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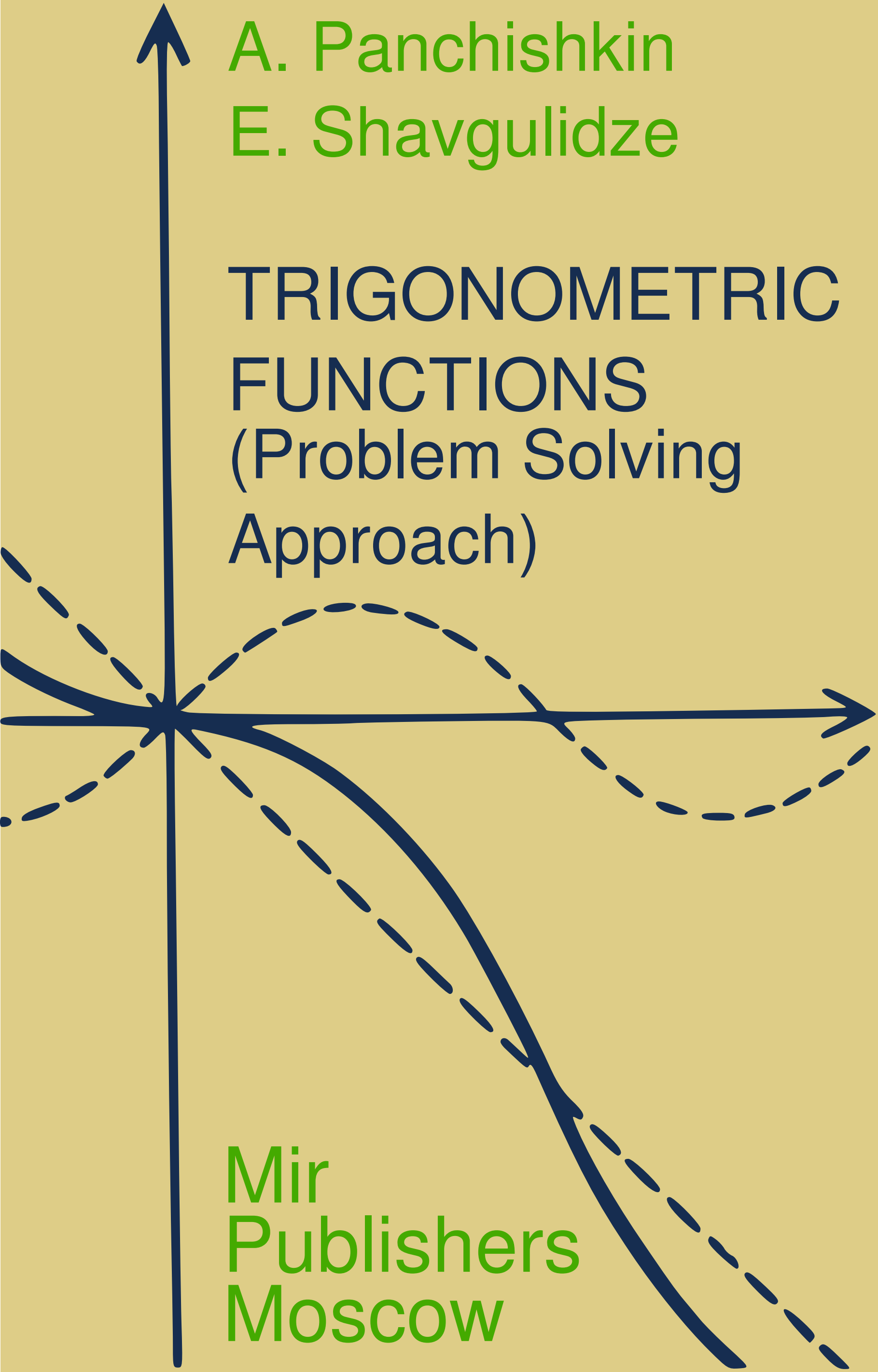
TRIGONOMETRIC FUNCTIONS

(Problem Solving Approach)



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TRIGONOMETRIC FUNCTIONS

А. А. Панчишкин, Е. Т. Шавгулидзе

ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ В ЗАДАЧАХ

Издательство «Наука» Москва

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From the Authors

By tradition, trigonometry is an important component of mathematics courses at high school, and trigonometry questions are always set at oral and written examinations to those entering universities, engineering colleges, and teacher-training institutes.

The aim of this study aid is to help the student to master the basic techniques of solving difficult problems in trigonometry using appropriate definitions and theorems from the school course of mathematics. To present the material in a smooth way, we have enriched the text with some theoretical material from the textbook *Algebra and Fundamentals of Analysis* edited by Academician A. N. Kolmogorov and an experimental textbook of the same title by Professors N.Ya. Vilenkin, A.G. Mordkovich, and V.K. Smyshlyaev, focussing our attention on the application of theory to solution of problems. That is why our book contains many worked competition problems and also some problems to be solved independently (they are given at the end of each chapter, the answers being at the end of the book).

Some of the general material is taken from *Elementary Mathematics* by Professors G.V. Dorofeev, M.K. Potapov, and N.Kh. Rozov (Mir Publishers, Moscow, 1982), which is one of the best study aids on mathematics for pre-college students.

We should like to note here that geometrical problems which can be solved trigonometrically and problems involving integrals with trigonometric functions are not considered.

At present, there are several problem books on mathematics (trigonometry included) for those preparing to pass their entrance examinations (for instance, *Problems* *at Entrance Examinations in Mathematics* by Yu.V. Nesterenko, S.N. Olekhnik, and M.K. Potapov (Moscow, Nauka, 1983); *A Collection of Competition Problems in Mathematics with Hints and Solutions* edited by A.I. Prilepko (Moscow, Nauka, 1986); *A Collection of Problems in Mathematics for Pre-college Students* edited by A.I. Prilepko (Moscow, Vysshaya Shkola, 1983); *A Collection of Competition Problems in Mathematics for Those Entering Engineering Institutes* edited by M.I. Skanavi (Moscow, Vysshaya Shkola, 1980). Some problems have been borrowed from these for our study aid and we are grateful to their authors for the permission to use them.

The beginning of a solution to a worked example is marked by the symbol ◀ and its end by the symbol ▶. The symbol ▶ indicates the end of the proof of a statement.

Our book is intended for high-school and pre-college students. We also hope that it will be helpful for the school children studying at the "smaller" mechanico-mathematical faculty of Moscow State University.

Contents

Chapter 1

Definitions and Basic Properties of Trigonometric Functions

1.1. Radian Measure of an Arc. Trigonometric Circle

1. The first thing the student should have in mind when studying trigonometric functions consists in that the arguments of these functions are real numbers. The pre-college student is sometimes afraid of expressions such as sin 1, cos 15 (but not sin 1°, cos 15°), cos (sin 1), and so cannot answer simple questions whose answer becomes obvious if the sense of these expressions is understood.

