ON FUNCTIONS OF THREE VARIABLES*

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In the present paper there is indicated a method of proof of a theorem which yields a complete solution of the 13th problem of Hilbert (in the sense of a denial of the hypothesis expressed by Hilbert).

Theorem 1. *Every real, continuous function $f(x_1,x_2,x_3)$ of three variables which is defined on the unit cube E^3 can be represented in the form*

$$f(x_1,x_2,x_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} h_{ij}[\varphi_{ij}(x_1,x_2), x_3], \tag{1}$$

where the functions h_{ij} and φ_{ij} of two variables are real and continuous.

A.N. Kolmogorov [1] obtained recently the representation

$$f(x_1,x_2,x_3) = \sum_{i=1}^{3} h_i[\varphi_i(x_1,x_2), x_3], \tag{2}$$

where the functions h_i and φ_i are continuous, the function h_i is real, and the function φ_i takes on values which belong to some tree Ξ. In the construction of A.N. Kolmogorov (for the case of functions of three variables), the tree Ξ can be taken not as a universal tree, but such that all of its points have a branching index not greater than 3. For this, the functions u^r_{km} of the fundamental lemma [1] (for $n = 2$) must be chosen so that in addition to the indicated five properties they must have the following properties.

(6) *The boundary of each level set of each function u^r_{km} divides the plane into not more than 3 parts.*

(7) *For every r, $G^r_{11} \supset E^2$.*

On the basis of this remark, Theorem 1 is a consequence of the existence of the representation (2) and of the next theorem.

Theorem 2. *Let F be any family of real equicontinuous functions $f(\xi)$ defined on a tree Ξ all of whose points have a branching index $\leqslant 3$. One can realize the tree as a subset X of the three-dimensional cube E^3 in such a way that any function of the family F can be represented in the form*

$$f(\xi) = \sum_{k=1}^{3} f_k(x_k),$$

where $x = (x_1,x_2,x_3)$ is the image of $\xi \in \Xi$ in the tree X; the $f_k(x_k)$ are continuous real functions of one variable, while the f_k depend continuously

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on f *(in the sense of uniform convergence).*

We will introduce certain auxiliary concepts. Let K be a finite complex of segments contained in E^3 and consisting of segments which are not parallel to any coordinate plane.

Definition 1. A system of points

$$a_0 \neq a_1 \neq \dots \neq a_{n-1} \neq a_n$$

belonging to K will be called a *zigzag* (lightning) if the segments $\overline{a_{i-1}a_i}$ are perpendicular to the axes X_{α_i}, respectively, and

$$\alpha_1 \neq \alpha_2 \neq \dots \neq \alpha_{n-1} \neq \alpha_n .$$

The finite system of the pairwise distinct points $a_{i_1 i_2 \dots i_n}$ tagged by the corteges of indices $i_1 i_2 \dots i_n$, will be called a *branching scheme* if (1) there exists only one point a_0 tagged with one index; (2) the presence of $a_{i_1 i_2 \dots i_{n-1} i_n}$ in the system implies the presence of $a_{i_1 \dots i_{n-1}}$ in the system.

Definition 2. A branching system of points $a_{i_1 \dots i_n}$ contained in K will be called a *generating scheme* if for a given cortege $i_1 \dots i_n$ the set of points of the form $a_{i_1 \dots i_n i_{n+1}}$ lies on the plane passing through $a_{i_1 \dots i_n}$ and perpendicular to some coordinate axis $x_{\alpha_{i_1 \dots i_n}}$, and contains all points of intersection of this plane with K, that are distinct from $a_{i_1 \dots i_n}$.

The tree Ξ can be represented in the form

$$\Xi = \overline{\bigcup_{n=1}^{\infty} D_n}, \quad D_n \subset D_{n+1},$$

where D_n is a finite tree, D_1 is a simple arc, and D_{n+1} is obtained from D_n by attaching segments S_n at certain points p_n that are not branch points or endpoints of d_n [2].

We will denote by ω_n the upper boundary of the oscillations of the functions $f \in F$ on the components of the difference $\Xi \setminus D_n$. It is easy to see that

$$\omega_n \to 0 \quad \text{when} \quad n \to \infty .$$

Therefore, one can select a sequence

$$n_1 < n_2 < \dots < n_r < \dots ,$$

so that

$$\omega_n \leqslant \frac{1}{r^2} \quad \text{when} \quad n \geqslant n_r .$$

The realization X of the tree Ξ in E^3 is constructed in the form

$$X = \overline{\bigcup_{n=1}^{\infty} D_n'},$$

where D_n' is a complex of segments which realize D_n in such a way that the images S_n' of the arcs S_n are segments that are not perpendicular to the coordinate axes.

The inductive construction of D_n' is performed so that $\overline{\bigcup_{n=1}^{\infty} D_n'}$ is a tree [2], and that the following conditions are satisfied.

(1) Every function $f \in F$ can be represented on D_n in the form

$$f(\xi) = \sum_{k=1}^{3} f_k^n(x_k), \tag{3}$$

where the $f_k^n(x_k)$ depend continuously on f.

(2) The tree D_n' has for every point a_0 a generating system issuing from a_1, and whose initial direction α_0 can be chosen arbitrarily.

(3) Let A_n be the set of points D_n' which is the image of the branch points of Ξ. There exists a denumerable set $B_n \subseteq D_n'$, $B_n \cap A_n = 0$ such that the zigzag $a_0 \ldots a_m$, which begins at $a_0 \in D_n' \setminus B_n$, has no points in common with A_n and no coincident points $a_i = a_j$, $i \neq j$.

(4) If $n_r < n \leqslant n_{r+1}$, then

$$|f_k^n(x_k) - f_k^{n_r}(x_k)| \leqslant \left(3 + \frac{n - n_r}{n_{r+1} - n_r}\right) \frac{1}{r^2}. \tag{4}$$

This proof of the possibility of the inductive construction of the trees D_n', and of the functions f_k^n with properties (1) to (4), is too complicated to be given here. Roughly speaking, at each step the attached segment S_{n+1}' is chosen of very short length; its direction, and the way of mapping of S_{n+1} on S_{n+1}' are selected so as to guarantee the fulfillment of properties (2) and (3) by D_{n+1}'. The preservation of equality (3), in the transition from n to $n + 1$, on the newly attached segment S_{n+1}, requires the introduction of a correction $f_k^{n+1} - f_k^n$, for at least one of the functions f_k^n, on the projection S_{n+1}' on the axis x_k. For the preservation of equality (3) on the earlier constructed tree D_n', it is necessary to compensate for this correction by means of new corrections for the functions f_k^n on a number of other segments. The exact method of the introduction of these corrections, we will not present here. We only note the following: these corrections must be such that inequality (4) will be preserved for $n' = n + 1$; if S_{n+1}' is chosen small enough, and if its direction is chosen appropriately, it must be possible to produce it for every function f_k^n on a finite system of non-intersecting segments of the axis x_k. In the proof of this possibility one makes use of the fact that the tree D_n' has properties (2) and (3).

The proof of the existence of the continuous function

$$f_k(x_k) = \lim_{n \to \infty} f_k^n(x_k)$$

and of the validity of the equation

$$f(\xi) = \sum_{k=1}^{3} f_k(x_k)$$

on the entire X, is not complicated.

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