetilde{\mathcal{F}}$ in place of $\mathcal{F}$. For $a > 0$, set $f(x) = e^{-ax^2}$. Applying $\widetilde{\mathcal{F}}$ to both sides of the equation $f'(x) = -2axf(x)$, $2\pi iy(\widetilde{\mathcal{F}}f)(y) = -2a\frac{1}{-(2\pi i)}(\mathcal{F}f)'(y)$, and that is equivalent to $(\widetilde{\mathcal{F}}f)'(y) = -\frac{2\pi^2}{a}y(\mathcal{F}f)(y)$. Solutions of $g'(y) = -\frac{2\pi^2}{a}yg(y)$ all look like $Ce^{-(\pi^2/a)y^2}$, so

$$f(x) = e^{-ax^2} \Longrightarrow (\widetilde{\mathcal{F}}f)(y) = Ce^{-(\pi^2/a)y^2}$$

for a constant C. We want $\pi^2/a = \pi$ so that $e^{-(\pi^2/a)y^2} = e^{-\pi y^2} = f(y)$, which occurs for $a = \pi$. Thus when $f(x) = e^{-\pi x^2}$ we have

$$(\widetilde{\mathcal{F}}f)(y) = Ce^{-\pi y^2} = Cf(y). \tag{11.4}$$

When $y = 0$ in (11.4), this becomes $\int_{-\infty}^{\infty} e^{-\pi x^2}\,\mathrm{d}x = C$, so $C = K$: see the top of the first page for the definition of K as the integral of $e^{-\pi x^2}$ over $\mathbf{R}$.

Applying $\widetilde{\mathcal{F}}$ to both sides of (11.4) with $C = K$ and using $(\widetilde{\mathcal{F}}^2 f)(x) = f(-x)$, we get $f(-x) = K(\widetilde{\mathcal{F}}f)(x) = K^2 f(x)$. At $x = 0$ this becomes $1 = K^2$, so $K = 1$ since $K > 0$. That $K = 1$, or in more explicit form $\int_{-\infty}^{\infty} e^{-\pi x^2}\,\mathrm{d}x = 1$, is equivalent to the evaluation of the Gaussian integral I with the change of variables $y = \sqrt{2\pi}x$ in the integral for K.

12. History of the Gaussian integral

The function $e^{-x^2/2}$, more commonly written as $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ ("normal distribution") so it has total integral 1 over $\mathbf{R}$, plays an essential role in probability and statistics, and it was in such settings that this function was initially found. Three separate investigations led to it.

(1) Large sample approximation to binomial: De Moivre [1, pp. 235-237] in 1733 found that a large number of samples from a binomial distribution with $p = 1/2$ can be approximated by a normal distribution. He did not write the Gaussian integral directly, but in [1, Cor. 2, p. 237] he estimated that the probability a binomial random variable with $p = 1/2$ is within a standard deviation of the mean from above is .341344. This is $\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-x^2/2}\,\mathrm{d}x$. Such a role for the normal distribution did not make it stand out. De Moivre's work was generalized by Laplace in 1812 to binomial distributions with any p in $(0,1)$. The normal approximation to the binomial for large sample sizes is a mainstay in probability and statistics courses.

(2) Distribution of Errors: Gauss in 1809 (based on work starting in 1801) was led to $\frac{a}{\sqrt{\pi}}e^{-a^2x^2}$ in his astronomical work on locating the lost asteroid Ceres by the method of least squares. (At $a = 1/\sqrt{2}$ this is $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.) To Gauss, such a function for suitable a describes the distribution of errors in measurements: it is "the error curve". The difficulty in finding such a curve to model errors is hard to appreciate today, when the answer is known. In the 1770s, for example, Laplace proposed other error curves with corners or asymptotes at $x = 0$, as in Figure 4 below. In the second picture, a is the largest imagined error.

(3) Central Limit Theorem: Laplace, in his 1812 book *Théorie analytique des probabilités* [8], developed the first version of the Central Limit Theorem, which assigns a special role to the normal distribution.