When teaching a school course of geometry, trigonometric functions are first introduced as functions of an angle (even only of an acute angle). In the subsequent study, the notion of trigonometric function is generalized when functions of an arc are considered. Here the study is not confined to the arcs enclosed within the limits of one complete revolution, that is, from 0° to 360°; the student is encountered with arcs whose measure is expressed by any number of degrees, both positive and negative. The next essential step consists in that the degree (or sexagesimal) measure is converted to a more natural radian measure. Indeed, the division of a complete revolution into 360 parts (degrees) is done by tradition (the division into other number of parts, say into 100 parts, is also used). Radian measure of angles is based on measuring the length of arcs of a circle. Here, the unit of measurement is one radian which is defined as a central angle subtended in a circle by an arc whose length is equal to the radius of the circle. Thus, the radian measure of an angle is the ratio of the arc it subtends to the radius of the circle in which it is the central angle; also called circular measure. Since the circumference of a circle of a unit radius is equal to 2π, the length of the arc of 360° is equal to 2π radians. Consequently, to 180° there correspond π radians. To change from degrees to radians and vice versa, it suffices to remember that the relation between the degree and radian measures of an arc is of proportional nature.

Example 1.1.1. How many degrees are contained in the arc of one radian?

◀ We write the proportion:

$$\text{If } \pi \text{ radians} = 180°,$$
$$\text{and } 1 \text{ radian} = x,$$

then $x = \frac{180°}{\pi} \approx 57.29578°$ or $57°17'44.8''$. ▶

Example 1.1.2. How many degrees are contained in the arc of $\frac{35\pi}{12}$ radians?

◀
$$\text{If } \pi \text{ radians} = 180°,$$
$$\text{and } \frac{35\pi}{12} \text{ radians} = x,$$

then $x = \left(\frac{35\pi}{12} \cdot 180°\right) \Big/ \pi = 525°$. ▶

Example 1.1.3. What is the radian measure of the arc of 1984°?

$$\text{If } \pi \text{ radians} = 180°,$$
$$\text{and } y \text{ radians} = 1984°,$$

then $y = \frac{\pi \cdot 1984}{180} = \frac{496\pi}{45} = 11\frac{1}{45}\pi$. ▶

2. Trigonometric Circle. When considering either the degree or the radian measure of an arc, it is of importance to know how to take into account the direction in which the arc is traced from the initial point A_1 to the terminal point A_2. The direction of tracing the arc anticlockwise is usually said to be positive (see Fig. 1*a*), while the direction of tracing the arc clockwise is said to be negative (Fig. 1*b*).

We should like to recall that a circle of unit radius with a given reference point and positive direction is called the *trigonometric* (or *coordinate*) circle.

Usually, the right-hand end point of the horizontal diameter is chosen as the reference point. We arrange the trigonometric circle on a coordinate plane with the

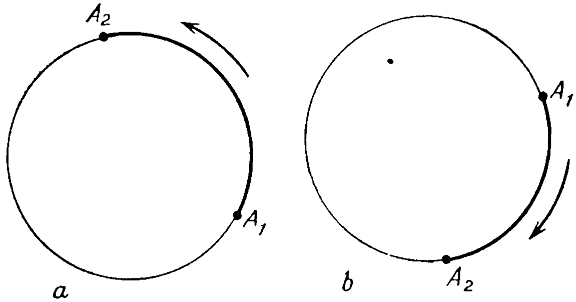

Fig. 1

rectangular Cartesian coordinate system introduced (Fig. 2), placing the centre of the circle into the origin. Then the reference point has the coordinates (1, 0). We denote: $A = A\ (1,\ 0)$. Also, let B, C, D denote the points $B\ (0,\ 1)$, $C\ (-1,\ 0)$, $D\ (0,\ -1)$, respectively.

The trigonometric circle will be denoted by S. According to the aforesaid,

$$S = \{(x,\ y): x^2 + y^2 = 1\}.$$

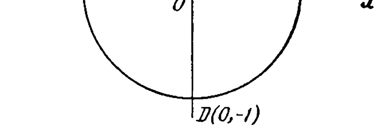

Fig. 2

3. Winding the Real Axis on the Trigonometric Circle. In the theory of trigonometric functions the fundamental role is played by the mapping P: $\mathbf{R} \to S$ of the set $\mathbf{R}$ of real numbers on the coordinate circle which is constructed as follows:

(1) The number $t = 0$ on the real axis is associated with the point A: $A = P_0$.

(2) If $t > 0$, then, on the trigonometric circle, we consider the arc AP_t, taking the point $A = P_0$ as the intitial point of the arc and tracing the path of length t

round the circle in the positive direction. We denote the terminal point of this path by P_t and associate the number t with the point P_t on the trigonometric circle. Or in other words: the point P_t is the image of the point $A = P_0$ when the coordinate plane is rotated about the origin through an angle of t radians.

(3) If $t < 0$, then, starting from the point A round the circle in the negative direction, we shall cover the path of length $|t|$. Let P_t denote the terminal point of this path which will just be the point corresponding to the negative number t.

As is seen, the sense of the constructed mapping P: $\mathbf{R} \to S$ consists in that the positive semiaxis is wound onto S in the positive direction, while the negative semiaxis is wound onto S in the negative direction. This mapping is not one-to-one: if a point $F \in S$ corresponds to a number $t \in \mathbf{R}$, that is, $F = P_t$, then this point also corresponds to the numbers $t + 2\pi$, $t - 2\pi$: $F = P_{t+2\pi} = P_{t-2\pi}$. Indeed, adding to the path of length t the path of length 2π (either in the positive or in the negative direction) we shall again find ourselves at the point F, since 2π is the circumference of the circle of unit radius. Hence it also follows that all the numbers going into the point P_t under the mapping P have the form $t + 2\pi k$, where k is an arbitrary integer. Or in a briefer formulation: the full inverse image $P^{-1}(P_t)$ of the point P_t coincides with the set

$$\{t + 2\pi k: k \in \mathbf{Z}\}.$$

Remark. The number t is usually identified with the point P_t corresponding to this number, however, when solving problems, it is useful to find out what object is under consideration.

Example 1.1.4. Find all the numbers $t \in \mathbf{R}$ corresponding to the point $F \in S$ with coordinates $(-\sqrt{2}/2, -\sqrt{2}/2)$ under the mapping P.

◀ The point F actually lies on S, since

$$\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1.$$

Let X, Y denote the feet of the perpendiculars dropped from the point F on the coordinate axes Ox and Oy (Fig. 3). Then $|XO| = |YO| = |XF|$, and $\triangle XFO$ is a right isosceles triangle, $\angle XOF = 45° = \pi/4$ radian. Therefore the magnitude of the arc AF is equal to $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$, and to the point F there correspond the numbers $\frac{5\pi}{4} + 2\pi k$, $k \in \mathbf{Z}$, and only they. ▶

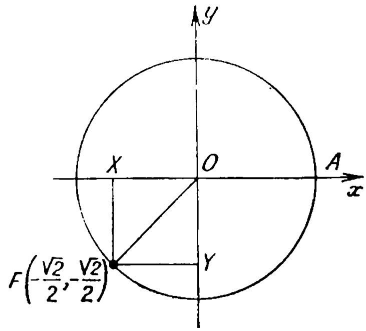

Fig. 3

Example 1.1.5. Find all the numbers corresponding to the vertices of a regular N-gon inscribed in the trigonometric circle so that one of the vertices coincides with the point P_1 (see Fig. 4 in which $N = 5$).

