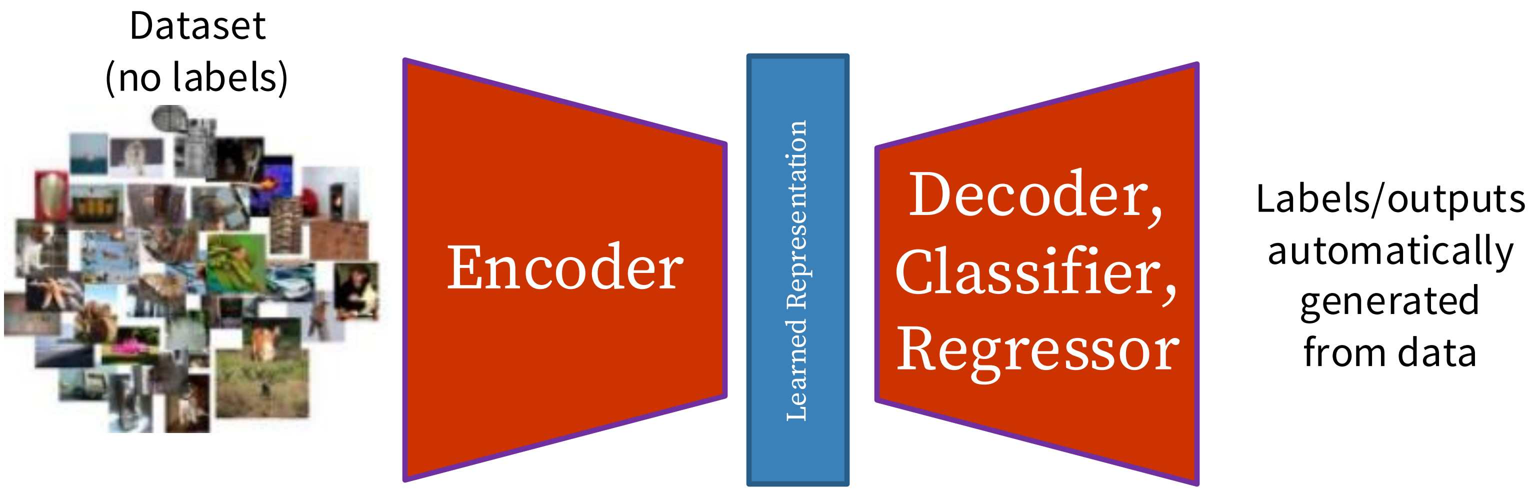
# Lecture 13:
# Generative Models (part 1)

# Administrative

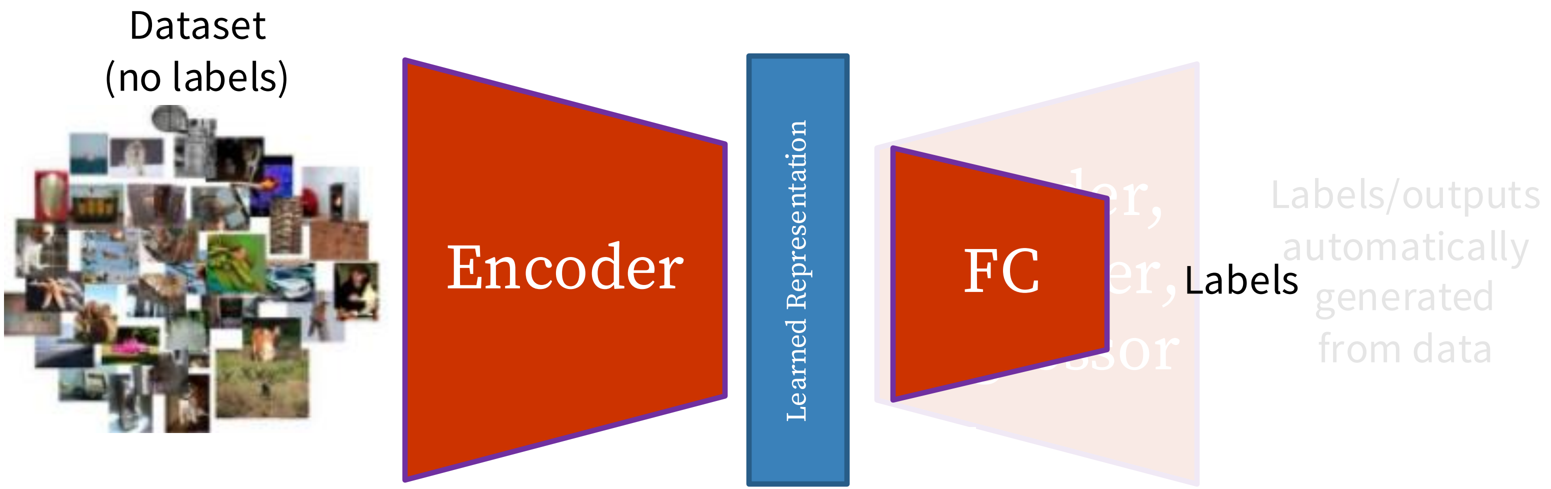
Tomorrow 5/16:

- Assignment 3 out
- Project milestone due

# Last Time: Self-Supervised Learning



# Last Time: Self-Supervised Learning



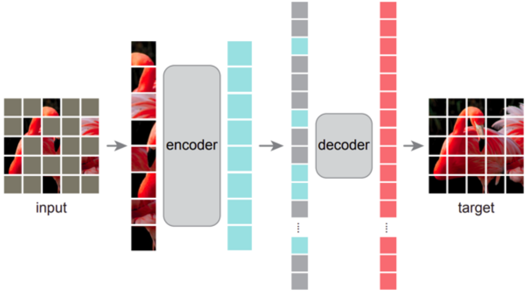
# Last Time: Self-Supervised Learning

**Pretext tasks from image transformations**

- Rotation, inpainting, rearrangement, coloring
- Reconstruction-based learning (MAE)


Rotation


Rearrangement



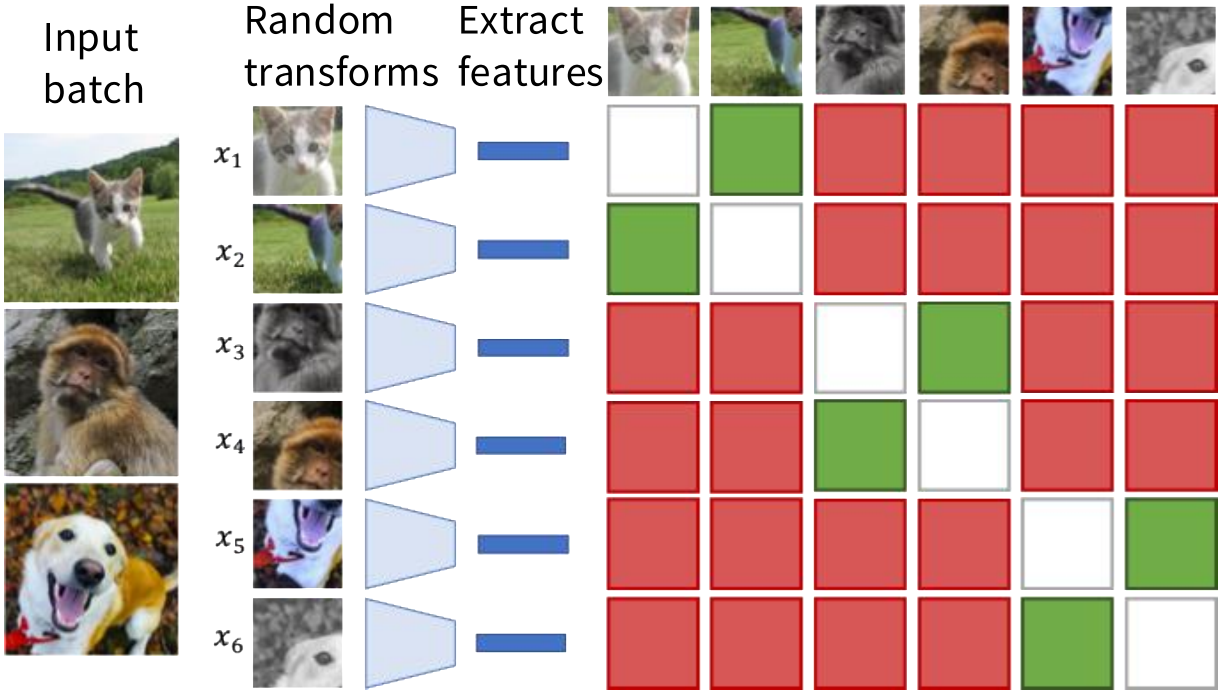

Reconstruction

# Last Time: Self-Supervised Learning

**Contrastive representation learning**

- Intuition and formulation
- Instance contrastive learning: SimCLR and MOCO
- Sequence contrastive learning: CPC
- Self-Distillation Without Labels, DINO

# Last Time: Contrastive Learning (SimCLR)

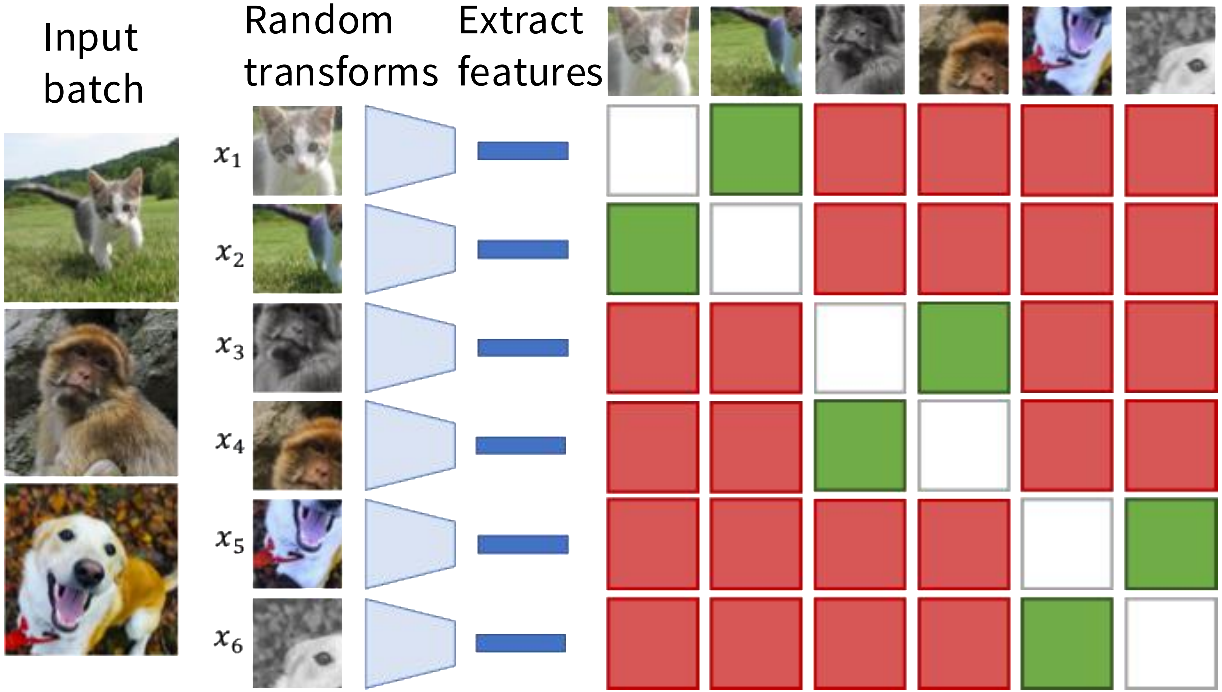



Corresponding pairs should have similar features

Other pairs should have dissimilar features

Chen et al, “A simple framework for contrastive learning of visual representations”, ICML 2020

# Last Time: Contrastive Learning (SimCLR)



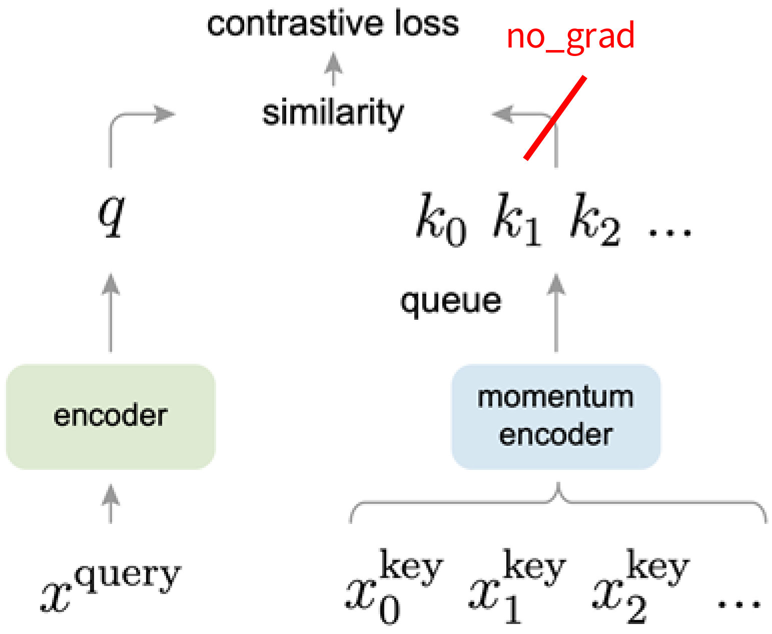

Corresponding pairs should have similar features

Other pairs should have dissimilar features

**Problem**: Need large batch size with lots of negatives

Chen et al, "A simple framework for contrastive learning of visual representations", ICML 2020

# Contrastive Learning: MoCo




Key differences to SimCLR:

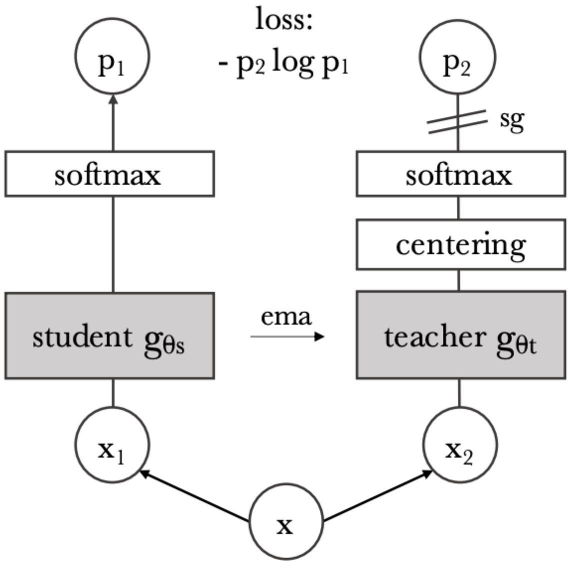
- Keep a running queue of keys (negative samples).
- Compute gradients and update the encoder only through the queries.
- Decouple min-batch size with the number of keys: can support a large number of negative samples.
- The key encoder is slowly progressing through the momentum update rules:

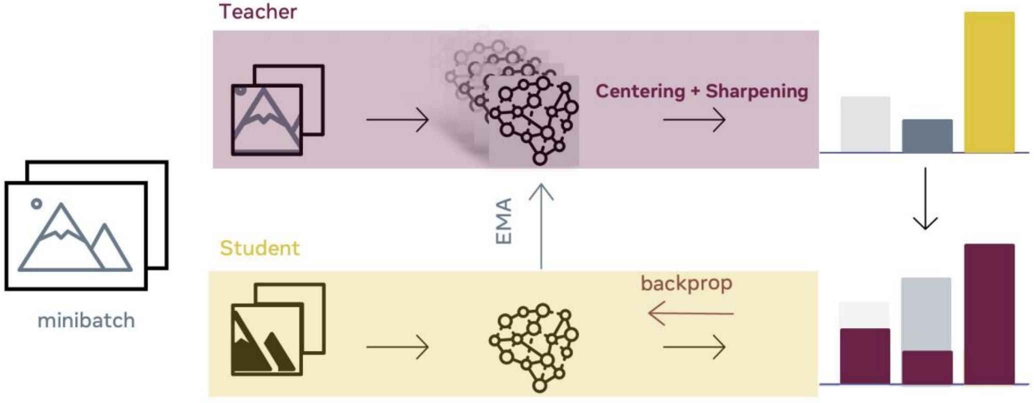
$$\theta_{\mathrm{k}} \leftarrow m\theta_{\mathrm{k}} + (1-m)\theta_{\mathrm{q}}$$

He et al, "Momentum Contrast for Unsupervised Visual Representation Learning", CVPR 2020

# Self-Supervised Learning: DINO

Similar in spirit to MoCo, but matches features using KL divergence instead of dot product, and uses Vision Transformers instead of ResNets