The discovery of normally distributed data in the social sciences by Quetelet and others in the mid-19th century along with more general versions of the Central Limit Theorem solidified its role

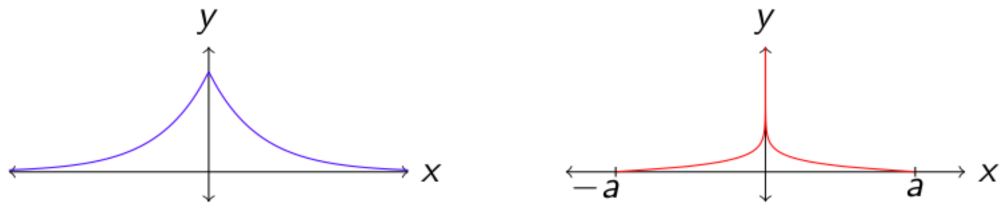

Figure 4. Laplace's error curves: $\frac{m}{2}e^{-m|x|}$ in 1774 and $\frac{1}{2a}\log(a/|x|)$ when $|x| \leq a$ in 1777.

from then on. See [18] for more details. In that era, it was called the "Law of Error" or "the Error Curve". The name Central Limit Theorem is due to George Polya in 1920,[2] and the widespread use of the name "normal distribution" is due to Karl Pearson.[3]

Appendix A. Redoing Section 2 without improper integrals in Fubini's theorem

In this appendix we will work out the calculation of the Gaussian integral in Section 2 without relying on Fubini's theorem for improper integrals. The key equation is (2.1), which we recall:

$$\int_0^\infty \left(\int_0^\infty ye^{-(t^2+1)y^2}\,\mathrm{d}t\right)\mathrm{d}y = \int_0^\infty \left(\int_0^\infty ye^{-(t^2+1)y^2}\,\mathrm{d}y\right)\mathrm{d}t.$$

The calculation in Section 2 that the iterated integral on the right is $\pi/4$ does not need Fubini's theorem in any form. It is going from the iterated integral on the left to $\pi/4$ that used Fubini's theorem for improper integrals. The next theorem could be used as a substitute, and its proof will only use Fubini's theorem for integrals on rectangles.

Theorem A.1. *For $b > 1$ and $c > 1$,*

$$\int_0^\infty \left(\int_0^\infty ye^{-(t^2+1)y^2}\,\mathrm{d}t\right)\mathrm{d}y = \frac{\pi}{4} + O\left(\frac{1}{b}\right) + O\left(\frac{1}{\sqrt{c}}\right).$$

Having $b \to \infty$ and $c \to \infty$ in Theorem A.1 makes the right side $\pi/4$ without changing the left side.

Lemma A.2. (1) *For all $x \in \mathbf{R}$, $e^{-x^2} \leq \dfrac{1}{x^2+1}$.*

(2) *For $a > 0$ $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{a^2x^2+1} = \frac{\pi}{2a}$.*

(3) *For $a > 0$ and $c > 0$, $\displaystyle\int_c^\infty \frac{\mathrm{d}x}{a^2x^2+1} = \frac{1}{a}\left(\frac{\pi}{2} - \arctan(ac)\right)$.*

(4) *For $a > 0$ and $c > 0$, $\displaystyle\int_c^\infty \frac{\mathrm{d}x}{a^2x^2+1} < \frac{1}{a^2c}$.*

(5) *For $a > 0$, $\dfrac{\pi}{2} - \arctan a < \dfrac{1}{a}$.*

Proof. The proofs of (1), (2), and (3) are left to the reader. To prove (4), replace $1 + a^2t^2$ by the smaller value a^2t^2. To prove (5), write the difference as $\int_a^\infty dx/(x^2+1)$ and then bound $1/(x^2+1)$ above by $1/x^2$. □

[2] See https://mathoverflow.net/questions/44132.

[3] See https://mathshistory.st-andrews.ac.uk/Miller/mathword/n/.

Now we prove Theorem A.1.