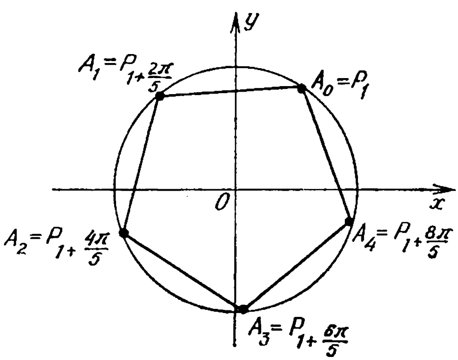

Fig. 4

◀The vertices of a regular N-gon divide the trigonometric circle into N equal arcs of length $2\pi/N$ each. Consequently, the vertices of the given N-gon coincide with the points $A_l = P_{1+\frac{2\pi l}{N}}$, where $l = 0, 1, \ldots, N-1$. Therefore the sought-for numbers $t \in \mathbf{R}$ have the form

$1+\frac{2\pi k}{N}$, where $k \in \mathbf{Z}$. The last assertion is verified in the following way: any integer $k \in \mathbf{Z}$ can be uniquely written in the form $k = Nm + l$, where $0 \leqslant l \leqslant N - 1$ and m, $l \in \mathbf{Z}$, l being the remainder of the division of the integer k by N. It is now obvious that the equality $1+\frac{2\pi k}{N} = 1+\frac{2\pi l}{N}+2\pi m$ is true since its right-hand side con-

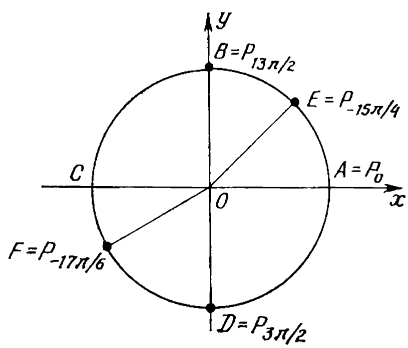

Fig. 5

tains the numbers which correspond to the points $P_{1+\frac{2\pi l}{N}}$ on the trigonometric circle. ▶

Example 1.1.6. Find the points of the trigonometric circle which correspond to the following numbers: (a) $3\pi/2$, (b) $13\pi/2$, (c) $-15\pi/4$, (d) $-17\pi/6$.

◀(a) $\frac{3\pi}{2} = \frac{3}{4}\cdot 2\pi$, therefore, to the number $\frac{3\pi}{2}$ there corresponds the point D with coordinates $(0, -1)$, since the arc AD traced in the positive direction has the measure equal to $\frac{3}{4}$ of a complete revolution (Fig. 5).

(b) $\frac{13\pi}{2} = 3\cdot 2\pi + \frac{\pi}{2}$, consequently, to the number $\frac{13\pi}{2}$ there corresponds the point B $(0, 1)$: starting from the point A we can reach the point B by tracing the trigonometric circle in the positive direction three times and then covering a quarter of revolution ($\pi/2$ radian) in the same direction.

(c) Let us represent the number $-15\pi/4$ in the form $2\pi k + t_0$, where k is an integer, and t_0 is a number such that $0 \leqslant t_0 < 2\pi$. To do so, it is necessary and sufficient that the following inequalities be fulfilled:

$$2\pi k \leqslant -15\pi/4 \leqslant 2\pi\ (k + 1).$$

Let us write the number $-15\pi/4$ in the form $-3\frac{3}{4}\pi = -4\pi + \frac{\pi}{4}$, whence it is clear that $k = -2$, $t_0 = \pi/4$, and to the number $t = -15\pi/4$ there corresponds a point $E = P_{\pi/4}$ such that the size of the angle EOA is $\pi/4$ (or 45°). Therefore, to construct the point $P_{-15\pi/4}$, we have to trace the trigonometric circle twice in the negative direction and then to cover the path of length $\pi/4$ corresponding to the arc of 45° in the positive direction. The point E thus obtained has the coordinates $(\sqrt{2}/2, \sqrt{2}/2)$.

(d) Similarly, $-\frac{17\pi}{6} = -2\frac{5}{6}\pi = -2\pi - \frac{5\pi}{6} = -3\pi + \frac{\pi}{6}$, and in order to reach the point $F = P_{-17\pi/6}$ (starting from A), we have to cover one and a half revolutions (3π radians) in the negative direction (as a result, we reach the point C $(-1, 0)$) and then to return tracing an arc of length $\pi/6$ in the positive direction. The point F has the coordinates $(-\sqrt{3}/2, -1/2)$. ▶

Example 1.1.7. The points $A = P_0$, $B = P_{\pi/2}$, $C = P_{\pi}$, $D = P_{3\pi/2}$ divide the trigonometric circle into four equal arcs, that is, into four quarters called quadrants. Find in what quadrant each of the following points lies: (a) P_{10}, (b) P_8, (c) P_{-8}.

◀To answer this question, one must know the approximate value of the number π which is determined as half the circumference of unit radius. This number has been computed to a large number of decimal places (here are the first 24 digits: $\pi \approx 3.141\ 592\ 653\ 589\ 793\ 238\ 462\ 643$).

To solve similar problems, it is sufficient to use far less accurate approximations, but they should be written in the form of strict inequalities of type

$$3.1 < \pi < 3.2 \tag{1.1}$$

$$3.14 < \pi < 3.15, \tag{1.2}$$

$$3.141 < \pi < 3.142. \tag{1.3}$$

Inaccurate handling of approximate numbers is a flagrant error when solving problems of this kind. Such problems are usually reduced to a rigorous proof of some inequalities. The proof of an inequality is, in turn, reduced to a certain obviously true estimate using equivalent transformations, for instance, to one of the estimates (1.1)-(1.3) if from the hypothesis it is clear that such an estimate is supposed to be known. In such cases, some students carry out computations with unnecessarily high accuracy forgetting about the logic of the proof. Many difficulties also arise in the cases when we have to prove some estimate for a quantity which is usually regarded to be approximately known to some decimal digits; for instance, to prove that $\pi > 3$ or that $\pi < 4$. The methods for estimating the number π are connected with approximation of the circumference of a circle with the aid of the sum of the lengths of the sides of regular N-gons inscribed in, and circumscribed about, the trigonometric circle. This will be considered later on (in Sec. 5.1); here we shall use inequality (1.1) to solve the problem given in Example 1.1.7.

Let us find an integer k such that

$$\frac{\pi k}{2} < 10 < \frac{\pi (k+1)}{2}. \tag{1.4}$$

Then the number of the quadrant in which the point P_{10} is located will be equal to the remainder of the division of the number $k+1$ by 4 since a complete revolution consists of four quadrants. Making use of the upper estimate $\pi < 3.2$, we find that $\frac{\pi \cdot 6}{2} < 9.6$ for $k = 6$; at the same time, $\pi > 3.1$ and $\frac{\pi (k+1)}{2} = \frac{\pi \cdot 7}{2} > 3.1 \cdot 3.5 = 10.85$. Combining these inequalities with the obvious inequality

$$9.6 < 10 < 10.85,$$

we get a rigorous proof of the fact that inequality (1.4) is fulfilled for $k = 6$, and the point P_{10} lies in the third quadrant since the remainder of the division of 7 by 4 is equal to 3.