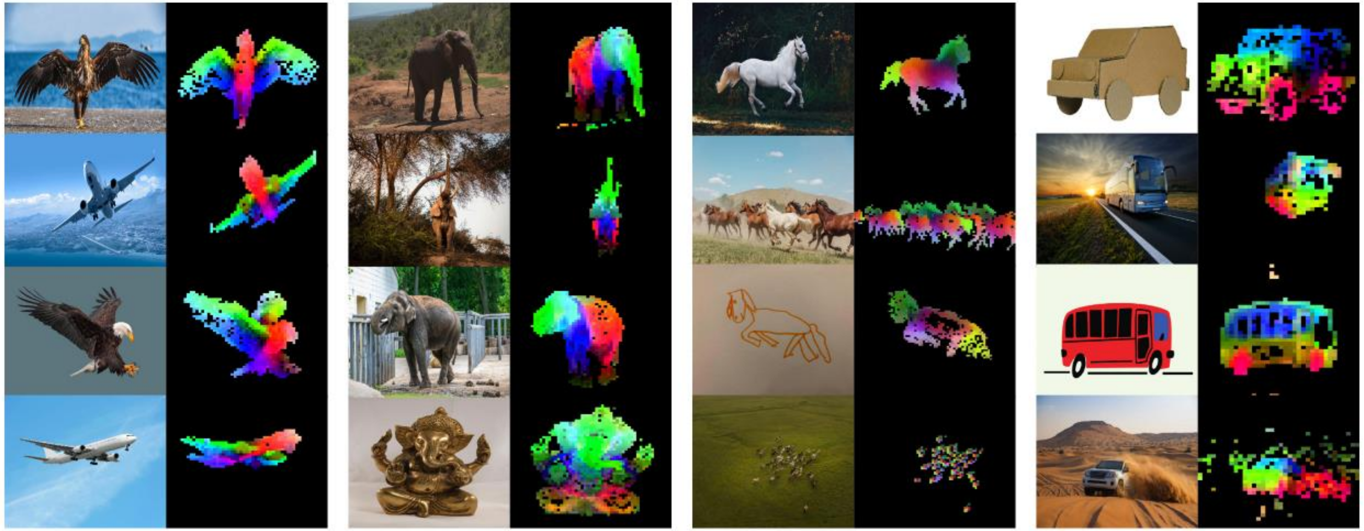




$$\theta_{\text{teacher}} \leftarrow \tau \cdot \theta_{\text{teacher}} + (1-\tau) \cdot \theta_{\text{student}}$$

Caron et al. 2021 Emerging Properties in Self-Supervised Vision Transformers

# Self-Supervised Learning: DINOv2

Scales up training data from 1M ImageNet images to 142M images
Very strong image features, commonly used in practice



PCA feature visualization

Oquab et al, "DINOv2: Learning Robust Visual Features without Supervision", arXiv 2023; Darcet et al, "Vision Transformers Need Registers", arXiv 2023

# Today:
# Generative Models (part 1)

# Supervised vs Unsupervised Learning

**Supervised Learning**

**Data**: (x, y)
x is data, y is label

**Goal**: Learn a function to map x -> y

**Examples**: Classification, regression, object detection, semantic segmentation, image captioning, etc.

# Supervised vs Unsupervised Learning

**Supervised Learning**

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→ Cat

Classification

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**Examples**: Classification, regression, object detection, semantic segmentation, image captioning, etc.

A cat sitting on a suitcase on the floor

Image captioning

Caption generated using neuraltalk2
Image is CC0 Public domain.

# Supervised vs Unsupervised Learning

**Supervised Learning**

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DOG, DOG, CAT

Object Detection

# Supervised vs Unsupervised Learning

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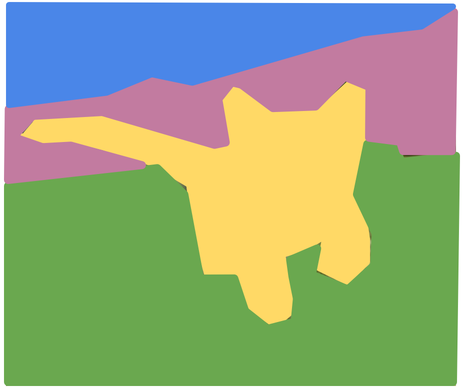

GRASS, CAT, TREE, SKY

Semantic Segmentation

# Supervised vs Unsupervised Learning

**Supervised Learning**

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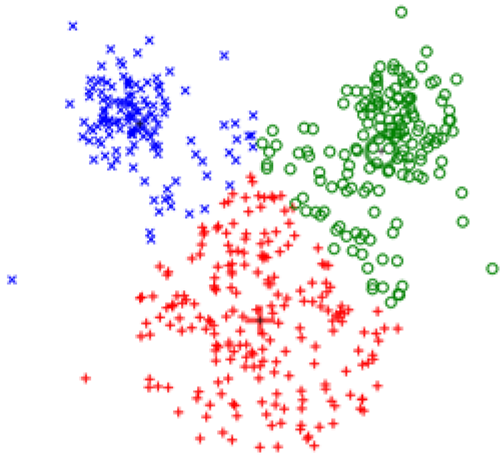
**Unsupervised Learning**

**Data**: x
Just data, no labels!

**Goal**: Learn hidden structure in data

**Examples**: Clustering, dimensionality reduction, density estimation, etc.

# Supervised vs Unsupervised Learning



K-means clustering

**Unsupervised Learning**

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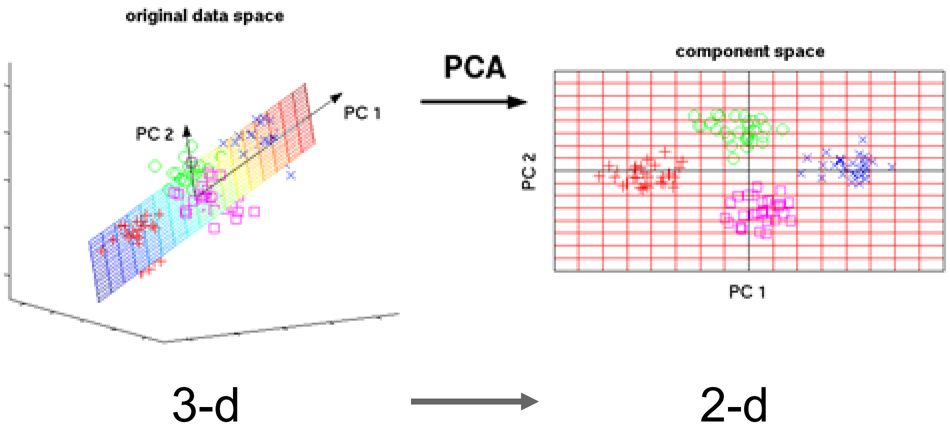
# Supervised vs Unsupervised Learning




Principal Component Analysis
(Dimensionality reduction)



**Unsupervised Learning**

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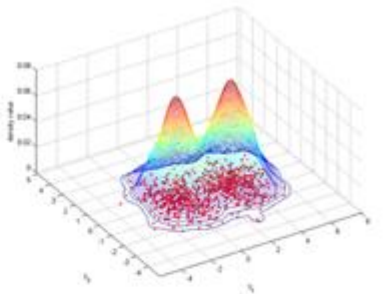
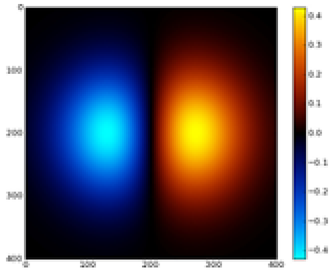
**Examples**: Clustering, dimensionality reduction, density estimation, etc.

# Supervised vs Unsupervised Learning



1-d density estimation





2-d density estimation

Modeling p(x)

**Unsupervised Learning**

**Data**: x
Just data, no labels!

**Goal**: Learn hidden structure in data

**Examples**: Clustering, dimensionality reduction, density estimation, etc.

# Generative vs Discriminative Models

**Discriminative Model**: Learn a probability distribution p(y|x)

**Generative Model**: Learn a probability distribution p(x)

**Conditional Generative Model**: Learn p(x|y)

**Data:** x

**Label:** y
Cat

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**Probability Recap:**

**Density Function**
p(x) assigns a positive number to each possible x; higher numbers mean x is more likely.

Density functions are **normalized:**

$$\int_X p(x)dx = 1$$

Different values of x **compete** for density

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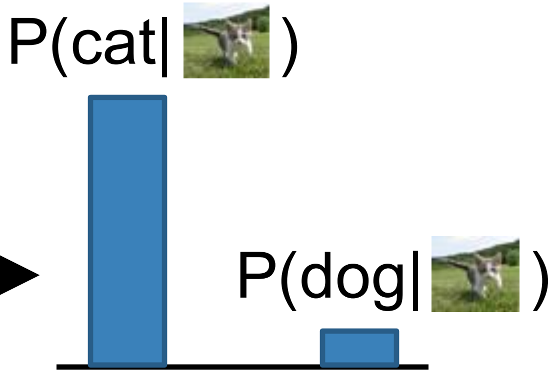


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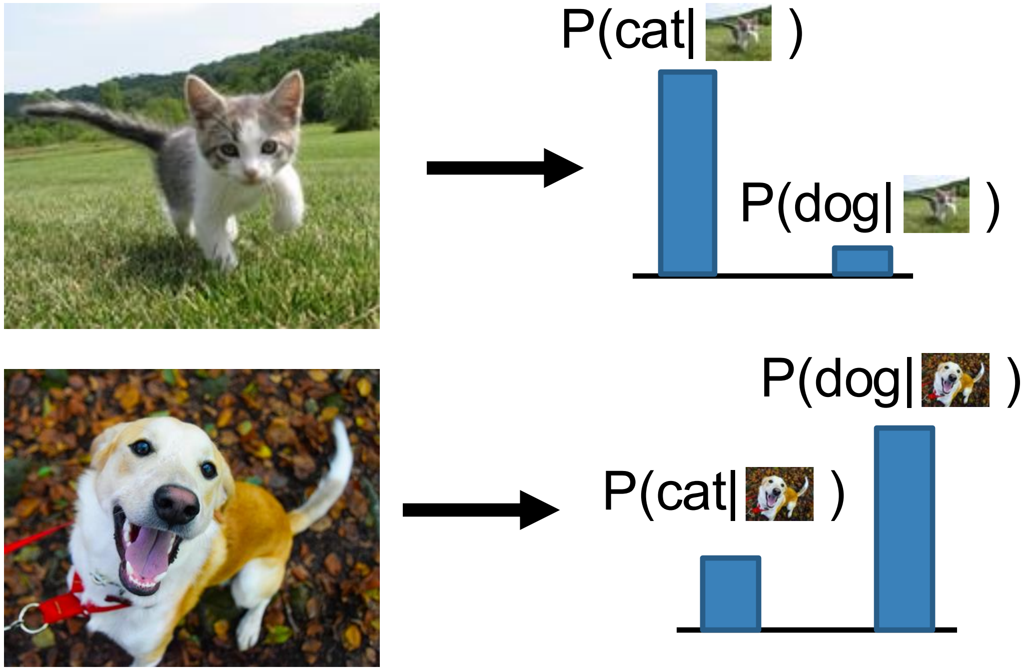


Possible **labels** for each image compete for probability.
No competition between **images**

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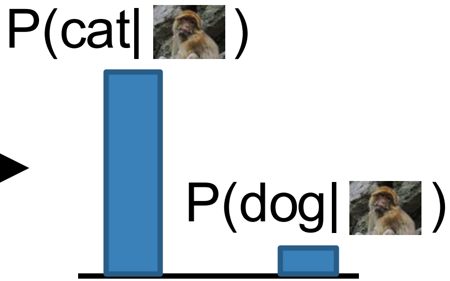


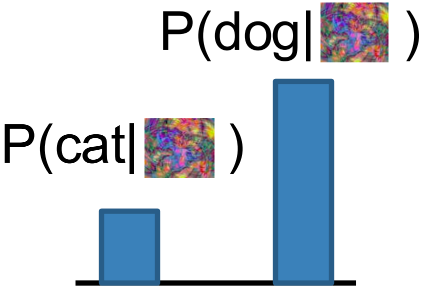


No way to handle unreasonable inputs; must give a label distribution for all possible inputs

# Generative vs Discriminative Models

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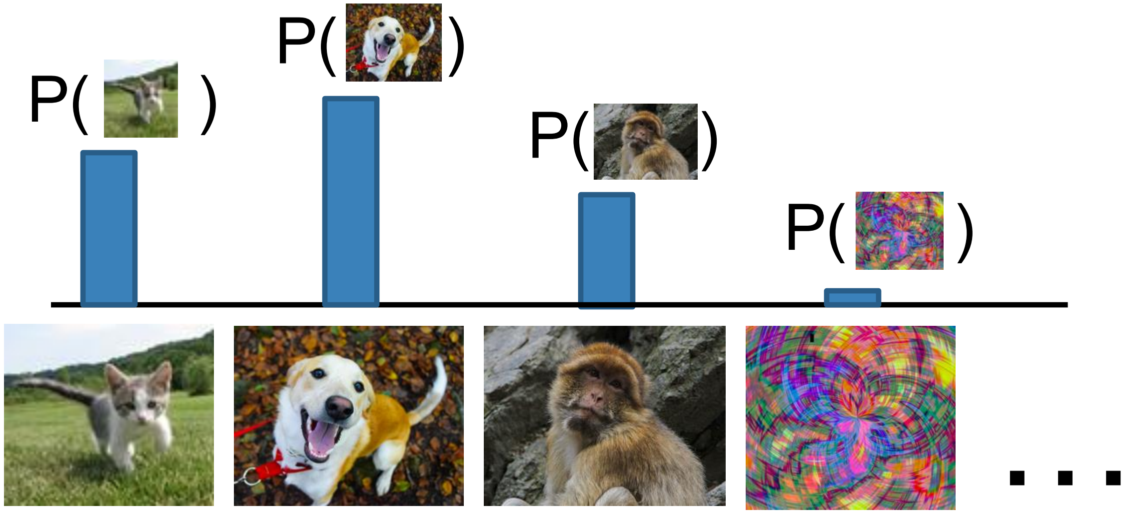


All possible **images** compete for probability mass

# Generative vs Discriminative Models

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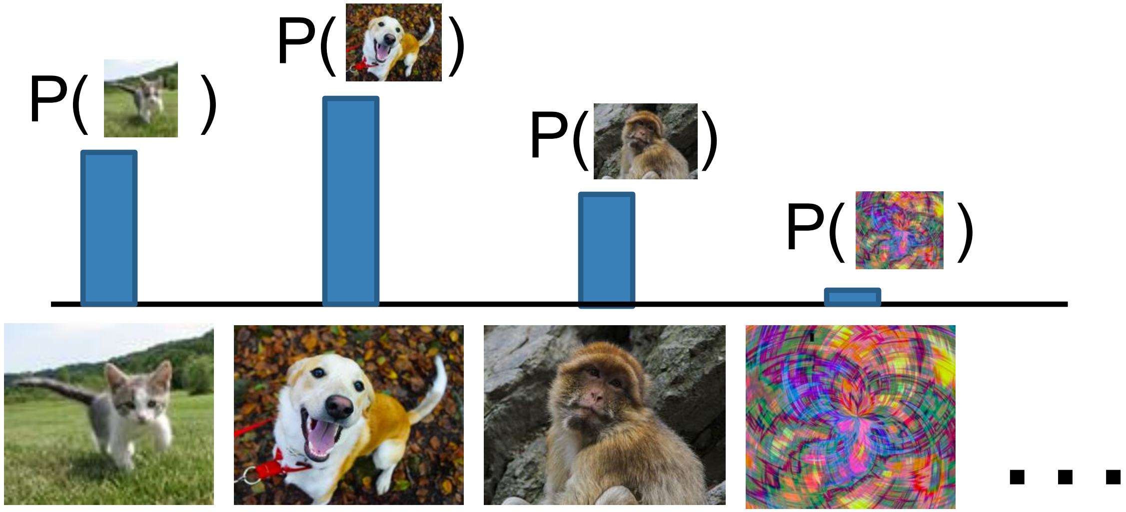


All possible **images** compete for probability mass

Requires deep understanding: Is a dog more likely to sit or stand? Is a 3-legged dog more likely than a 3-armed monkey?