Proof. Step 1. For $b > 1$ and $c > 1$, we'll show the improper integral can be truncated to an integral over $[0, b] \times [0, c]$ plus error terms:

$$\int_0^\infty \left(\int_0^\infty y e^{-(t^2+1)y^2}\, \mathrm{d}t \right) \mathrm{d}y = \int_0^b \left(\int_0^c y e^{-(t^2+1)y^2}\, \mathrm{d}t \right) \mathrm{d}y + O\left(\frac{1}{\sqrt{c}}\right) + O\left(\frac{1}{b}\right).$$

Subtract the integral on the right from the integral on the left and split the outer integral $\int_0^\infty$ into $\int_0^b + \int_b^\infty$:

$$\int_0^\infty \left(\int_0^\infty y e^{-(t^2+1)y^2}\, \mathrm{d}t \right) \mathrm{d}y - \int_0^b \left(\int_0^c y e^{-(t^2+1)y^2}\, \mathrm{d}t \right) \mathrm{d}y = \int_0^b \left(\int_c^\infty y e^{-(t^2+1)y^2}\, \mathrm{d}t \right) \mathrm{d}y + \int_b^\infty \left(\int_0^\infty y e^{-(t^2+1)y^2}\, \mathrm{d}t \right) \mathrm{d}y.$$

On the right side, we will show the first iterated integral is $O(1/\sqrt{c})$ and the second iterated integral is $O(1/b)$. The second iterated integral is simpler:

$$\begin{aligned}
\int_b^\infty \left(\int_0^\infty y e^{-(t^2+1)y^2}\, \mathrm{d}t \right) \mathrm{d}y &= \int_b^\infty \left(\int_0^\infty e^{-(yt)^2}\, \mathrm{d}t \right) y e^{-y^2}\, \mathrm{d}y \\
&\le \int_b^\infty \left(\int_0^\infty \frac{\mathrm{d}t}{y^2t^2+1} \right) y e^{-y^2}\, \mathrm{d}y \quad \text{by Lemma A.2(1)} \\
&= \int_b^\infty \frac{\pi}{2y} y e^{-y^2}\, \mathrm{d}y \quad \text{by Lemma A.2(2)} \\
&= \frac{\pi}{2} \int_b^\infty e^{-y^2}\, \mathrm{d}y \\
&\le \frac{\pi}{2} \int_b^\infty \frac{\mathrm{d}y}{y^2+1} \quad \text{by Lemma A.2(1)} \\
&= \frac{\pi}{2b} \quad \text{since } \frac{1}{y^2+1} < \frac{1}{y^2},
\end{aligned}$$

and this is $O(1/b)$. Returning to the first iterated integral,

$$\begin{aligned}
\int_0^b \left(\int_c^\infty y e^{-(t^2+1)y^2}\, \mathrm{d}t \right) \mathrm{d}y &= \int_0^b \left(\int_c^\infty e^{-(yt)^2}\, \mathrm{d}t \right) y e^{-y^2}\, \mathrm{d}y \\
&\le \int_0^b \left(\int_c^\infty \frac{\mathrm{d}t}{y^2t^2+1} \right) y e^{-y^2}\, \mathrm{d}y \quad \text{by Lemma A.2(1)} \\
&= \int_0^1 \left(\int_c^\infty \frac{\mathrm{d}t}{y^2t^2+1} \right) y e^{-y^2}\, \mathrm{d}y + \int_1^b \left(\int_c^\infty \frac{\mathrm{d}t}{y^2t^2+1} \right) y e^{-y^2}\, \mathrm{d}y \\
&\le \int_0^1 \left(\int_c^\infty \frac{\mathrm{d}t}{y^2t^2+1} \right) y e^{-y^2}\, \mathrm{d}y + \int_1^b \frac{1}{y^2c} y e^{-y^2}\, \mathrm{d}y \quad \text{by Lemma A.2(4)} \\
&= \int_0^1 \left(\frac{\pi}{2} - \arctan(yc) \right) e^{-y^2}\, \mathrm{d}y + \frac{1}{c} \int_1^b \frac{\mathrm{d}y}{y e^{y^2}} \quad \text{by Lemma A.2(3)} \\
&\le \int_0^1 \left(\frac{\pi}{2} - \arctan(yc) \right) \mathrm{d}y + \frac{1}{c} \int_1^\infty \frac{\mathrm{d}y}{y e^{y^2}}.
\end{aligned}$$

The last term is $O(1/c)$. We will show the first term is $O(1/\sqrt{c})$ by carefully splitting up $\int_0^1$.