In similar fashion, we find that the inequalities

$$\frac{\pi k}{2} < 8 < \frac{\pi(k+1)}{2}$$

are valid for $k = 5$, since $\frac{\pi \cdot 5}{2} < \frac{3.2 \cdot 5}{2} = 8$ and $\frac{\pi \cdot 6}{2} > \frac{3.1 \cdot 6}{2} = 9.3$. Consequently, the point P_8 lies in the second quadrant, since the remainder of the division of the number $k + 1 = 6$ by 4 is equal to 2. The point P_{-8}, symmetric to the point P_8 with respect to the x-axis, lies in the third quadrant. ▶

Example 1.1.8. Find in which quadrant the point $P_{-\sqrt{5}-\sqrt[3]{7}}$ lies.

◀ Let us find an integer k such that

$$\pi k/2 < -\sqrt{5} - \sqrt[3]{7} < \pi(k+1)/2. \qquad (1.5)$$

To this end, we use the inequalities

$$2.2 < \sqrt{5} < 2.3,$$
$$1.9 < \sqrt[3]{7} < 2,$$

whose validity is ascertained by squaring and cubing both sides of the respective inequality (let us recall that if both sides of an inequality contain nonnegative numbers, then raising to a positive power is a reversible transformation). Consequently,

$$-4.3 < -\sqrt{5} - \sqrt[3]{7} < -4.1. \qquad (1.6)$$

Again, let us take into consideration that $3.1 < \pi < 3.2$. Therefore the following inequalities are fulfilled:

$$\pi(-3)/2 < -4.65 < -4.3, \qquad (1.7)$$
$$\pi(-2)/2 > -3.2 > -4.1. \qquad (1.8)$$

From inequalities (1.6)-(1.8) it follows that (1.5) is valid for $k = -3$, consequently, the point $P_{-\sqrt{5}-\sqrt[3]{7}}$ lies in the second quadrant, since the remainder after the division of the number $-3 + 1$ by 4 is equal to 2. ▶

1.2. Definitions of the Basic Trigonometric Functions

1. The Sine and Cosine Defined. Here, recall that in school textbooks the sine and cosine of a real number $t \in \mathbf{R}$ is defined with the aid of a trigonometric mapping

$$P: \mathbf{R} \to S.$$

Definition. Let the mapping P associate a number $t \in \mathbf{R}$ with the point P_t on the trigonometric circle. Then the ordinate y of P_t is called the *sine* of the number t and is symbolized $\sin t$, and the abscissa x of P_t is called the *cosine* of the number t and is denoted by $\cos t$.

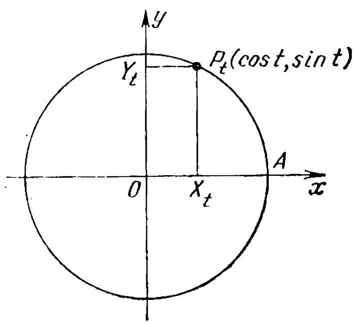

Fig. 6

Let us drop perpendiculars from the point P_t on the coordinate axes Ox and Oy. Let X_t and Y_t denote the feet of these perpendiculars. Then the coordinate of the point Y_t on the y-axis is equal to $\sin t$, and the coordinate of the point X_t on the x-axis is equal to $\cos t$ (Fig. 6).

The lengths of the line segments OY_t and OX_t do not exceed 1, therefore $\sin t$ and $\cos t$ are functions defined throughout the number line whose values lie in the closed interval $[-1, 1]$:

$$D(\sin t) = D(\cos t) = \mathbf{R},$$

$$E(\sin t) = E(\cos t) = [-1, 1].$$

The important property of the sine and cosine (the **fundamental trigonometric identity**): for any $t \in \mathbf{R}$

$$\sin^2 t + \cos^2 t = 1.$$

Indeed, the coordinates (x, y) of the point P_t on the trigonometric circle satisfy the relationship $x^2 + y^2 = 1$, and consequently $\cos^2 t + \sin^2 t = 1$.

Example 1.2.1. Find $\sin t$ and $\cos t$ if: (a) $t = 3\pi/2$, (b) $t = 13\pi/2$, (c) $t = -15\pi/4$, (d) $t = -17\pi/6$.

◀ In Example 1.1.6, it was shown that

$$P_{3\pi/2}=D(0,\ -1),\quad P_{13\pi/2}=B(0,\ 1),$$

$$P_{-15\pi/4}=E(\sqrt{2}/2,\ \sqrt{2}/2),\quad P_{-17\pi/6}=F(-\sqrt{3}/2,\ -1/2).$$

Consequently, $\sin(3\pi/2) = -1$, $\cos(3\pi/2) = 0$; $\sin(13\pi/2)=1$, $\cos(13\pi/2)=0$; $\sin(-15\pi/4)=\sqrt{2}/2$, $\cos(-15\pi/4)=\sqrt{2}/2$; $\sin(-17\pi/6) = -1/2$, $\cos(-17\pi/6)=-\sqrt{3}/2$. ▶

Example 1.2.2. Compare the numbers sin 1 and sin 2.

◀ Consider the points P_1 and P_2 on the trigonometric circle: P_1 lies in the first quadrant and P_2 in the second quadrant since $\pi/2<2<\pi$. Through the point P_2, we pass a line parallel to the x-axis to intersect the circle at a point E. Then the points E and P_2 have equal ordinates. Since $\angle AOE = \angle P_2OC$, $E = P_{\pi-2}$ (Fig. 7), consequently, $\sin 2 = \sin(\pi - 2)$ (this is a particular case of the reduction formulas considered below). The inequality $\pi - 2 > 1$ is valid, therefore $\sin(\pi - 2) > \sin 1$, since both points P_1 and $P_{\pi-2}$ lie in the first quadrant, and when a movable point traces the arc of the first quadrant from A to B the ordinate of this point increases from 0 to 1 (while its abscissa decreases from 1 to 0). Consequently, $\sin 2 > \sin 1$. ▶

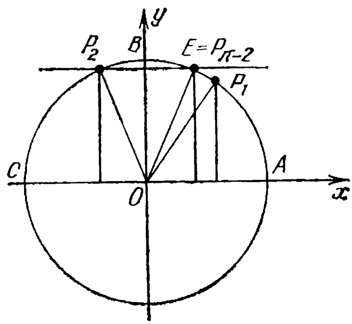

Fig. 7

Example 1.2.3. Compare the numbers cos 1 and cos 2.

◀ The point P_2 lying in the second quadrant has a negative abscissa, whereas the abscissa of the point P_1 is positive; consequently, $\cos 1 > 0 > \cos 2$. ▶

Example 1.2.4. Determine the signs of the numbers sin 10, cos 10, sin 8, cos 8.

◀ It was shown in Example 1.1.7 that the point P_{10} lies in the third quadrant, while the point P_8 is in the second quadrant. The signs of the coordinates of a point on the trigonometric circle are completely determined by the position of a given point, that is, by the quadrant in which the point is found. For instance, both coordinates of any point lying in the third quadrant are negative, while a point lying in the second quadrant has a negative abscissa and a positive ordinate. Consequently, $\sin 10 < 0$, $\cos 10 < 0$, $\sin 8 > 0$, $\cos 8 < 0$. ▶

Example 1.2.5. Determine the signs of the numbers $\sin(\sqrt{5}+\sqrt[3]{7})$ and $\cos(\sqrt{5}+\sqrt[3]{7})$.