# Generative vs Discriminative Models

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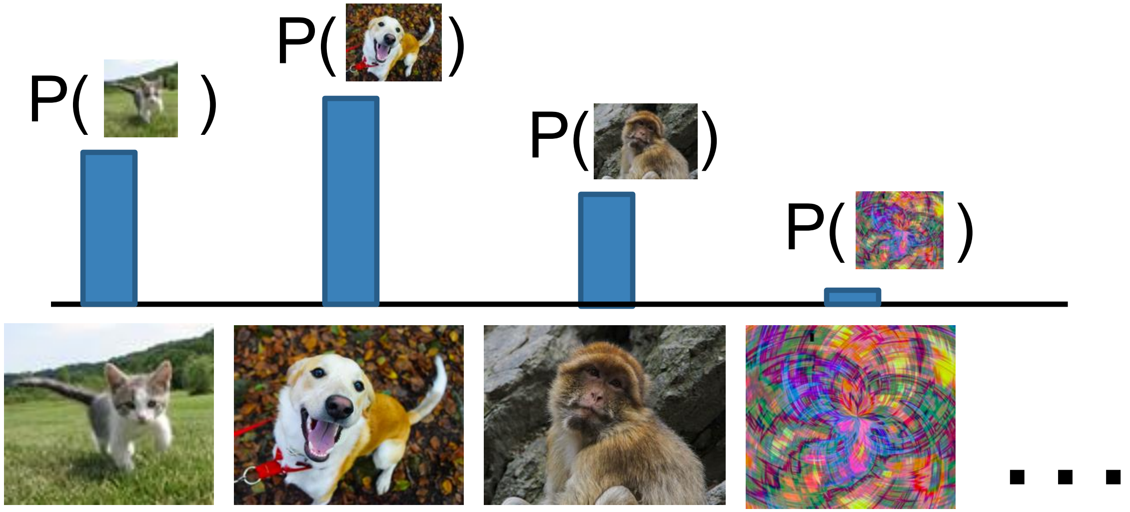


All possible **images** compete for probability mass

Model can "reject" unreasonable inputs by giving them small probability mass

# Generative vs Discriminative Models

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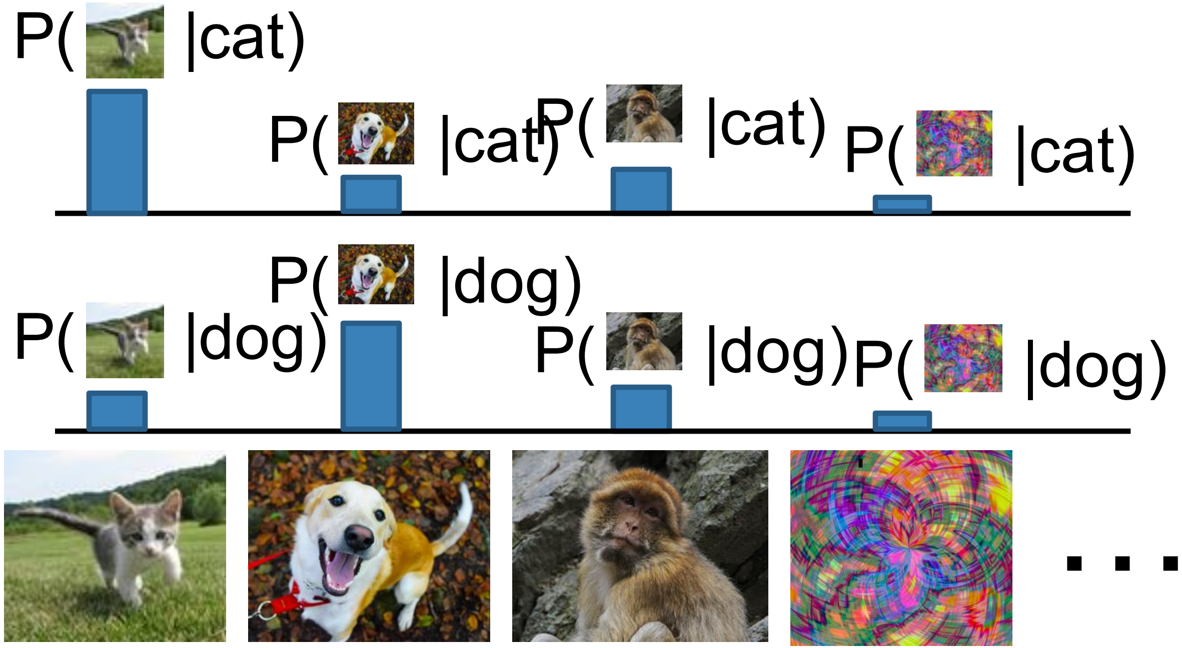


Each possible **label** induces a competition across all possible **images**

# Generative vs Discriminative Models

**Discriminative Model**: Learn a probability distribution p(y|x)

**Generative Model**: Learn a probability distribution p(x)

**Conditional Generative Model**: Learn p(x|y)

Recall **Bayes' Rule**:

$$P(x \mid y) = \frac{P(y \mid x)}{P(y)} P(x)$$

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$$P(x \mid y) = \frac{P(y \mid x)}{P(y)} P(x)$$

- $P(x \mid y)$: Conditional Generative Model
- $P(y \mid x)$: Discriminative Model
- $P(y)$: Prior over labels
- $P(x)$: (Unconditional) Generative Model

We can build a conditional generative model from other components … but not common in practice

# Generative vs Discriminative Models

**Discriminative Model**: Learn a probability distribution $p(y|x)$

→ Assign labels to data
Feature learning (with labels)

**Generative Model**: Learn a probability distribution $p(x)$

**Conditional Generative Model**: Learn $p(x|y)$

# Generative vs Discriminative Models

**Discriminative Model**: Learn a probability distribution p(y|x) → Assign labels to data; Feature learning (with labels)

**Generative Model**: Learn a probability distribution p(x) → Detect outliers; Feature learning (without labels); **Sample** to generate new data

**Conditional Generative Model**: Learn p(x|y)

# Generative vs Discriminative Models

**Discriminative Model**: Learn a probability distribution p(y|x) → Assign labels to data; Feature learning (with labels)

**Generative Model**: Learn a probability distribution p(x) → Detect outliers; Feature learning (without labels); **Sample** to generate new data

**Conditional Generative Model**: Learn p(x|y) → Assign labels while rejecting outliers; **Sample** to generate data from labels

# Generative vs Discriminative Models

**Discriminative Model**: Learn a probability distribution p(y|x)

**Generative Model**: Learn a probability distribution p(x)

**Conditional Generative Model**: Learn p(x|y)

"Generative models" means either of these; conditional generative models are most common in practice

# Why Generative Models?

**Modeling ambiguity:** If there are many possible outputs x for an input y, we want to model P(x | y)

**Language Modeling**: Produce output text x from input text y

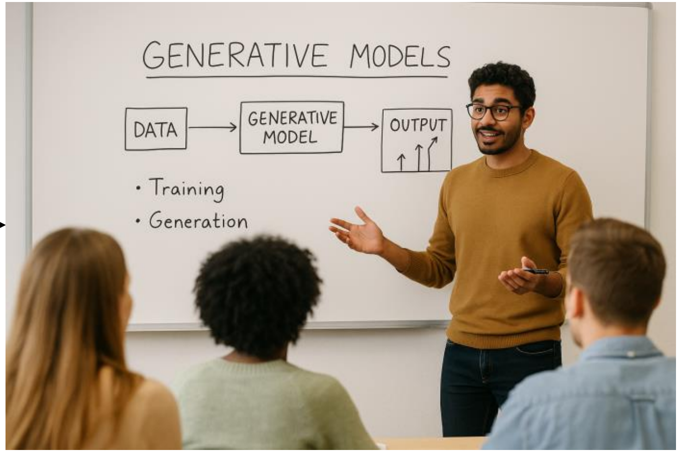
*Write me a short rhyming poem about generative models* → *They sample from a learned **P**, A distribution—structured, free. Each token comes conditionally, On all the ones that used to be.*

# Why Generative Models?

**Modeling ambiguity:** If there are many possible outputs x for an input y, we want to model P(x | y)

**Text to Image**: Produce output image x from input text y

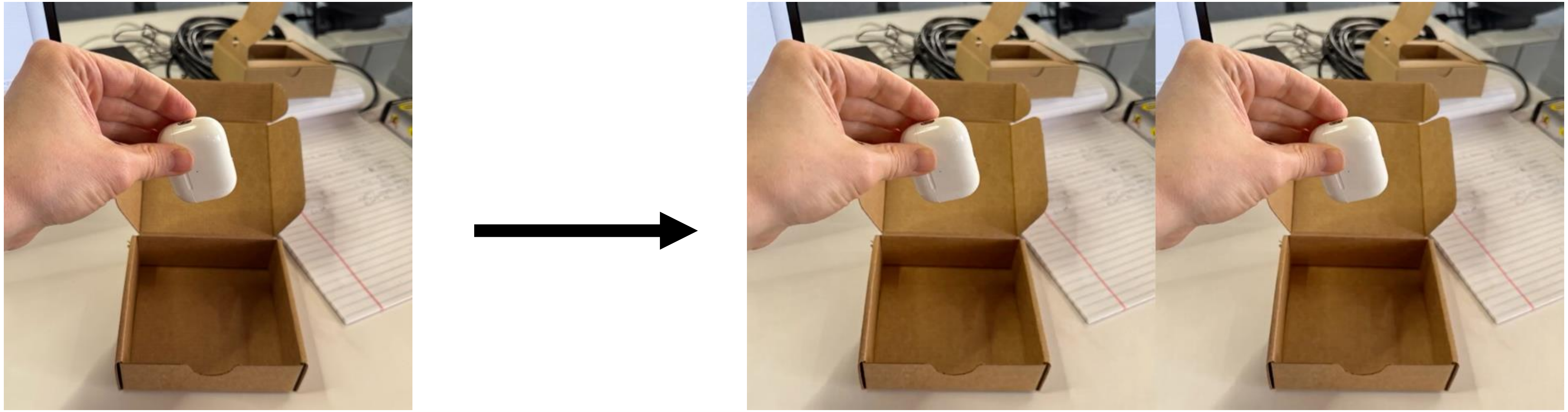
*Make me an image showing a person teaching a class on generative models in front of a whiteboard*



# Why Generative Models?

**Modeling ambiguity:** If there are many possible outputs x for an input y, we want to model P(x | y)

**Image to Video**: What happens next?



# Taxonomy of Generative Models

Figure adapted from Ian Goodfellow, Tutorial on Generative Adversarial Networks, 2017

Generative models

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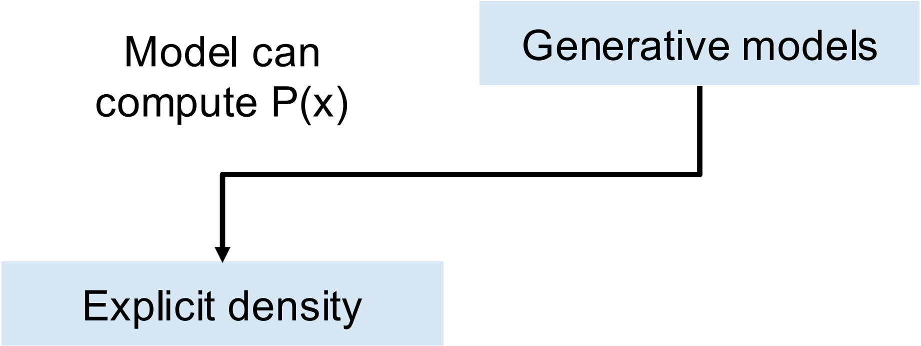

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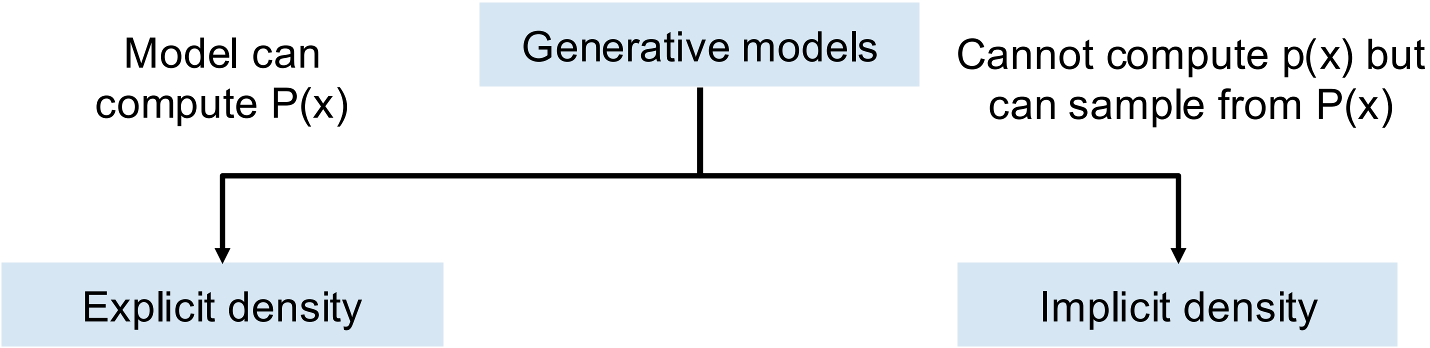

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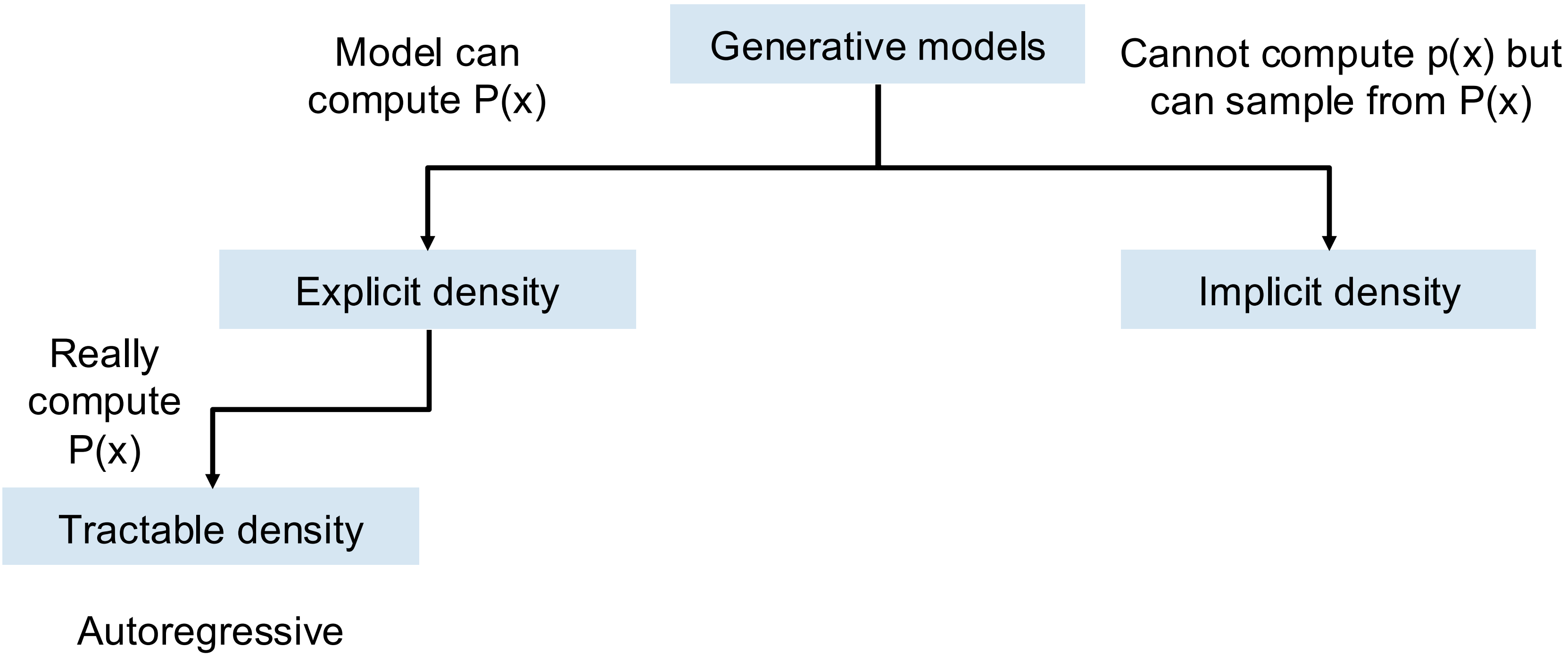