For $0 < \varepsilon < 1$,

$$\int_0^1 \left(\frac{\pi}{2} - \arctan(yc)\right) \mathrm{d}y = \int_0^\varepsilon \left(\frac{\pi}{2} - \arctan(yc)\right) \mathrm{d}y + \int_\varepsilon^1 \left(\frac{\pi}{2} - \arctan(yc)\right) \mathrm{d}y.$$

Both integrals are positive, and the first one is less than $(\pi/2)\varepsilon$. The integrand of the second integral is less than $1/(yc)$ by Lemma A.2(5), so

$$\int_\varepsilon^1 \left(\frac{\pi}{2} - \arctan(yc)\right) \mathrm{d}y < \int_\varepsilon^1 \frac{\mathrm{d}y}{yc} < \frac{1-\varepsilon}{\varepsilon c} < \frac{1}{\varepsilon c}.$$

Therefore

$$0 < \int_0^1 \left(\frac{\pi}{2} - \arctan(yc)\right) \mathrm{d}y < \frac{\pi}{2}\varepsilon + \frac{1}{\varepsilon c}$$

for each ε in $(0, 1)$. Use $\varepsilon = 1/\sqrt{c}$ to get

$$0 < \int_0^1 \left(\frac{\pi}{2} - \arctan(yc)\right) \mathrm{d}y < \frac{\pi}{2\sqrt{c}} + \frac{1}{\sqrt{c}} = O\left(\frac{1}{\sqrt{c}}\right).$$

That proves the first iterated integral is $O(1/\sqrt{c}) + O(1/c) = O(1/\sqrt{c})$ as $c \to \infty$.

Step 2. For $b > 0$ and $c > 0$, we will show

$$\int_0^b \left(\int_0^c ye^{-(t^2+1)y^2} \,\mathrm{d}t\right) \mathrm{d}y = \frac{\pi}{4} + O\left(\frac{1}{e^{b^2}}\right) + O\left(\frac{1}{c}\right).$$

By Fubini's theorem for continuous functions on a *rectangle* in $\mathbf{R}^2$,

$$\int_0^b \left(\int_0^c ye^{-(t^2+1)y^2} \,\mathrm{d}t\right) \mathrm{d}y = \int_0^c \left(\int_0^b ye^{-(t^2+1)y^2} \,\mathrm{d}y\right) \mathrm{d}t.$$

For the inner integral on the right, the formula $\int_0^b ye^{-ay^2} \,\mathrm{d}y = 1/(2a) - 1/(2ae^{ab^2})$ for $a > 0$ tells us

$$\int_0^b ye^{-(t^2+1)y^2} \,\mathrm{d}y = \frac{1}{2(t^2+1)} - \frac{1}{2(t^2+1)e^{(t^2+1)b^2}},$$

so

$$\begin{aligned}\int_0^c \left(\int_0^b ye^{-(t^2+1)y^2} \,\mathrm{d}y\right) \mathrm{d}t &= \frac{1}{2}\int_0^c \frac{\mathrm{d}t}{t^2+1} - \frac{1}{2}\int_0^c \frac{\mathrm{d}t}{(t^2+1)e^{(t^2+1)b^2}} \\ &= \frac{1}{2}\arctan(c) - \frac{1}{2}\int_0^c \frac{\mathrm{d}t}{(t^2+1)e^{(t^2+1)b^2}}.\end{aligned} \tag{A.1}$$

Let's estimate these last two terms. Since

$$\arctan(c) = \int_0^\infty \frac{\mathrm{d}t}{t^2+1} - \int_c^\infty \frac{\mathrm{d}t}{t^2+1} = \frac{\pi}{2} + O\left(\int_c^\infty \frac{\mathrm{d}t}{t^2}\right) = \frac{\pi}{2} + O\left(\frac{1}{c}\right)$$

and

$$\int_0^c \frac{\mathrm{d}t}{(t^2+1)e^{(t^2+1)b^2}} \le \int_0^c \frac{\mathrm{d}t}{t^2+1}\frac{1}{e^{b^2}} \le \int_0^\infty \frac{\mathrm{d}t}{t^2+1}\frac{1}{e^{b^2}} = O\left(\frac{1}{e^{b^2}}\right),$$

feeding these error estimates into (A.1) finishes Step 2. □

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