◀ From what was proved in Example 1.1.8, it follows that

$$\pi < \sqrt{5}+\sqrt[3]{7} < 3\pi/2.$$

Consequently, the point $P_{\sqrt{5}+\sqrt[3]{7}}$ lies in the third quadrant; therefore

$$\sin(\sqrt{5}+\sqrt[3]{7}) < 0, \quad \cos(\sqrt{5}+\sqrt[3]{7}) < 0. \ ▶$$

Note for further considerations that $\sin t = 0$ if and only if the point P_t has a zero ordinate, that is, $P_t = A$ or C, and $\cos t = 0$ is equivalent to that $P_t = B$ or D (see Fig. 2). Therefore all the solutions of the equation $\sin t = 0$ are given by the formula

$$t = \pi n, \quad n \in \mathbf{Z},$$

and all the solutions of the equation $\cos t = 0$ have the form

$$t = \frac{\pi}{2} + \pi n, \quad n \in \mathbf{Z}.$$

2. The Tangent and Cotangent Defined.

Definition. The ratio of the sine of a number $t \in \mathbf{R}$ to the cosine of this number is called the *tangent* of the number t and is symbolized $\tan t$. The ratio of the cosine of the number t to the sine of this number is termed the *cotangent* of t and is denoted by $\cot t$.

By definition,

$$\tan t = \frac{\sin t}{\cos t}, \quad \cot t = \frac{\cos t}{\sin t}.$$

The expression $\frac{\sin t}{\cos t}$ has sense for all real values of t, except those for which $\cos t = 0$, that is, except for the values $t = \frac{\pi}{2} + \pi k$, $k \in \mathbf{Z}$, and the expression $\cot t$ has sense for all values of t, except those for which $\sin t = 0$, that is, except for $t = \pi k$, $k \in \mathbf{Z}$. Thus, the function $\tan t$ is defined on the set of all real numbers except the numbers $t = \frac{\pi}{2} + \pi k, k \in \mathbf{Z}$. The function $\cot t$ is defined on the set of all real numbers except the numbers $t = \pi k$, $k \in \mathbf{Z}$.

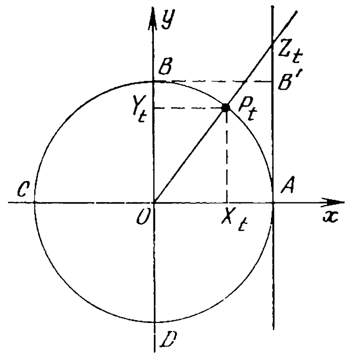

Fig. 8

Graphical representation of the numbers $\tan t$ and $\cot t$ with the aid of the trigonometric circle is very useful. Draw a tangent AB' to the trigonometric circle through the point $A = P_0$, where $B' = (1, 1)$. Draw a straight line through the origin O and the point P_t and denote the point of its intersection with the tangent AB' by Z_t (Fig. 8). The tangent AB' can be regarded as a coordinate axis with the origin A so that the point B' has the coordinate 1 on this axis. Then the ordinate of the point Z_t on this axis is equal to $\tan t$. This follows from the similarity of the triangles OX_tP_t and OAZ_t and the definition of the function $\tan t$. Note that the point of intersection is absent exactly for those values of t for which $P_t = B$ or D, that is, for $t = \frac{\pi}{2} + \pi n$, $n \in \mathbf{Z}$, when the function $\tan t$ is not defined.

Now, draw a tangent BB' to the trigonometric circle through the point B and consider the point of intersection W_t of the line OP_t and the tangent. The abscissa of W_t is equal to $\cot t$. The point of intersection W_t is absent exactly for those t for which $P_t = A$ or C, that is, when $t = \pi n$, $n \in \mathbf{Z}$, and the function $\cot t$ is not defined (Fig. 9).

In this graphical representation of tangent and cotangent, the tangent lines AB' and BB' to the trigonometric circle are called the *line* (or *axis*) *of tangents* and the *line* (*axis*) *of cotangents*, respectively.

Example 1.2.6. Determine the signs of the numbers: tan 10, tan 8, cot 10, cot 8.

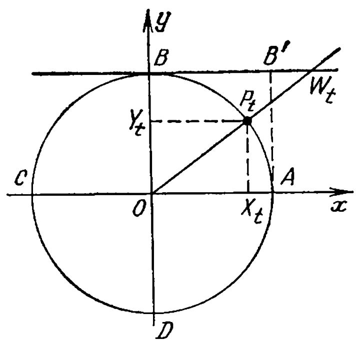

Fig. 9

◀ In Example 1.2.4, it was shown that $\sin 10 < 0$ and $\cos 10 < 0$, $\sin 8 > 0$ and $\cos 8 < 0$, consequently, $\tan 10 > 0$, $\cot 10 > 0$, $\tan 8 < 0$, $\cot 8 < 0$.

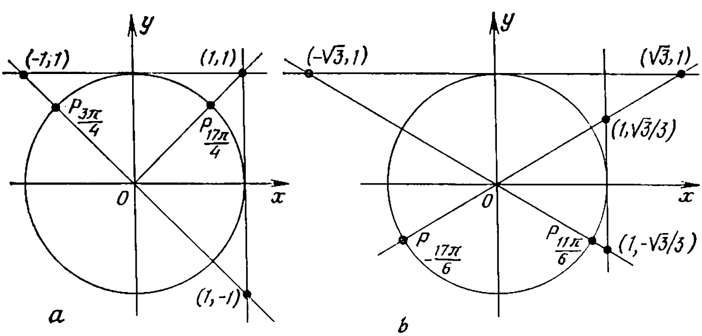

Fig. 10

Example 1.2.7. Determine the sign of the number $\cot(\sqrt{5} + \sqrt[3]{7})$.

◀ In Example 1.2.5, it was shown that $\sin(\sqrt{5} + \sqrt[3]{7}) < 0$, and $\cos(\sqrt{5} + \sqrt[3]{7}) < 0$, therefore $\cot(\sqrt{5} + \sqrt[3]{7}) > 0$. ▶

Example 1.2.8. Find $\tan t$ and $\cot t$ if $t = \frac{3\pi}{4}, \frac{17\pi}{4}, -\frac{17\pi}{6}, \frac{11\pi}{6}$.

◀ As in Item 3 of Sec. 1.1, we locate the points $P_{3\pi/4}$, $P_{17\pi/4}$ (Fig. 10*a*), $P_{-17\pi/6}$, $P_{11\pi/6}$ (Fig. 10*b*) on the trigonometric circle and compute their coordinates:

$$P_{3\pi/4}(-\sqrt{2}/2, \sqrt{2}/2), \quad P_{17\pi/4}(\sqrt{2}/2, \sqrt{2}/2),$$
$$P_{-17\pi/6}(-\sqrt{3}/2, -1/2), \quad P_{11\pi/6}(\sqrt{3}/2, -1/2),$$

therefore $\tan(3\pi/4) = \cot(3\pi/4) = -1$, $\tan(17\pi/4) = \cot(17\pi/4) = 1$, $\tan(-17\pi/6) = 1/\sqrt{3} = \sqrt{3}/3$, $\cot(-17\pi/6) = \sqrt{3}$, $\tan(11\pi/6) = -\sqrt{3}/3$, $\cot(11\pi/6) = -\sqrt{3}$. ▶

1.3. Basic Properties of Trigonometric Functions

1. Periodicity. A function f with domain of definition $X = D(f)$ is said to be *periodic* if there is a nonzero number T such that for any $x \in X$

$$x + T \in X \quad \text{and} \quad x - T \in X,$$

and the following equality is true:

$$f(x - T) = f(x) = f(x + T).$$

The number T is then called the *period* of the function $f(x)$. A periodic function has infinitely many periods since, along with T, any number of the form nT, where n is an integer, is also a period of this function. The smallest positive period of the function (if such period exists) is called the *fundamental period.*

Theorem 1.1. *The functions* $f(x) = \sin x$ *and* $f(x) = \cos x$ *are periodic with fundamental period* 2π.