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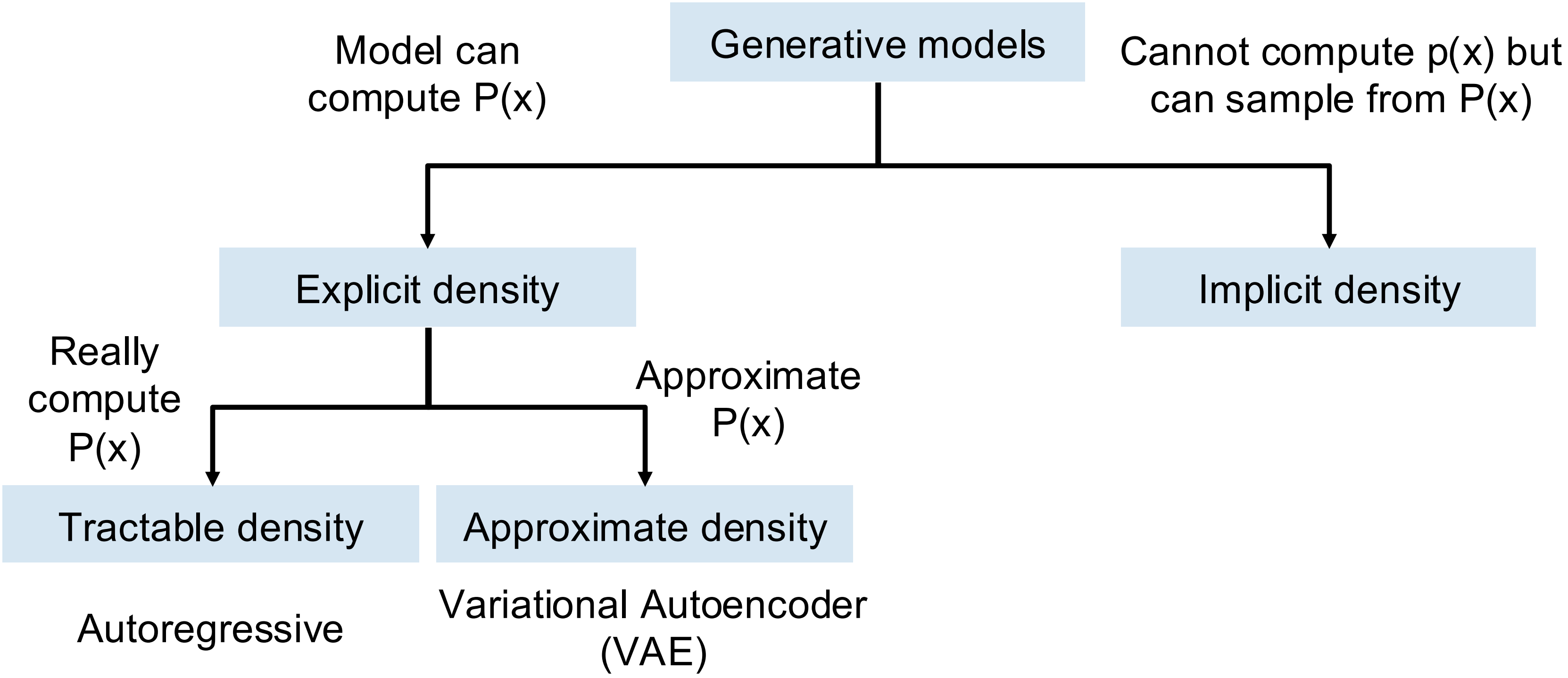

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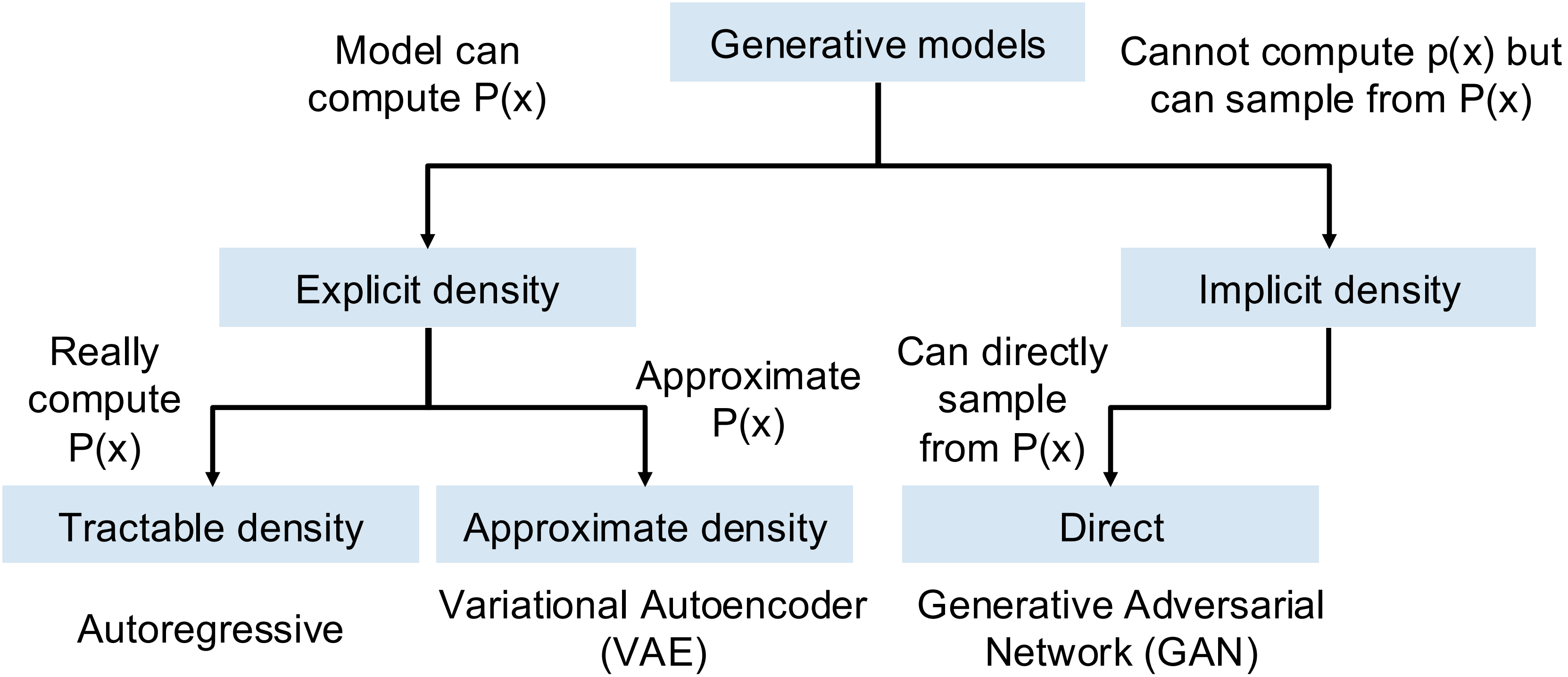

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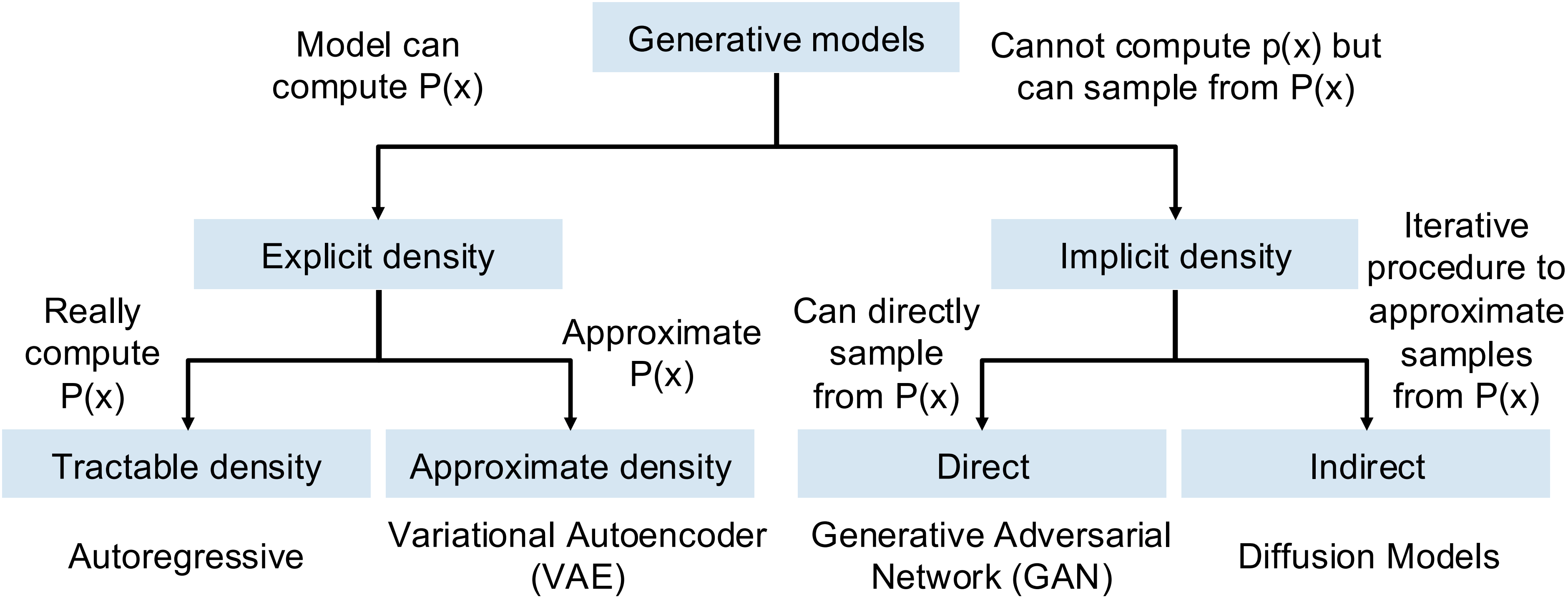

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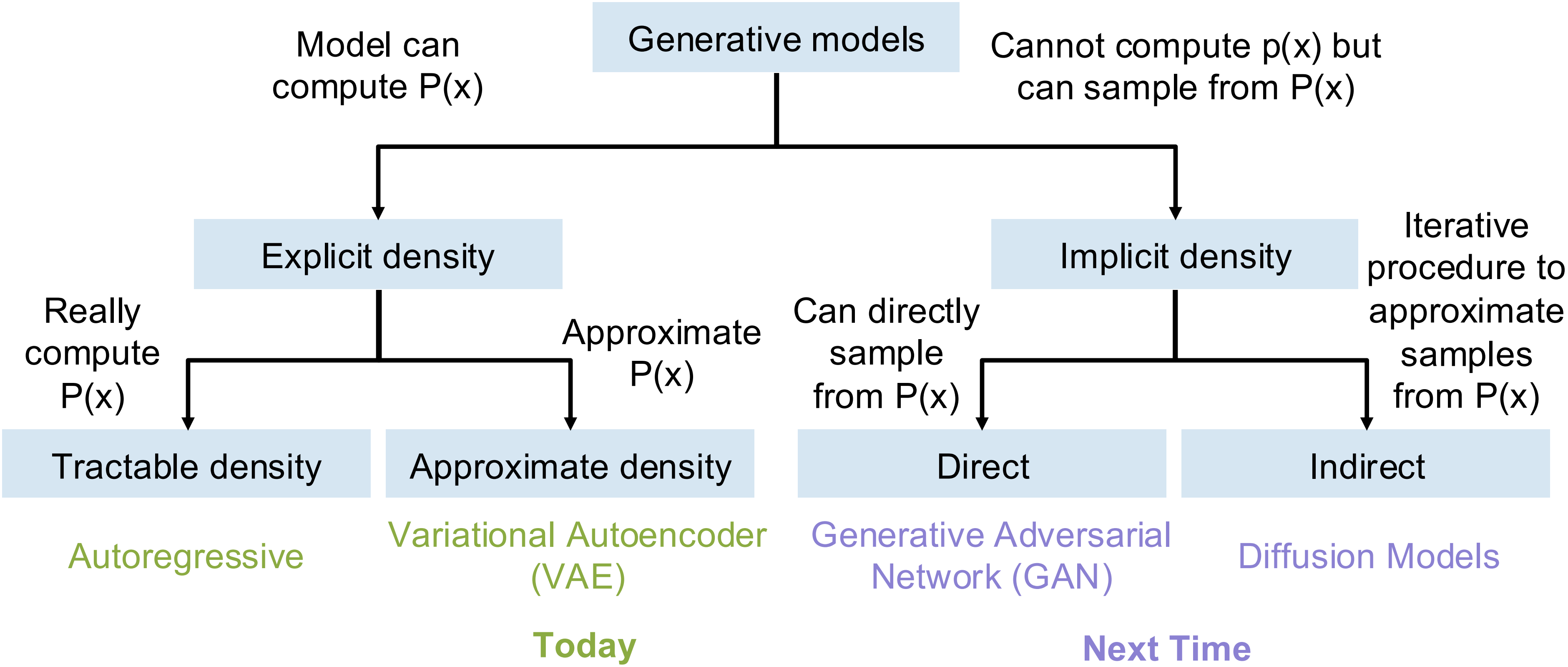

# Autoregressive Models

# Maximum Likelihood Estimation

**Goal**: Write down an explicit function for $p(x) = f(x, W)$

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Given dataset $x^{(1)}, x^{(2)}, \dots x^{(N)}$, train the model by solving:

$$W^* = \arg\max_W \prod_i p(x^{(i)})$$

Maximize probability of training data
(Maximum likelihood estimation)

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$$= \arg\max_W \sum_i \log p(x^{(i)})$$

Log trick: Swap product and sum

# Maximum Likelihood Estimation

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Maximize probability of training data
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$$= \arg\max_{W} \sum_i \log p(x^{(i)})$$

Log trick: Swap product and sum

$$= \arg\max_{W} \sum_i \log f(x^{(i)}, W)$$

This is our loss function.
maximize it with gradient descent

# Autoregressive Models

**Goal**: Write down an explicit function for $p(x) = f(x, W)$

Assume x is a sequence: $x = (x_1, x_2, \dots, x_T)$

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Assume x is a sequence: $x = (x_1, x_2, \ldots, x_T)$

Use the chain rule of probability:

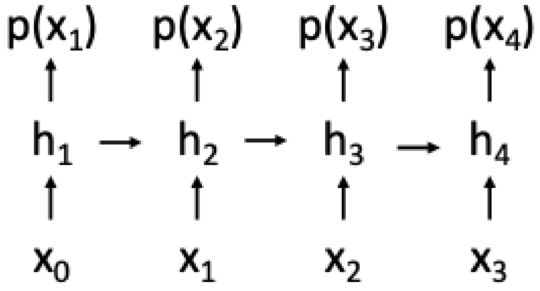
$$
\begin{aligned}
p(x) &= p(x_1, x_2, x_3, \ldots, x_T) \\
&= p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1, x_2) \ldots \\
&= \prod_{t=1}^{T} p(x_t \mid x_1, \ldots, x_{t-1})
\end{aligned}
$$

# Autoregressive Models

**Goal**: Write down an explicit function for $p(x) = f(x, W)$

Assume x is a sequence: $x = (x_1, x_2, \dots, x_T)$

We have already seen this!




Language modeling with RNN

Use the chain rule of probability:

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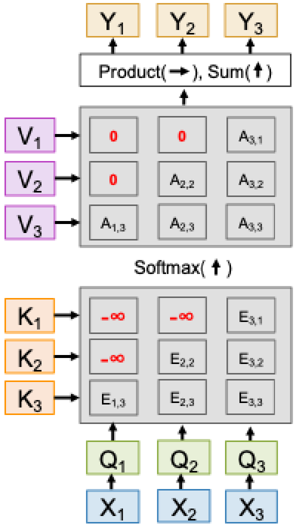
# LLMs are Autoregressive Models

**Goal**: Write down an explicit function for $p(x) = f(x, W)$

Assume x is a sequence: $x = (x_1, x_2, \dots, x_T)$

Language modeling with masked Transformer

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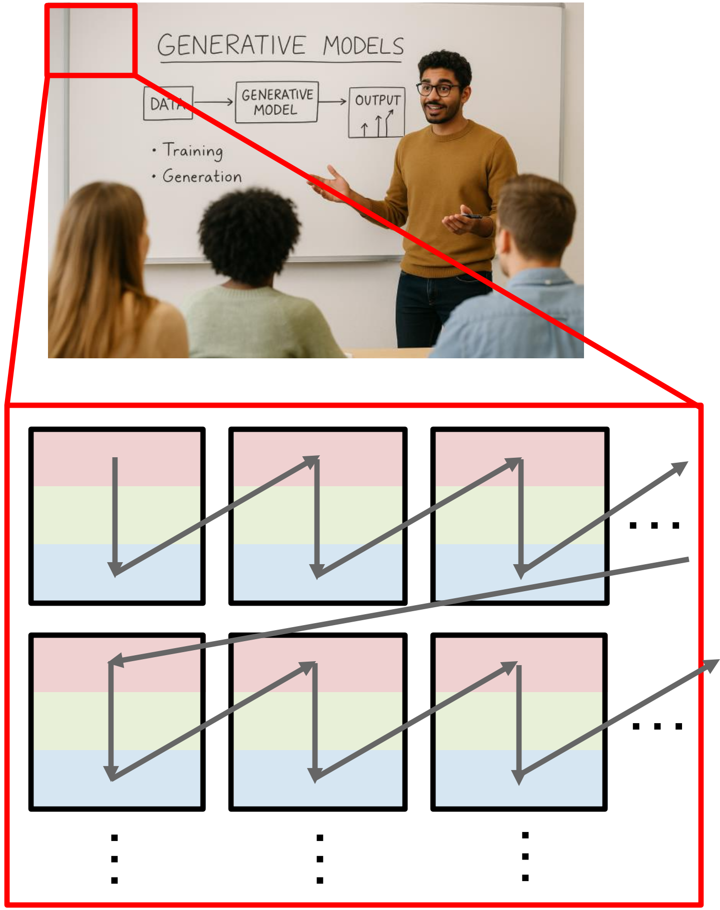
\end{aligned}
$$