Theorem 1.2. *The functions* $f(x) = \tan x$ *and* $f(x) = \cot x$ *are periodic with fundamental period* π.

It is natural to carry out the *proof* of Theorems 1.1 and 1.2 using the graphical representation of sine, cosine, tangent, and cotangent with the aid of the trigonometric circle. To the real numbers x, $x + 2\pi$, and $x - 2\pi$, there corresponds one and the same point P_x on the trigonometric circle, consequently, these numbers have the same sine and cosine. At the same time, no positive number less than 2π can be the period of the functions $\sin x$ and $\cos x$. Indeed, if T is the period of $\cos x$, then $\cos T = \cos(0 + T) = \cos 0 = 1$. Hence, to the number T, there corresponds the point P_T with coordinates $(1, 0)$, therefore the number T has the form $T = 2\pi n$ $(n \in \mathbf{Z})$; and since it is positive, we have $T \geqslant 2\pi$. Similarly, if T is the period of the function $\sin x$, then $\sin\left(\frac{\pi}{2} + T\right) = \sin\frac{\pi}{2} = 1$, and to the number $\frac{\pi}{2} + T$ there corresponds the point $P_{\frac{\pi}{2}+T}$ with coordinates $(0, 1)$. Hence, $\frac{\pi}{2} + T = \frac{\pi}{2} + 2\pi n$ $(n \in \mathbf{Z})$ or $T = 2\pi n$, that is, $T \geqslant 2\pi$. ▶

To prove Theorem 1.2, let us note that the points P_t and $P_{t+\pi}$ are symmetric with respect to the origin for any t (the number π specifies a half-revolution of the trigonometric circle), therefore the coordinates of the points P_t and $P_{t+\pi}$ are equal in absolute value and have unlike signs, that is, $\sin t = -\sin(t + \pi)$, $\cos t = -\cos(t + \pi)$. Consequently, $\tan t = \frac{\sin t}{\cos t} = \frac{-\sin(t+\pi)}{-\cos(t+\pi)} = \tan(t + \pi)$, $\cot t = \frac{\cos t}{\sin t} = \frac{-\cos(t+\pi)}{-\sin(t+\pi)} = \cot(t + \pi)$. Therefore π is the period of the functions $\tan t$ and $\cot t$. To make sure that π is the fundamental period, note that $\tan 0 = 0$, and the least positive value of t for which $\tan t = 0$ is equal to π. The same reasoning is applicable to the function $\cot t$. ▶

Example 1.3.1. Find the fundamental period of the function $f(t) = \cos^4 t + \sin t$.

◀The function f is periodic since

$$f(t + 2\pi) = \cos^4(t + 2\pi) + \sin(t + 2\pi) = \cos^4 t + \sin t.$$

No positive number T, smaller than 2π, is the period of the function $f(t)$ since $f\left(-\frac{T}{2}\right) \neq f\left(-\frac{T}{2} + T\right) = f\left(\frac{T}{2}\right)$. Indeed, the numbers $\sin\left(-\frac{T}{2}\right)$ and $\sin\frac{T}{2}$

are distinct from zero and have unlike signs, the numbers $\cos\left(-\frac{T}{2}\right)$ and $\cos\frac{T}{2}$ coincide, therefore $\cos^4\left(-\frac{T}{2}\right)+\sin\left(-\frac{T}{2}\right)\neq\cos^4\frac{T}{2}+\sin\frac{T}{2}$. ▶

Example 1.3.2. Find the fundamental period of the function $f(\sqrt{5}x)$ if it is known that T is the fundamental period of the function $f(x)$.

◀ First of all, let us note that the points $x-t$, x, $x+t$ belong to the domain of definition of the function $g(x)=f(\sqrt{5}x)$ if and only if the points $x\sqrt{5}-t\sqrt{5}$, $x\sqrt{5}$, $x\sqrt{5}+t\sqrt{5}$ belong to the domain of definition of the function $f(x)$. The definition of the function $g(x)$ implies that the equalities $g(x-t)=g(x)=g(x+t)$ and $f(x\sqrt{5}-t\sqrt{5})=f(x\sqrt{5})=f(x\sqrt{5}+t\sqrt{5})$ are equivalent. Therefore, since T is the fundamental period of the function $f(x)$, the number $T/\sqrt{5}$ is a period of the function $g(x)$; it is the fundamental period of $g(x)$, since otherwise the function $g(x)$ would have a period $t<T/\sqrt{5}$, and, hence, the function $f(x)$ would have a period $t\sqrt{5}$, strictly less than T. ▶

Note that a more general statement is valid: if a function $f(x)$ has the fundamental period T, then the function $g(x)=f(ax+b)$ $(a\neq 0)$ has the fundamental period $T/|a|$.

Example 1.3.3. Prove that the function $f(x)=\sin\sqrt{|x|}$ is not periodic.

◀ Suppose that T is the period of the function $f(x)$. We take a positive x satisfying the equality $\sin\sqrt{x}=1$. Then $\sin\sqrt{x+T}=\sin\sqrt{x}=1$, hence $\sqrt{x+T}-\sqrt{x}=2\pi n$ $(n\in\mathbf{Z})$. But $\sqrt{x+T}\geqslant\sqrt{x}$, therefore the following inequality holds true:

$$\sqrt{x+T}\geqslant 2\pi+\sqrt{x}.$$

Both sides of this inequality contain positive numbers, consequently, when squared, this inequality will be replaced by an equivalent one: $x+T\geqslant 4\pi^2+4\pi\sqrt{x}+x$ or

$$T\geqslant 4\pi^2+4\pi\sqrt{x}.$$

The obtained inequality leads to a contradiction since the number $\sqrt{x}$ can be chosen arbitrarily large and, in particular, so that the inequality is not valid for the given fixed number T. ▶

2. Evenness and Oddness. Recall that a function f is said to be *even* if for any x from its domain of definition $-x$ also belongs to this domain, and the equality $f(-x) = f(x)$ is valid. A function f is said to be *odd* if, under the same conditions, the equality $f(-x) = -f(x)$ holds true.

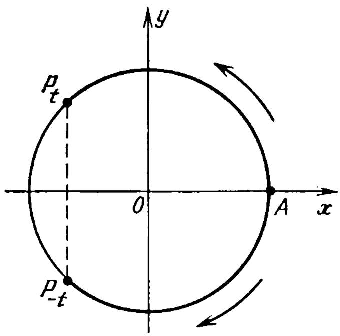

Fig. 11

A couple of examples of even functions: $f(x) = x^2$, $f(x) = x^4 + \sqrt{5}x^2 + \pi$.

A couple of examples of odd functions: $f(x) = -\sqrt{x^3}$, $f(x) = 2x^5 + \pi x^3$.