# Autoregressive Models of Images

Treat an image as a sequence of 8-bit subpixel values (scanline order)

Predict each subpixel as a classification among 256 values [0…255]

Model with an RNN or Transformer




Van den Oord et al, "Pixel Recurrent Neural Networks", ICML 2016
Van den Oord et al, "Conditional Image Generation with PixelCNN Decoders", NeurIPS 2016

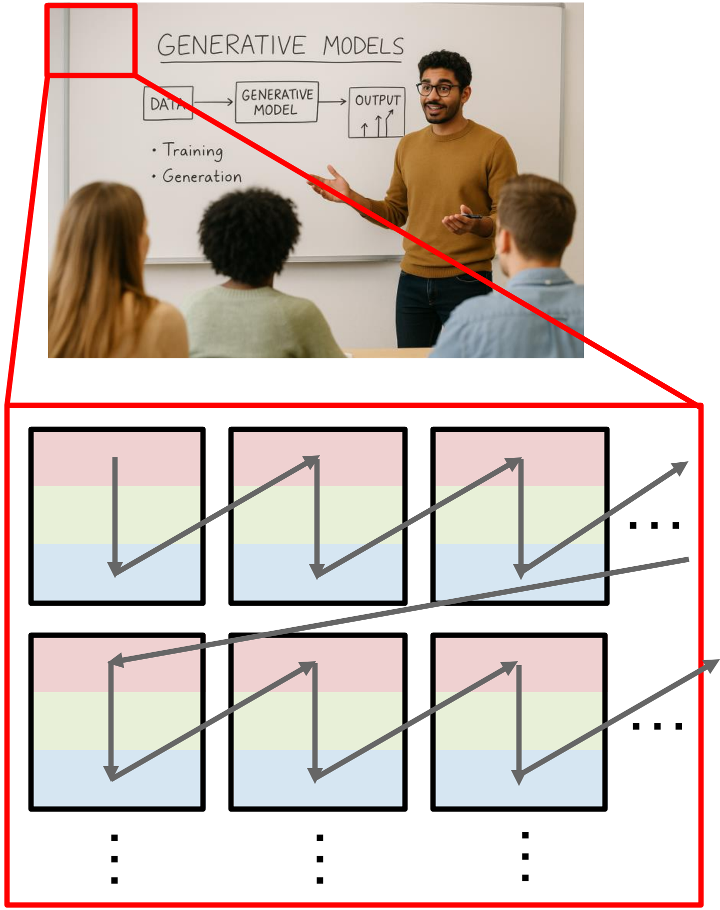
# Autoregressive Models of Images

Treat an image as a sequence of 8-bit subpixel values (scanline order)

Predict each subpixel as a classification among 256 values [0…255]

Model with an RNN or Transformer

**Problem**: Too expensive. 1024x1024 image is a sequence of 3M subpixels




Van den Oord et al, "Pixel Recurrent Neural Networks", ICML 2016
Van den Oord et al, "Conditional Image Generation with PixelCNN Decoders", NeurIPS 2016

# Autoregressive Models of Images

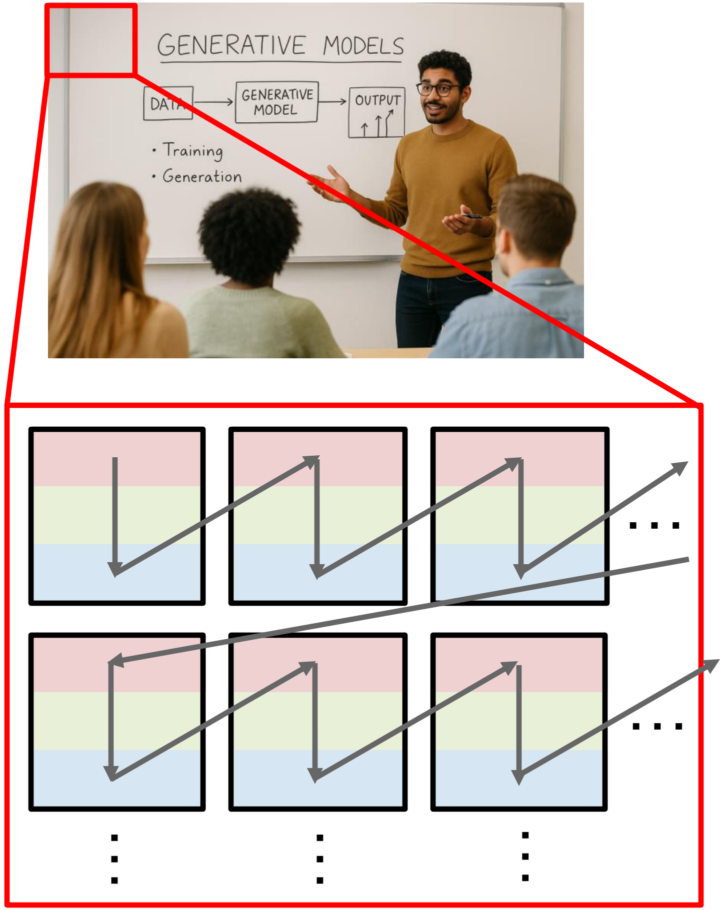
Treat an image as a sequence of 8-bit subpixel values (scanline order)

Predict each subpixel as a classification among 256 values [0…255]

Model with an RNN or Transformer

**Problem**: Too expensive. 1024x1024 image is a sequence of 3M subpixels

**Solution** (jumping ahead): Model as sequence of tiles, not sequence of subpixels



# Variational Autoencoders (VAEs)

# Variational Autoencoders

PixelRNN / PixelCNN explicitly parameterizes density function with a neural network, so we can train to maximize likelihood of training data:

$$\mathrm{p}_W(x) = \prod_{t=1}^{T} p_W(x_t \mid x_1, \ldots, x_{t-1})$$

Variational Autoencoders (VAE) define an **intractable density** that we cannot explicitly compute or optimize

But we will be able to directly optimize a **lower bound** on the density

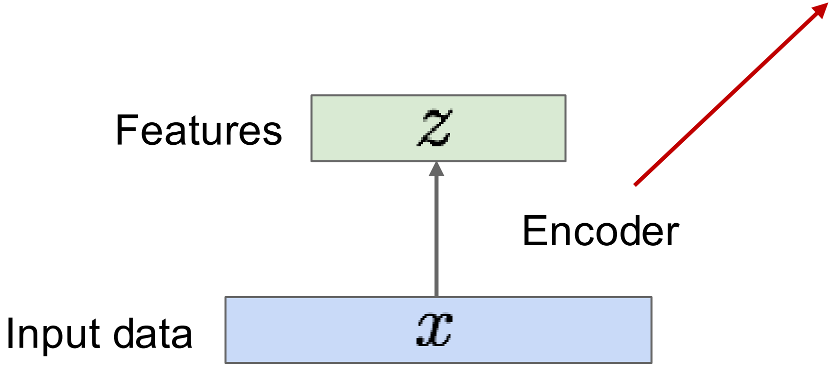
# Variational Autoencoders (VAEs)

# (Non-Variational) Autoencoders

**Idea**: Unsupervised method for learning to extract features z from inputs x, without labels

Features should extract useful information
(object identity, appearance, scene type, etc)
that can be used for downstream tasks

Encoder can be MLP,
CNN, Transformer, …

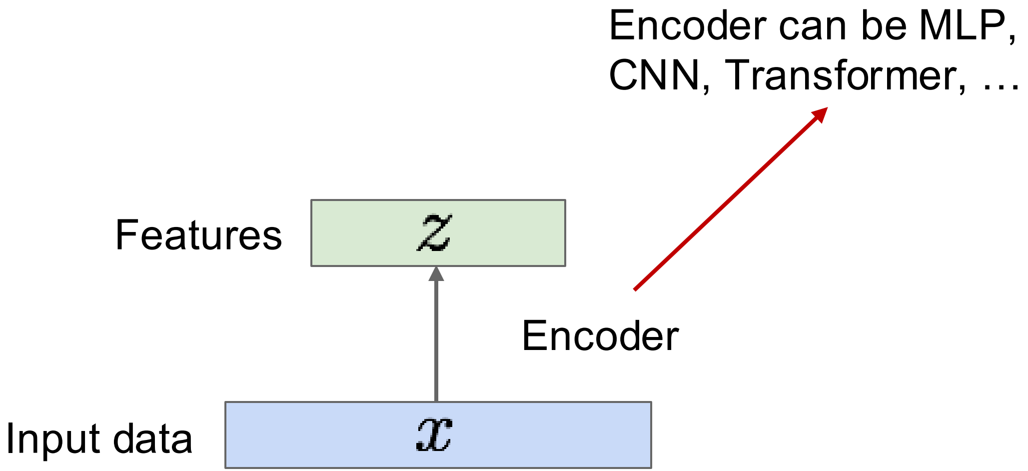



Input Data

# (Non-Variational) Autoencoders

**Problem**: How can we learn without labels?

Features should extract useful information (object identity, appearance, scene type, etc) that can be used for downstream tasks




Input Data

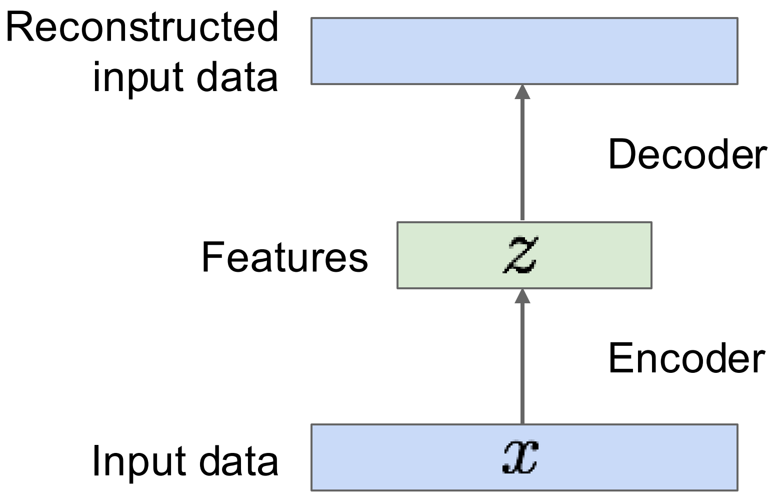
# (Non-Variational) Autoencoders

"Autoencoding" =
Encoding yourself

**Problem**: How can we learn without labels?

**Solution**: Reconstruct the input data with a decoder.

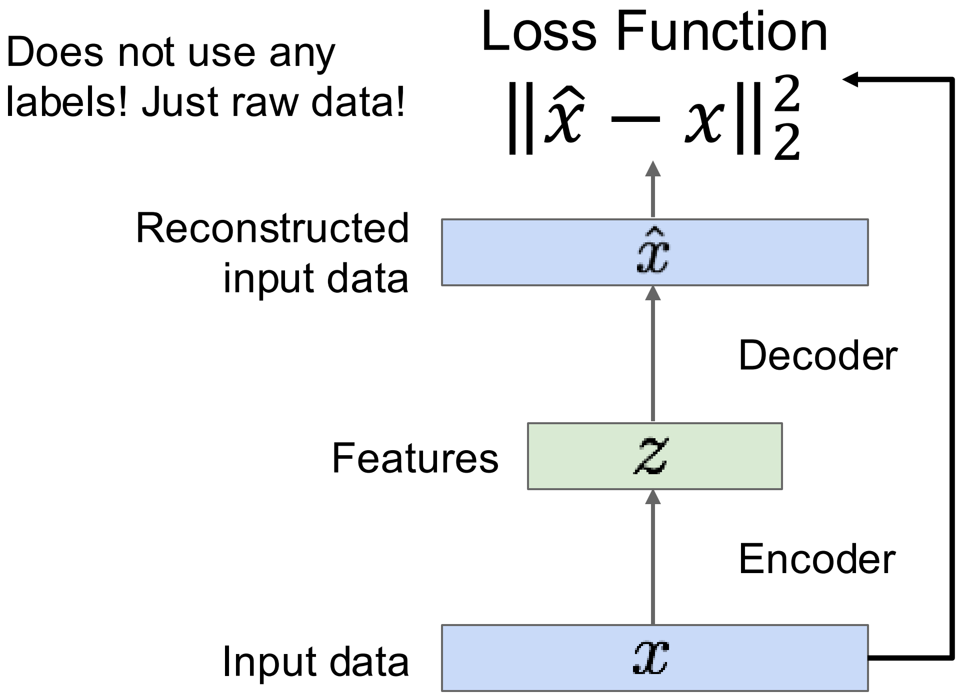
Decoder can be MLP, CNN, Transformer, …

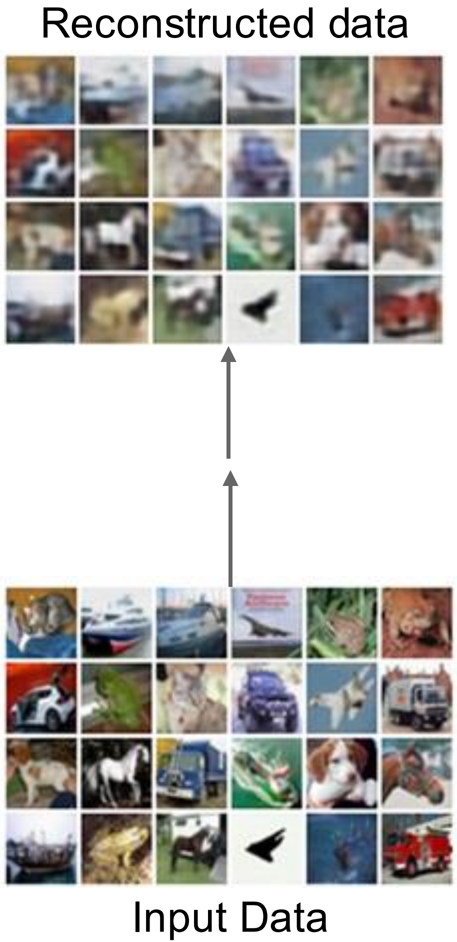



Input Data

# (Non-Variational) Autoencoders

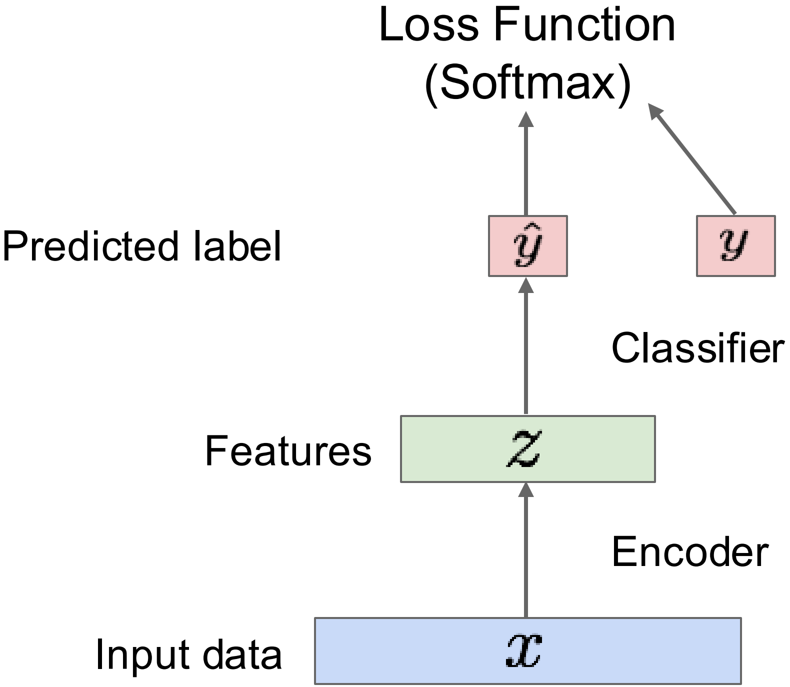
**Loss**: L2 distance between input and reconstructed data.