Note that many functions are neither even nor odd. For instance, the function $f(x) = x^3 + x^2 + 1$ is not even since $f(-x) = (-x)^3 + (-x)^2 + 1 = -x^3 + x^2 + 1 \neq f(x)$ for $x \neq 0$. Similarly, the function $f(x)$ is not odd since $f(-x) \neq -f(x)$.

Theorem 1.3. *The functions* $\sin x$, $\tan x$, $\cot x$ *are odd, and the function* $\cos x$ *is even.*

Proof. Consider the arcs AP_t and AP_{-t} of the trigonometric circle having the same length $|t|$ but opposite directions (Fig. 11). These arcs are symmetric with respect to the axis of abscissas, therefore their end points

$$P_t(\cos t, \sin t), \quad P_{-t}(\cos(-t), \sin(-t))$$

have equal abscissas but opposite ordinates, that is, $\cos(-t) = \cos t$, $\sin(-t) = -\sin t$. Consequently, the function $\sin t$ is odd, and the function $\cos t$ is even.

Further, by the definition of the tangent and cotangent,

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin t}{\cos t} = -\tan t$$

if $\cos t \neq 0$ (here also $\cos(-t) = \cos t \neq 0$), and

$$\cot(-t) = \frac{\cos(-t)}{\sin(-t)} = \frac{\cos t}{-\sin t} = -\cot t$$

if $\sin t \neq 0$ (here also $\sin(-t) = -\sin t \neq 0$). Thus, the functions $\tan t$ and $\cot t$ are odd. ▶

Example 1.3.4. Prove that the function $f(t) = \sin^3 2t \cos 4t + \tan 5t$ is odd.

◀ Note that $f(-t) = (\sin(-2t))^3 \cos(-2t) + \tan(-5t) = (-\sin 2t)^3 \cos 2t - \tan 5t = -f(t)$ for any t from the domain of definition of the function (that is, such that $\cos 5t \neq 0$). ▶

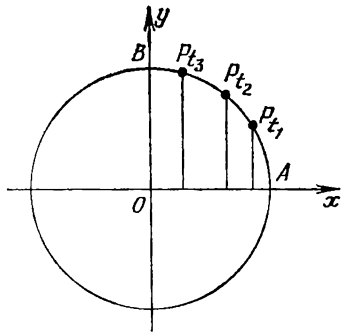

Fig. 12

3. Monotonicity. Recall that a function f defined in an interval X is said to be *increasing* in this interval if for any numbers x_1, $x_2 \in X$ such that $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds true; and if a weak inequality is valid, that is, $f(x_1) \leqslant f(x_2)$, then the function f is said to be *nondecreasing* on the interval X. The notion of a *decreasing* function and a *nonincreasing* function is introduced in a similar way. The properties of increasing or decreasing of a function are also called *monotonicity* of the function. The interval over which the function increases or decreases is called the *interval of monotonicity* of the function.

Let us test the functions $\sin t$ and $\cos t$ for monotonicity. As the point P_t moves along the trigonometric circle anticlockwise (that is, in the positive direction) from the point $A = P_0$ to the point $B(0, 1)$, it keeps rising and displacing to the left (Fig. 12), that is, with an increase in t the ordinate of the point increases, while the abscissa decreases. But the ordinate of P_t is equal to $\sin t$, its abscissa being equal to $\cos t$. Therefore, on the closed interval $[0, \pi/2]$, that is, in the first quadrant, the function $\sin t$ increases from 0 to 1, and $\cos t$ decreases from 1 to 0.

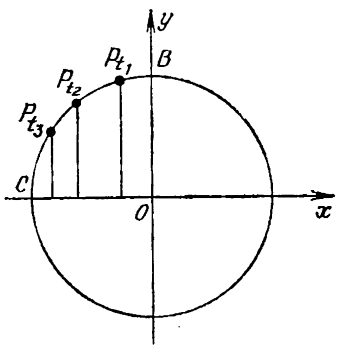

Fig. 13

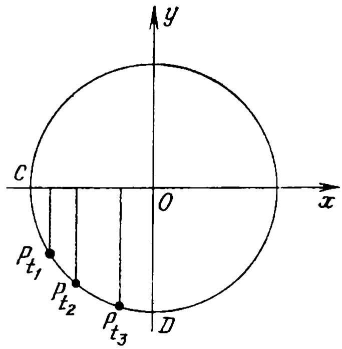

Fig. 14

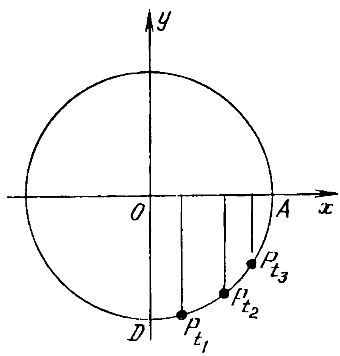

Fig. 15

Similarly, we can investigate the behaviour of these functions as the point P_t moves in the second, third, and fourth quadrants. Hence, we may formulate the following theorem.

Theorem 1.4. *On the interval* $[0, \pi/2]$ *the function* $\sin t$ *increases from* 0 *to* 1, *while* $\cos t$ *decreases from* 1 *to* 0. *On the interval* $[\pi/2, \pi]$ *the function* $\sin t$ *decreases from* 1 *to* 0, *while* $\cos t$ *decreases from* 0 to —1. *On the interval* $[\pi, 3\pi/2]$ *the function* $\sin t$ *decreases from* 0 *to* —1, *while* $\cos t$ *increases from* —1 *to* 0. *On the interval* $[3\pi/2, 2\pi]$ *the function* $\sin t$ *increases from* —1 *to* 0, *while* $\cos t$ *increases from* 0 *to* 1.

The *proof* of the theorem is graphically illustrated in Figs. 12-15, where points P_{t_1}, P_{t_2}, P_{t_3} are such that $t_1 < t_2 < t_3$. ▶

Theorem 1.5. *The function* tan t *increases on the interval* $(-\pi/2, \pi/2)$, *and the function* cot t *decreases on the interval* $(0, \pi)$.

Proof. Consider the function tan t. We have to show that for any numbers t_1, t_2 such that $-\pi/2 < t_1 < t_2 < \pi/2$, the inequality $\tan t_1 < \tan t_2$ holds.

Consider three cases:

(1) $0 \leqslant t_1 < t_2 < \pi/2$. Then, by virtue of Theorem 1.4,

$$0 \leqslant \sin t_1 < \sin t_2, \quad \cos t_1 > \cos t_2 > 0,$$

whence

$$\frac{\sin t_1}{\cos t_1} < \frac{\sin t_2}{\cos t_2}.$$

Consequently, $\tan t_1 < \tan t_2$.

(2) $-\pi/2 < t_1 < 0 < t_2 < \pi/2$. In this case, $\tan t_1 < 0$, and $\tan t_2 > 0$, therefore $\tan t_1 < \tan t_2$.

(3) $-\pi/2 < t_1 < t_2 \leqslant 0$. By virtue of Theorem 1.4,

$$\sin t_1 < \sin t_2 \leqslant 0, \quad 0 < \cos t_1 < \cos t_2,$$

therefore

$$\frac{\sin t_1}{\cos t_1} < \frac{\sin t_2}{\cos t_2},$$

that is, $\tan t_1 < \tan t_2$.