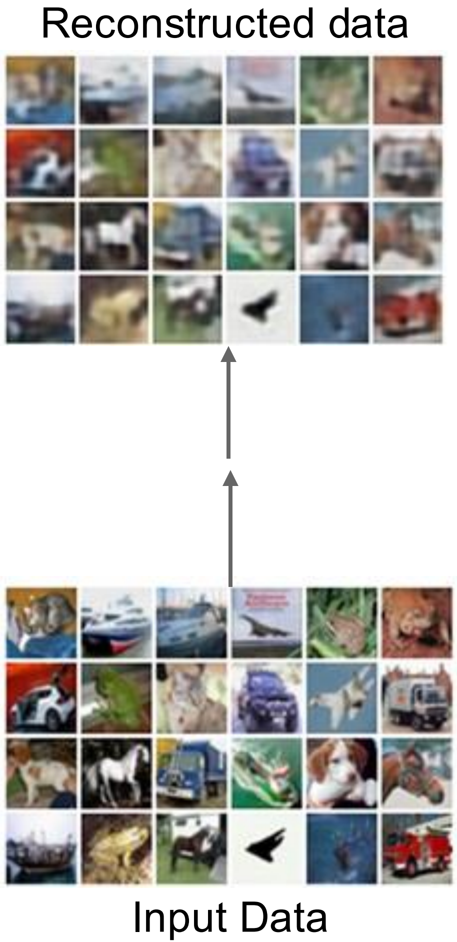


# (Non-Variational) Autoencoders

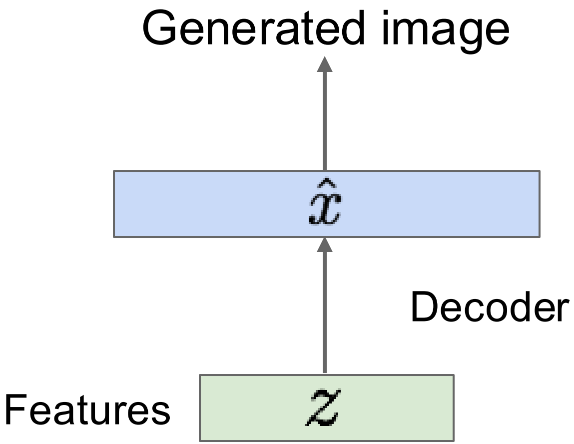
After training, can use encoder for downstream tasks



# (Non-Variational) Autoencoders

If we could generate new z, could use the decoder to generate images



# (Non-Variational) Autoencoders

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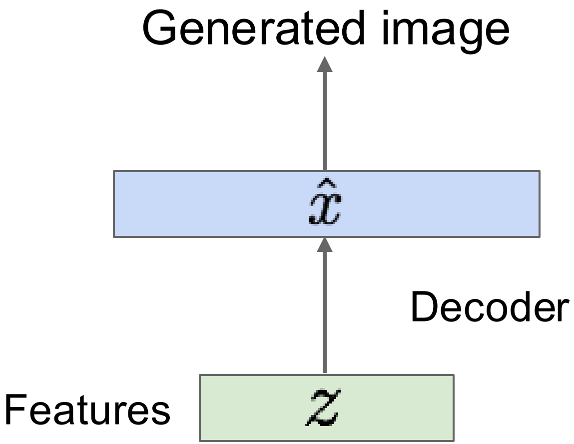
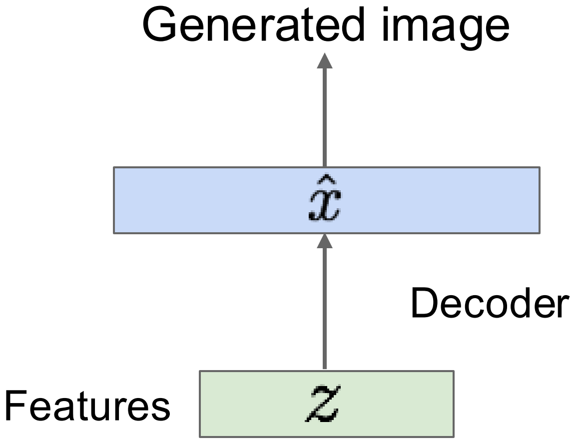



**Problem**: Generating new z is not any easier than generating new x

# (Non-Variational) Autoencoders

If we could generate new z, could use the decoder to generate images




**Problem**: Generating new z is not any easier than generating new x

**Solution**: What if we force all z to come from a known distribution?

# Variational Autoencoders (VAEs)

Kingma and Welling, Auto-Encoding Variational Beyes, ICLR 2014

# Variational Autoencoders

Probabilistic spin on autoencoders:

1. Learn latent features z from raw data
2. Sample from the model to generate new data

# Variational Autoencoders

Probabilistic spin on autoencoders:

1. Learn latent features z from raw data
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Assume training data $\{x^{(i)}\}_{i=1}^{N}$ is generated from unobserved (latent) representation **z**

**Intuition: x** is an image, **z** is latent factors used to generate **x:** attributes, orientation, etc.

# Variational Autoencoders

Probabilistic spin on autoencoders:

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Assume training data $\{x^{(i)}\}_{i=1}^{N}$ is generated from unobserved (latent) representation **z**

**Intuition: x** is an image, **z** is latent factors used to generate **x:** attributes, orientation, etc.

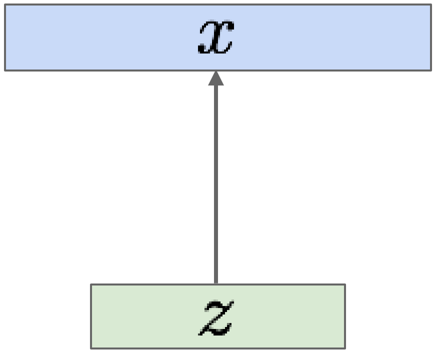
After training, sample new data like this:

Sample from conditional

$p_{\theta^*}(x \mid z^{(i)})$

Sample z from prior

$p_{\theta^*}(z)$



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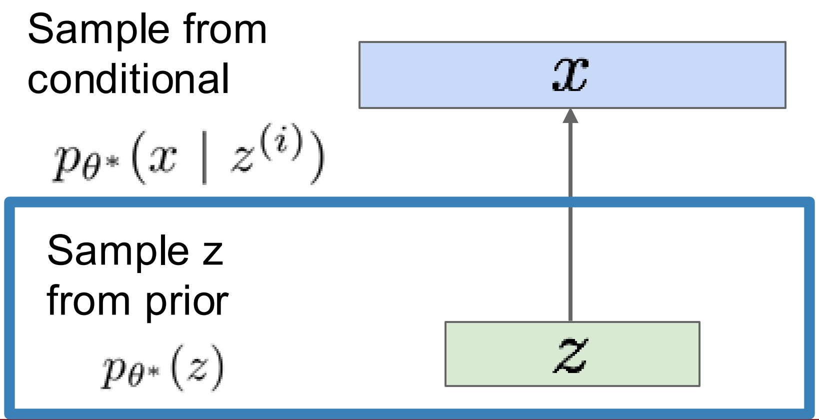


Assume training data $\{x^{(i)}\}_{i=1}^{N}$ is generated from unobserved (latent) representation **z**

**Intuition: x** is an image, **z** is latent factors used to generate **x:** attributes, orientation, etc.

Assume simple prior p(z), e.g. Gaussian

# Variational Autoencoders

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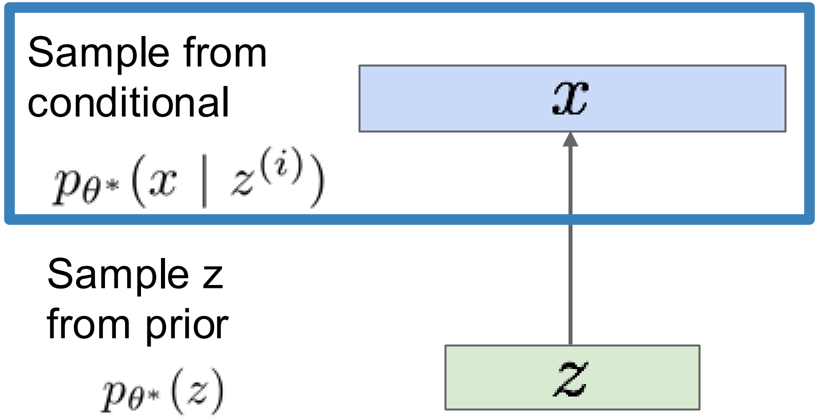


Assume training data $\{x^{(i)}\}_{i=1}^{N}$ is generated from unobserved (latent) representation **z**

How can we train this?

Basic idea: **maximum likelihood**

If we had a dataset of (x, z) then train a *conditional generative model* p(x | z)

# Variational Autoencoders

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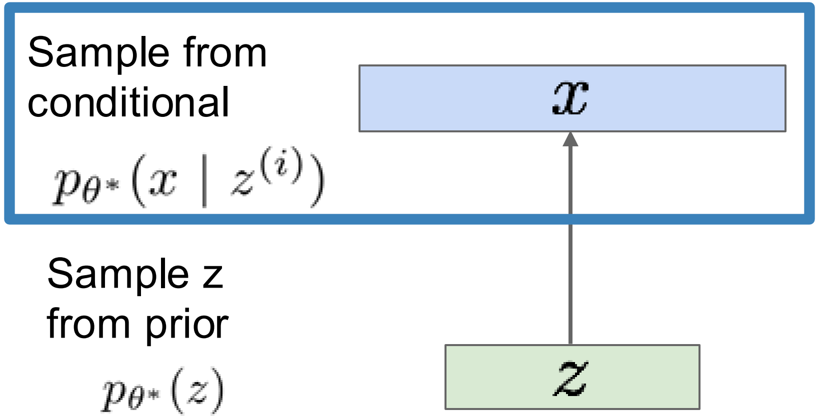


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How can we train this?

Basic idea: **maximum likelihood**

We don't observe z, so **marginalize**:

$$p_\theta(x) = \int p_\theta(x, z)dz = \int p_\theta(x|z)p_\theta(z)dz$$

# Variational Autoencoders

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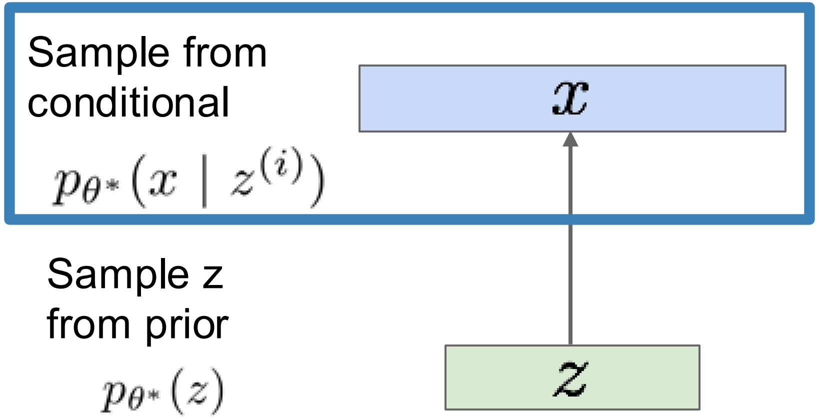


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Ok, we can compute this with the decoder

# Variational Autoencoders

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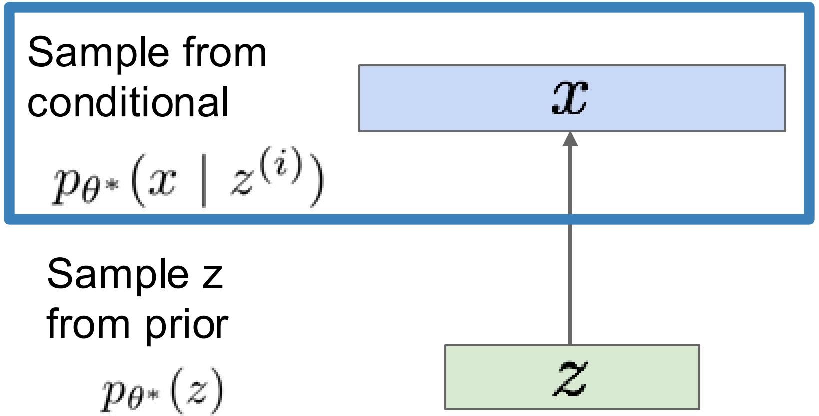


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Ok, we assumed Gaussian prior for z

# Variational Autoencoders

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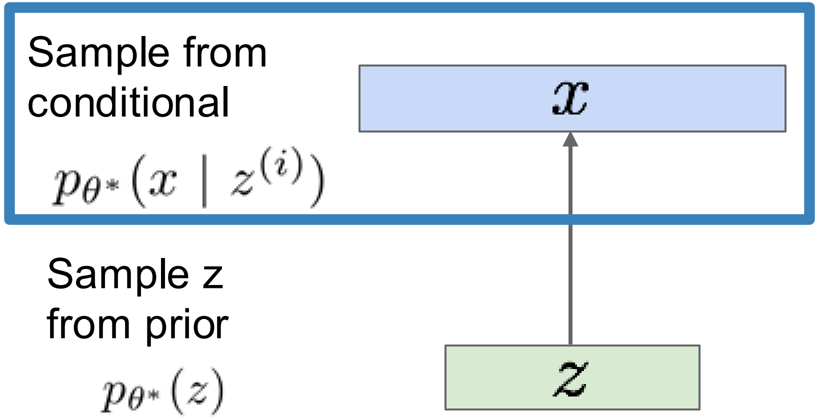


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Problem, we can't integrate over all z

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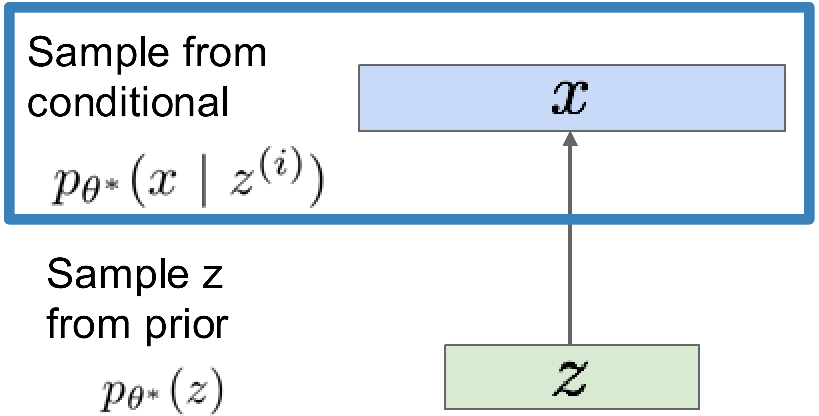


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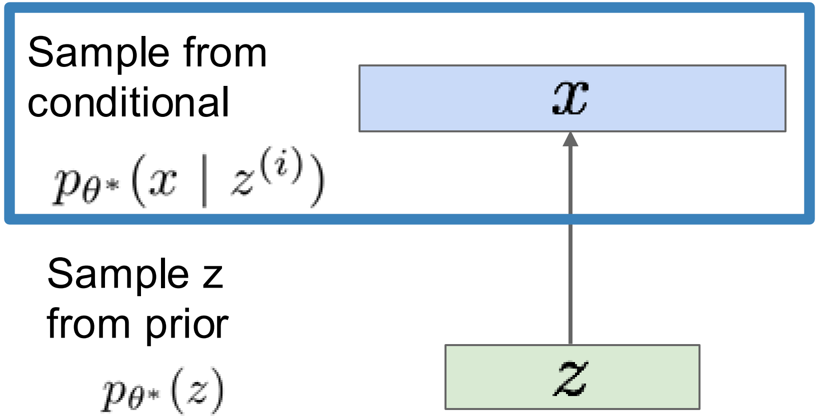


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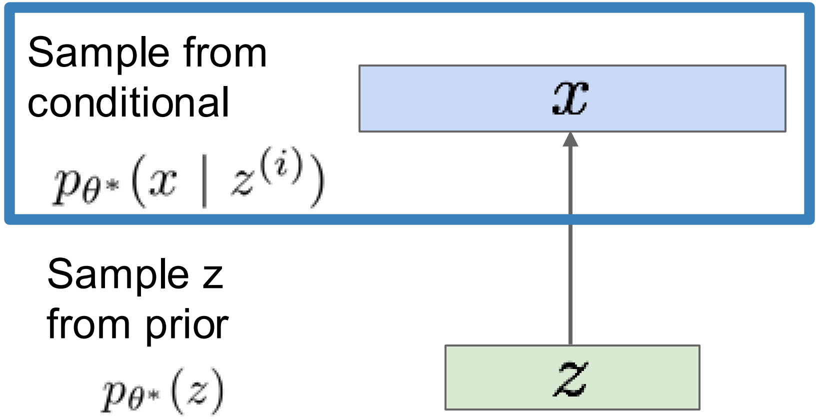


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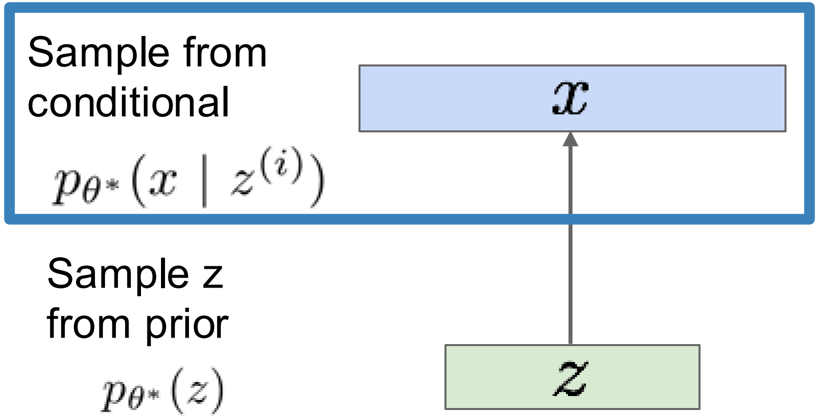


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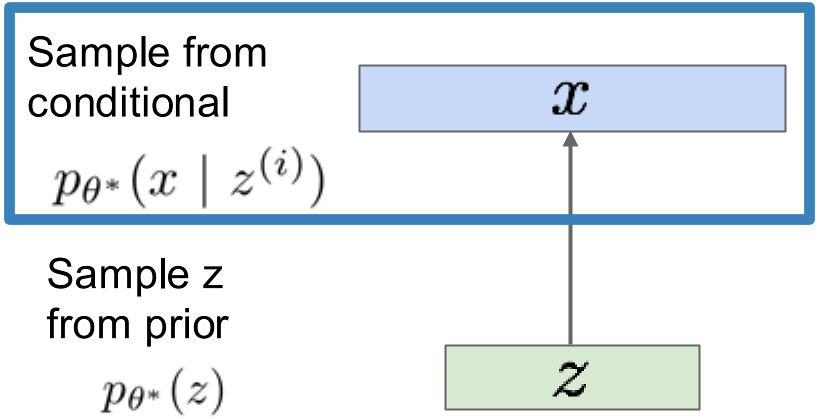


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**Problem**: no way to compute this

**Solution**: Train another network that learns $q_\phi(z \mid x) \approx p_\theta(z \mid x)$

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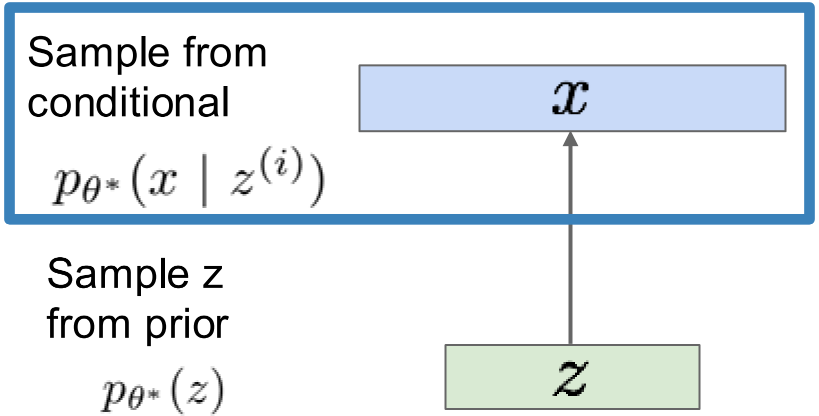


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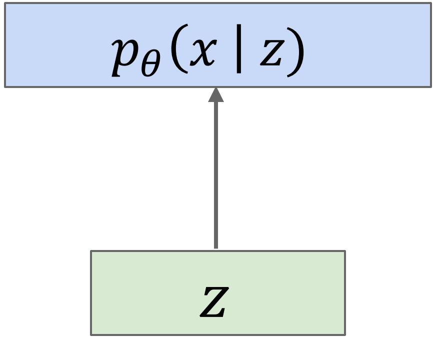
Another idea: Try Bayes' Rule:

$$p_\theta(x) = \frac{p_\theta(x \mid z)p_\theta(z)}{p_\theta(z \mid x)} \approx \frac{p_\theta(x \mid z)p_\theta(z)}{q_\phi(z \mid x)}$$

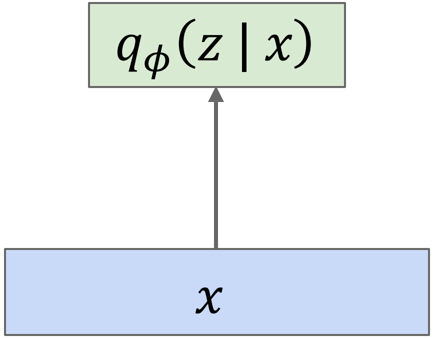
**Solution**: Train another network that learns $q_\phi(z \mid x) \approx p_\theta(z \mid x)$

# Variational Autoencoders

**Decoder Network**:
Input latent code z,
Output distribution over data x




**Encoder Network**:
Input data x,
Output distribution over latent codes z




If we can ensure that
$q_\phi(z \mid x) \approx p_\theta(z \mid x)$,

then we can approximate

$$p_\theta(x) \approx \frac{p_\theta(x \mid z)p(z)}{q_\phi(z \mid x)}$$

**Idea**: Jointly train both encoder and decoder

# Variational Autoencoders

**Decoder Network**:
Input latent code z,
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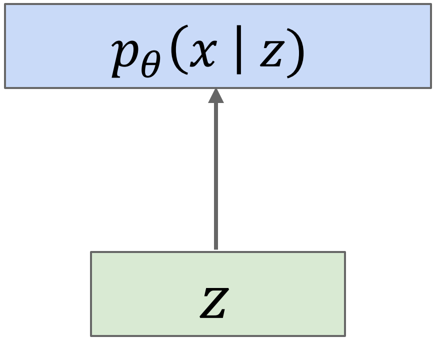
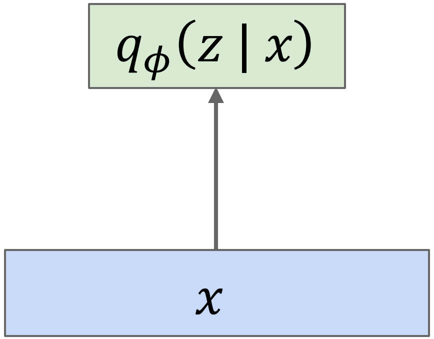



**Encoder Network**:
Input data x,
Output distribution over latent codes z

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**Idea**: Jointly train both encoder and decoder

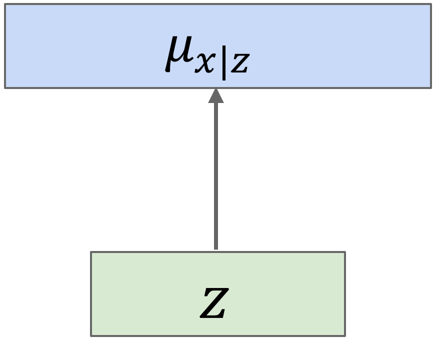
**Aside**: How to output probability distributions from neural networks?

Network outputs mean (and std) of a (diagonal) distribution

# Variational Autoencoders

**Decoder Network**:
Input latent code z,
Output distribution over data x

$$p_\theta(x \mid z) = N(\mu_{x|z}, \sigma^2)$$

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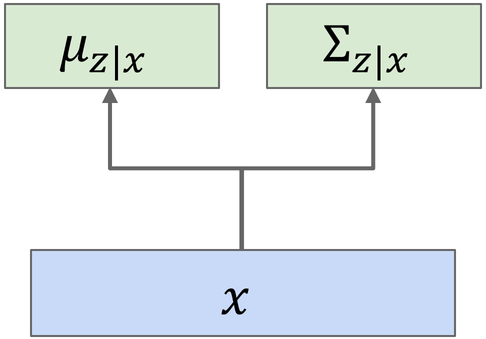


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# Variational Autoencoders

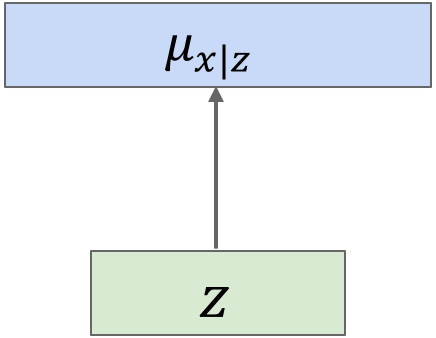
**Decoder Network**:
Input latent code z,
Output distribution over data x

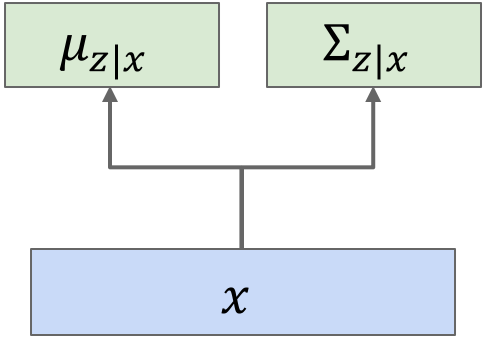
**Encoder Network**:
Input data x,
Output distribution over latent codes z

$$p_\theta(x \mid z) = N(\mu_{x|z}, \sigma^2)$$

$$q_\phi(z \mid x) = N(\mu_{z|x}, \Sigma_{z|x})$$

$$\log p_\theta(x \mid z) = -\frac{1}{2\sigma^2}\|x - \mu\|_2^2 + C_2$$

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$$p_\theta(x) \approx \frac{p_\theta(x \mid z)p(z)}{q_\phi(z \mid x)}$$

**Idea**: Jointly train both encoder and decoder

Maximizing $\log p_\theta(x \mid z)$ is equivalent to minimizing L2 distance between x and network output!

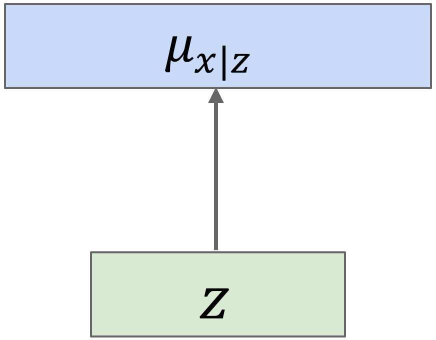
# Variational Autoencoders

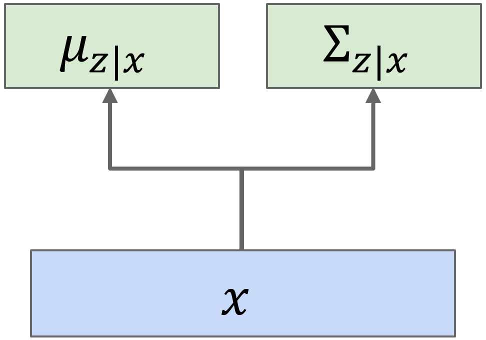
**Decoder Network**:
Input latent code z,
Output distribution over data x

**Encoder Network**:
Input data x,
Output distribution over latent codes z

$$p_\theta(x \mid z) = N(\mu_{x|z}, \sigma^2) \qquad q_\phi(z \mid x) = N(\mu_{z|x}, \Sigma_{z|x})$$

$$\log p_\theta(x \mid z) = -\frac{1}{2\sigma^2}\|x - \mu\|_2^2 + C_2$$

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**Idea**: Jointly train both encoder and decoder

**Q**: What's our training objective?

# Variational Autoencoders (ELBO)

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)}$$

Bayes' Rule

# Variational Autoencoders (ELBO)

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

Multiply top and bottom by $q_\Phi(z|x)$

# Variational Autoencoders (ELBO)

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

$$= \log p_\theta(x|z) - \log \frac{q_\phi(z|x)}{p(z)} + \log \frac{q_\phi(z|x)}{p_\theta(z|x)}$$

Logarithms + rearranging

# Variational Autoencoders (ELBO)

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

$$= \log p_\theta(x|z) - \log \frac{q_\phi(z|x)}{p(z)} + \log \frac{q_\phi(z|x)}{p_\theta(z|x)}$$

$$\log p_\theta(x) = E_{z\sim q_\phi(z|x)}[\log p_\theta(x)]$$

We can wrap in an expectation since it doesn't depend on z

# Variational Autoencoders (ELBO)

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

$$= E_z[\log p_\theta(x|z)] - E_z\left[\log \frac{q_\phi(z|x)}{p(z)}\right] + E_z\left[\log \frac{q_\phi(z|x)}{p_\theta(z|x)}\right]$$

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**Data reconstruction**: x => encoder => decoder should reconstruct x

Can compute in closed form for Gaussians.

# Variational Autoencoders (ELBO)

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

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$$= E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right) + D_{KL}(q_\phi(z|x), p_\theta(z|x))$$

**Prior**: Encoder output should match prior over z.
Can compute in closed for for Gaussians.

# Variational Autoencoders (ELBO)

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

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**Posterior Approximation**: Encoder output $q_\phi(z|x)$ should match $p_\theta(z|x)$

We cannot compute this for Gaussians…

# Variational Autoencoders (ELBO)

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

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**Posterior Approximation**: Decoder output $q_\phi(z|x)$ should match $p_\theta(z|x)$

KL is >= 0, so we can drop it to get a lower bound on likelihood

# Variational Autoencoders (ELBO)

$$\log p_\theta(x) = \log \frac{p_\theta(x \mid z)p(z)}{p_\theta(z \mid x)} = \log \frac{p_\theta(x|z)p(z)q_\phi(z|x)}{p_\theta(z|x)q_\phi(z|x)}$$

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$$\log p_\theta(x) \geq E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

This is our VAE training objective

# Variational Autoencoders

Jointly train **encoder** q and **decoder** p to maximize the **variational lower bound** on the data likelihood
Also called **Evidence Lower Bound** (**ELBo**)

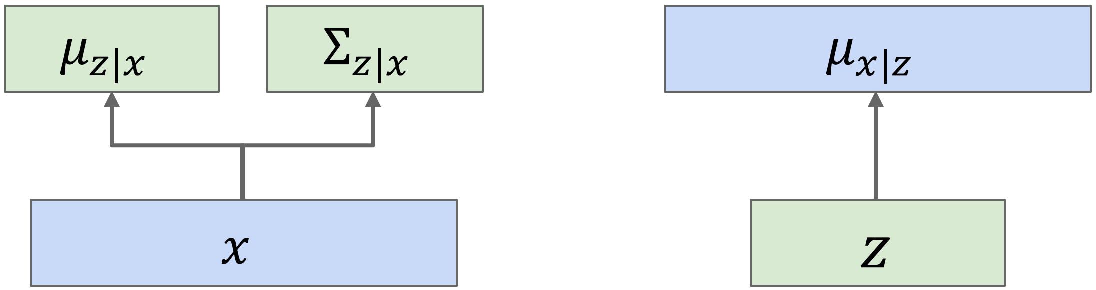
$$\log p_\theta(x) \geq E_{z\sim q_\phi(z|x)}[\log p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

**Encoder Network**

$$q_\phi(z \mid x) = N(\mu_{z|x}, \Sigma_{z|x})$$

**Decoder Network**

$$p_\theta(x \mid z) = N(\mu_{x|z}, \sigma^2)$$



# Variational Autoencoders: Training

Train by maximizing the
**variational lower bound**

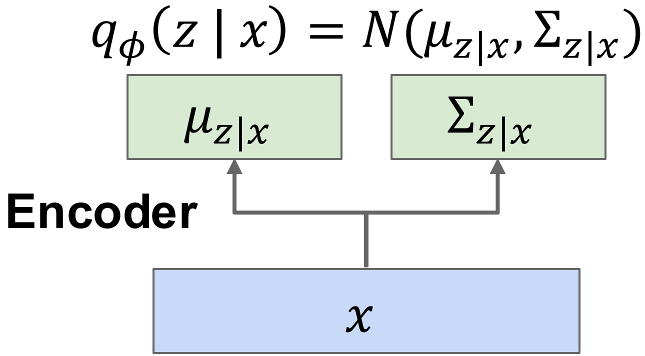
$$E_{z\sim q_\phi(z|x)}[log\, p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

# Variational Autoencoders: Training

Train by maximizing the
**variational lower bound**

$$E_{z\sim q_\phi(z|x)}[log\, p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

1. Run input data through **encoder** to get distribution over z



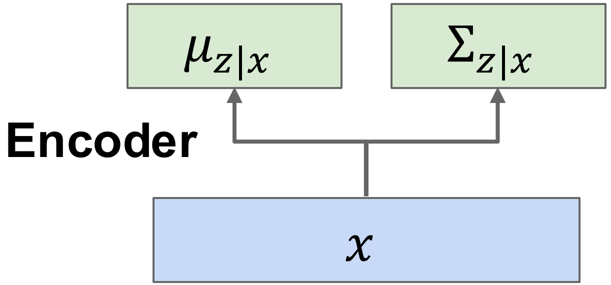
# Variational Autoencoders: Training

Train by maximizing the
**variational lower bound**

$$E_{z \sim q_\phi(z|x)}[log\, p_\theta(x|z)] - \boxed{D_{KL}\left(q_\phi(z|x), p(z)\right)}$$

1. Run input data through **encoder** to get distribution over z
2. Prior loss: Encoder output should be unit Gaussian (zero mean, unit variance)

$$q_\phi(z \mid x) = N(\mu_{z|x}, \Sigma_{z|x})$$



# Variational Autoencoders: Training

Train by maximizing the
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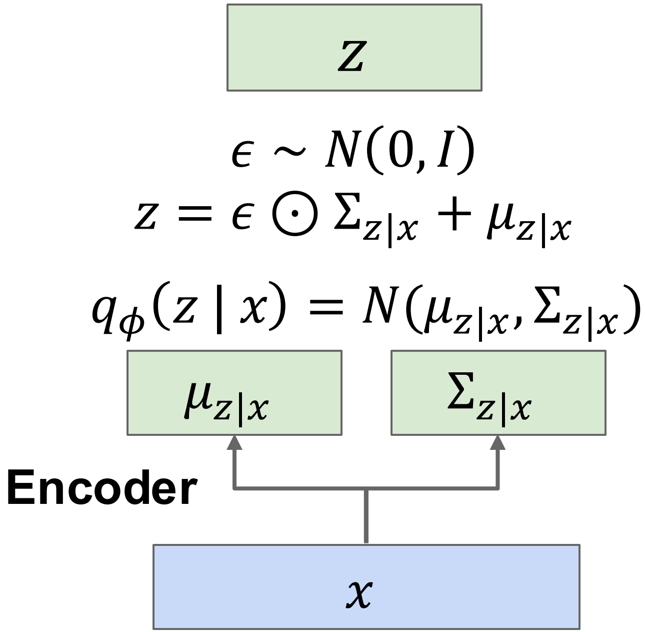
$$E_{z \sim q_\phi(z|x)}[log\, p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

1. Run input data through **encoder** to get distribution over z
2. Prior loss: Encoder output should be unit Gaussian (zero mean, unit variance)
3. Sample z from encoder output $q_\phi(z \mid x)$ (Reparameterization trick)



# Variational Autoencoders: Training

Train by maximizing the
**variational lower bound**

$$E_{z \sim q_\phi(z|x)}[log\, p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

1. Run input data through **encoder** to get distribution over z
2. Prior loss: Encoder output should be unit Gaussian (zero mean, unit variance)
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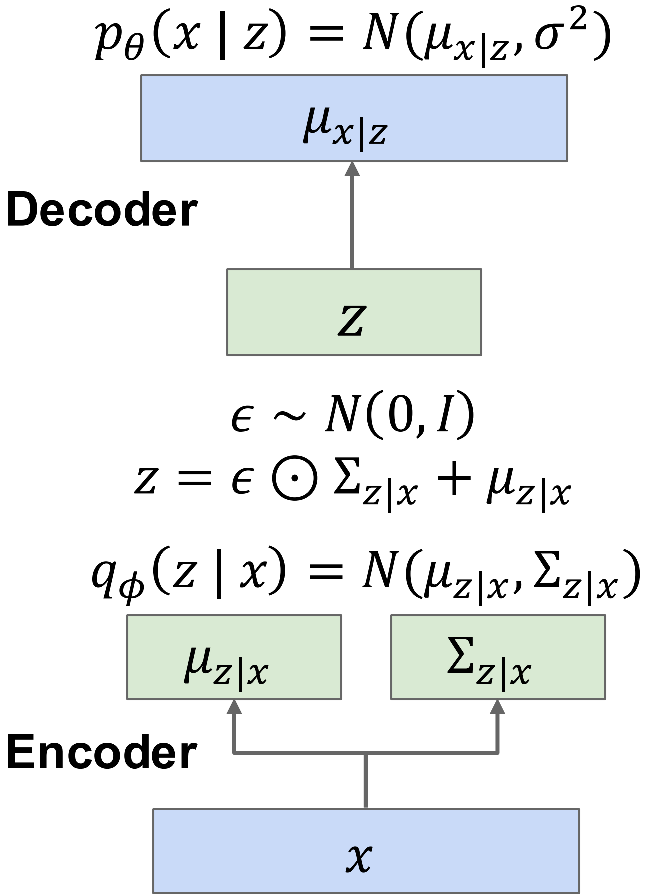
4. Run z through **decoder** to get predicted data mean

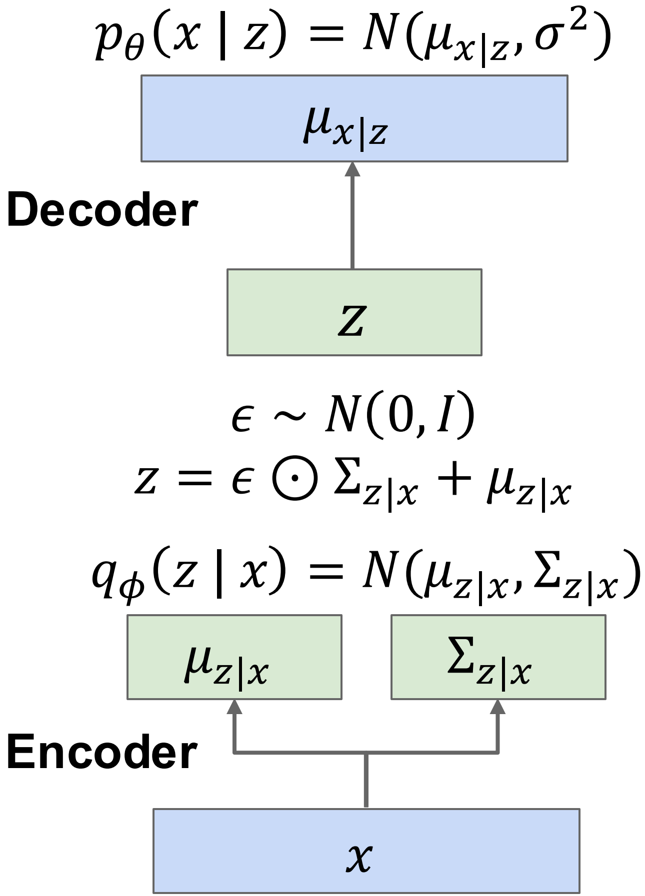


# Variational Autoencoders: Training

Train by maximizing the
**variational lower bound**

$$\boxed{E_{z\sim q_\phi(z|x)}[log\, p_\theta(x|z)]} - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

1. Run input data through **encoder** to get distribution over z
2. Prior loss: Encoder output should be unit Gaussian (zero mean, unit variance)
3. Sample z from encoder output $q_\phi(z \mid x)$ (Reparameterization trick)
4. Run z through **decoder** to get predicted data mean
5. Reconstruction loss: predicted mean should match x in L2



# Variational Autoencoders: Training

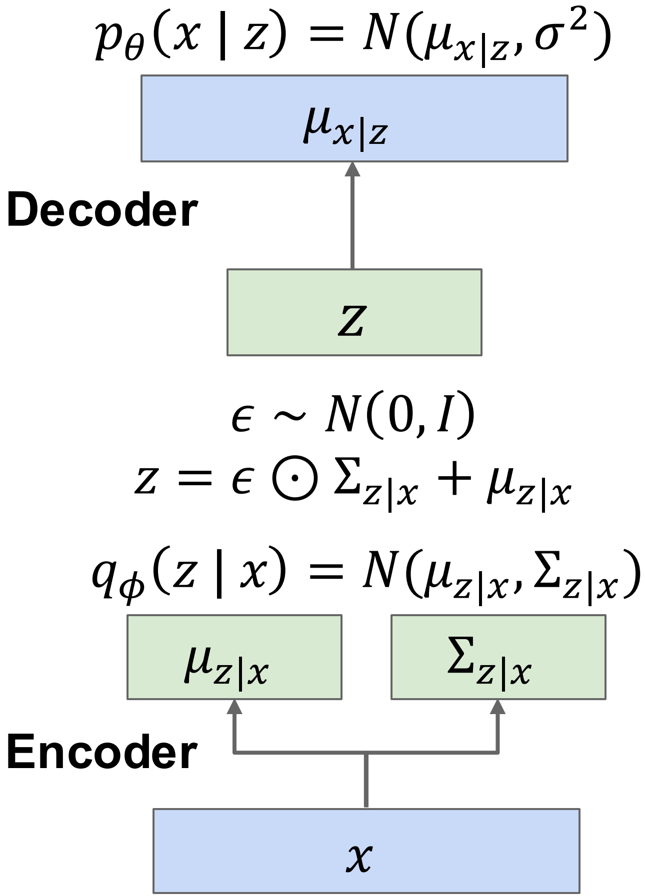
Train by maximizing the **variational lower bound**

$$E_{z\sim q_\phi(z|x)}[log\, p_\theta(x|z)] - D_{KL}\left(q_\phi(z|x), p(z)\right)$$

The loss terms fight against each other!

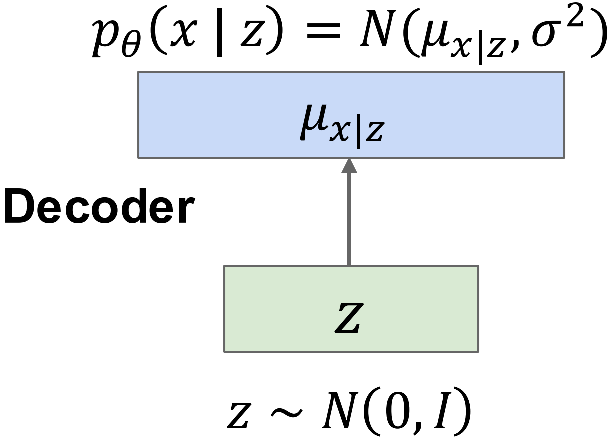
**Reconstruction loss** wants $\Sigma_{z|x} = 0$ and $\mu_{z|x}$ to be unique for each x, so decoder can deterministically reconstruct x

**Prior loss** wants $\Sigma_{z|x} = \mathbf{I}$ and $\mu_{z|x} = 0$ so encoder output is always a unit Gaussian



# Variational Autoencoders: Sampling

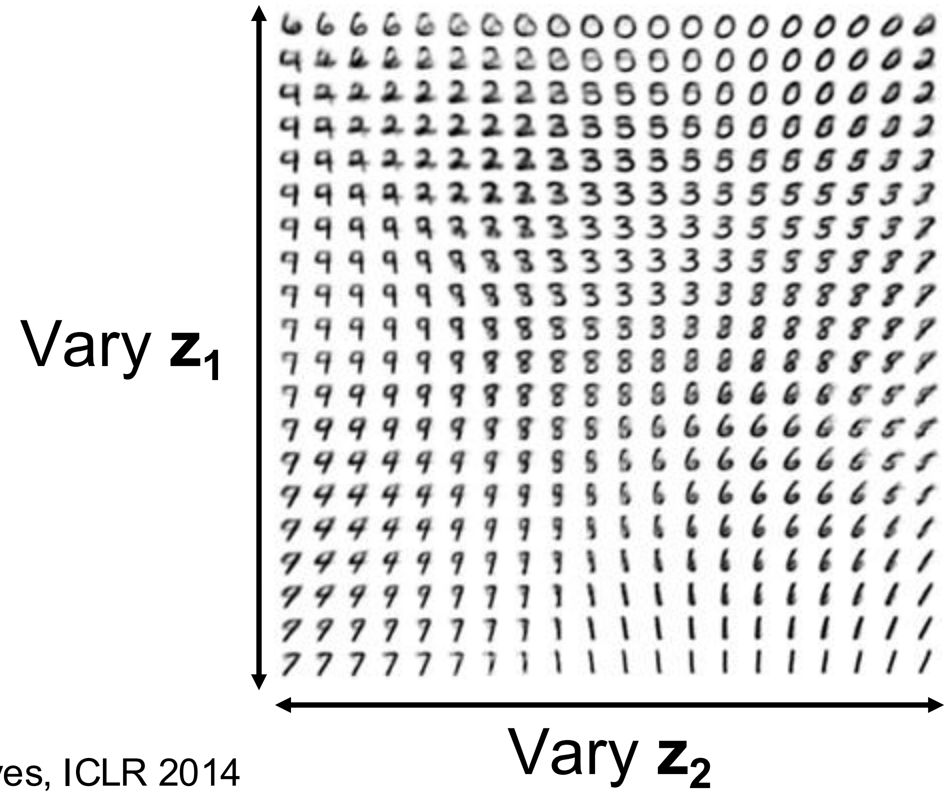
1. Sample z from the prior
2. Run through decoder to get an image



# Variational Autoencoders: Disentangling

The diagonal prior on p(z) causes dimensions of z to be independent

“Disentangling factors of variation”




Kingma and Welling, Auto-Encoding Variational Beyes, ICLR 2014

# Recap: Supervised vs Unsupervised Learning

**Supervised Learning**

**Data**: (x, y)
x is data, y is label

**Goal**: Learn a function to map x -> y

**Examples**: Classification, regression, object detection, semantic segmentation, image captioning, etc.

**Unsupervised Learning**

**Data**: x
Just data, no labels!

**Goal**: Learn hidden structure in data

**Examples**: Clustering, dimensionality reduction, density estimation, etc.

# Recap: Generative vs Discriminative Models

**Discriminative Model**: Learn a probability distribution p(y|x)

**Generative Model**: Learn a probability distribution p(x)

**Conditional Generative Model**: Learn p(x|y)

**Data:** x

**Label:** y
Cat

**Density Function**
p(x) assigns a positive number to each possible x; higher numbers mean x is more likely.

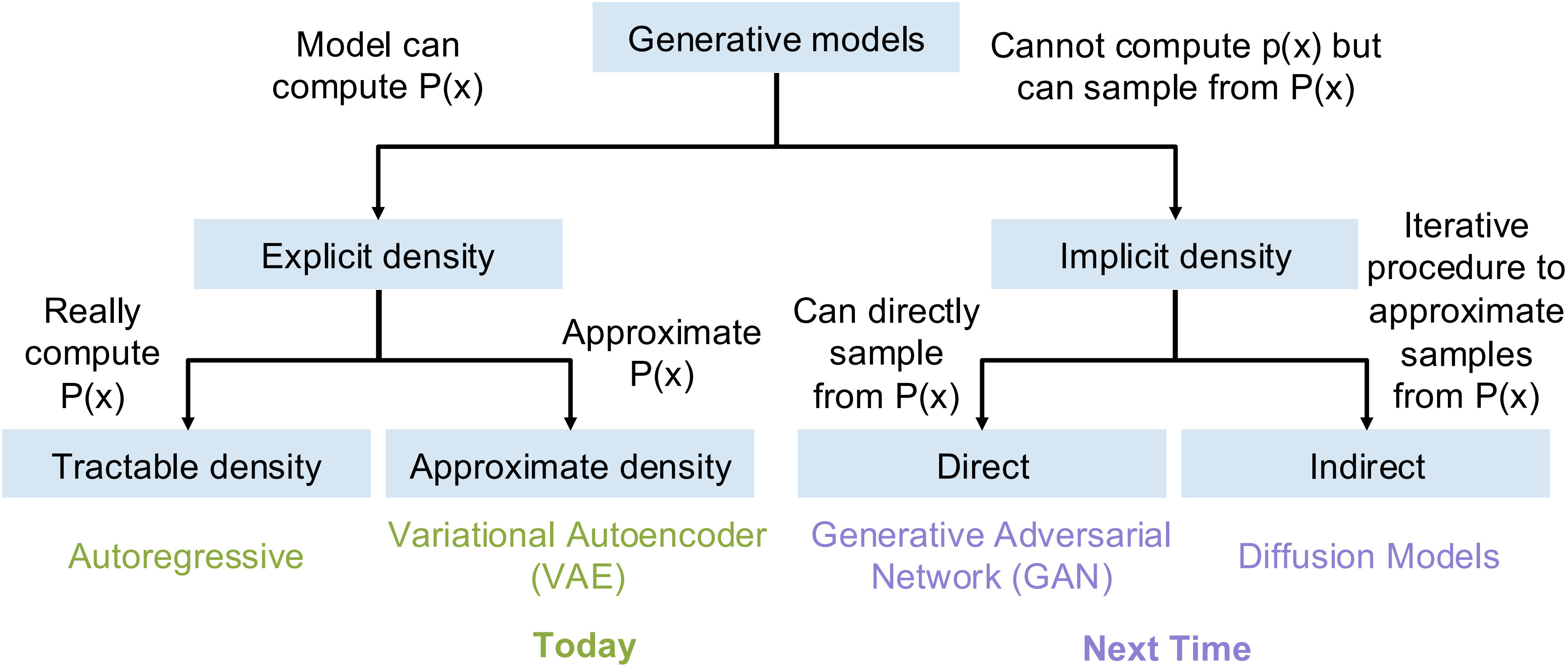
Density functions are **normalized:**

$$\int_X p(x)dx = 1$$

Different values of x **compete** for density

# Recap: Generative Models

Figure adapted from Ian Goodfellow, Tutorial on Generative Adversarial Networks, 2017



# Next Time:
# Generative Models (part 2)
# Generative Adversarial Networks
# Diffusion Models