The proof for the function cot t is carried out in a similar way. ▶

Note that the monotonicity properties of the basic trigonometric functions on other intervals can be obtained from the periodicity of these functions. For example, on the closed interval $[-\pi/2, 0]$ the functions sin t and cos t increase, on the open interval $(\pi/2, 3\pi/2)$ the function tan t increases, and on the open interval $(\pi, 2\pi)$ the function cot t decreases.

Example 1.3.5. Prove that the functions sin (cos t) and cos (sin t) decrease on the interval $[0, \pi/2]$.

◀ If $t_1, t_2 \in [0, \pi/2]$, where $t_1 < t_2$, then, by Theorem 1.4, $\sin t_1 < \sin t_2$, and $\cos t_2 < \cos t_1$. Note that the points on the trigonometric circle, corresponding to the real numbers $\sin t_1$, $\sin t_2$, $\cos t_1$, $\cos t_2$, are in the first quadrant since these numbers lie on the closed interval $[0, 1]$, and $1 < \pi/2$. Therefore, we may once again apply Theorem 1.4 which implies that for any t_1, $t_2 \in [0, \pi/2]$ such that $t_1 < t_2$, the following inequalities are valid:

$$\sin(\cos t_1) > \sin(\cos t_2), \quad \cos(\sin t_1) > \cos(\sin t_2),$$

that is, $\sin(\cos t)$ and $\cos(\sin t)$ are decreasing functions on the interval $[0, \pi/2]$. ▶

4. Relation Between Trigonometric Functions of One and the Same Argument. If for a fixed value of the argument the value of a trigonometric function is known, then, under certain conditions, we can find the values of other trigonometric functions. Here, the most important relationship is the principal trigonometric identity (see Sec. 1.2, Item 1):

$$\sin^2 t + \cos^2 t = 1. \tag{1.9}$$

Dividing this identity by $\cos^2 t$ termwise (provided $\cos t \neq 0$), we get

$$1 + \tan^2 t = \frac{1}{\cos^2 t}, \tag{1.10}$$

where $t \neq \frac{\pi}{2} + \pi k$, $k \in \mathbf{Z}$. Using this identity, it is possible to compute $\tan t$ if the value of $\cos t$ and the sign of $\tan t$ are known, and, vice versa, to compute $\cos t$ given the value of $\tan t$ and the sign of $\cos t$. In turn, the signs of the numbers $\tan t$ and $\cos t$ are completely determined by the quadrant in which the point P_t, corresponding to the real number t, lies.

Example 1.3.6. Compute $\cos t$ if it is known that $\tan t = \frac{5}{12}$ and $t \in \left(\pi, \frac{3\pi}{2}\right)$.

◀ From formula (1.10) we find:

$$\cos^2 t = \frac{1}{1 + \tan^2 t} = \frac{1}{1 + \frac{25}{144}} = \frac{144}{169}.$$

Consequently, $|\cos t| = 12/13$, and therefore either $\cos t = 12/13$ or $\cos t = -12/13$. By hypothesis, $t \in (\pi, 3\pi/2)$, that is, the point P_t lies in the third quadrant. In the third quadrant, $\cos t$ is negative; consequently, $\cos t = -12/13$. ▶

Dividing both sides of equality (1.9) by $\sin^2 t$ (for $\sin t \neq 0$) termwise, we get

$$1 + \cot^2 t = \frac{1}{\sin^2 t}, \qquad (1.11)$$

where $t \neq \pi k$, $k \in \mathbf{Z}$. Using this identity, we can compute $\cot t$ if the value of $\sin t$ and the sign of $\cot t$ are known and compute $\sin t$ knowing the value of $\cot t$ and the sign of $\sin t$.

Equalities (1.9)-(1.11) relate different trigonometric functions of one and the same argument.

1.4. Solving the Simplest Trigonometric Equations. Inverse Trigonometric Functions

1. Solving Equations of the Form $\sin t = m$. Arc Sine. To solve the equation of the form $\sin t = m$, it is necessary to find all real numbers t such that the ordinate of the corresponding point P_t is equal to m. To this end, we draw a straight line $y = m$ and find the points of its intersection with the trigonometric circle. There are two such points if $|m| < 1$ (points E and F in Fig. 16), one point if $|m| = 1$, and no points of intersection for $|m| > 1$. Let $|m| \leqslant 1$. One of the points of intersection lies necessarily in the right-hand half-plane, where

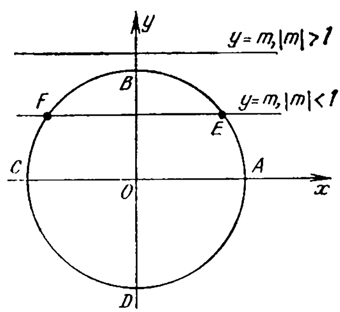

Fig. 16

$x \geqslant 0$. This point can be written in the form $E = P_{t_0}$, where t_0 is some number from the closed interval $[-\pi/2, \pi/2]$. Indeed, when the real axis is being wound on the trigonometric circle, the numbers from the interval $[-\pi/2, \pi/2]$ go into the points of the first and fourth quadrants on the trigonometric circle, the points B and D included. Note that the ordinate $\sin t_0$ of the point $E = P_t$ is equal to m: $\sin t_0 = m$.

Definition. The *arc sine* of a number m is a number t_0, $-\pi/2 \leqslant t_0 \leqslant \pi/2$, such that $\sin t_0 = m$. The following notation is used: $t_0 = \arcsin m$ (or $\sin^{-1} m$).

Obviously, the expression $\arcsin m$ has sense only for $|m| \leqslant 1$. By definition, we have:

$$\sin(\arcsin m) = m, \quad -\pi/2 \leqslant \arcsin m \leqslant \pi/2.$$

The following equality holds true:

$$\arcsin(-m) = -\arcsin m.$$

Note that the left-hand point F of intersection of the line $y = m$ with the trigonometric circle can be written in the form $F = P_{\pi - t_0}$, therefore all the solutions of the equation $\sin t = m$, $|m| \leqslant 1$, are given by the formulas

$$t = \arcsin m + 2\pi k, \quad k \in \mathbf{Z},$$

$$t = \pi - \arcsin m + 2\pi k, \quad k \in \mathbf{Z},$$

which are usually united into one formula:

$$t = (-1)^n \arcsin m + \pi n, \quad n \in \mathbf{Z}.$$

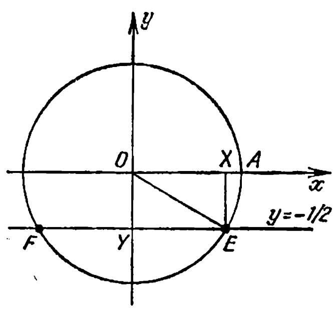

Fig. 17

Example 1.4.1. Solve the equation $\sin t = -1/2$.

◀Consider the points of intersection of the line $y = -1/2$ and the trigonometric circle S. Let X and Y be the feet of the perpendiculars dropped from the right-hand point E of intersection on the coordinate axes (Fig. 17). In the right triangle XOE, we have: $|EX| = 1/2$, $|OE| = 1$, that is, $\angle XOE = 30°$. Consequently, $\angle AOE$ is measured by an arc of $-\pi/6$ radian, and $E = P_{-\pi/6}$. Therefore $\arcsin(-1/2) = -\pi/6$, and the general solution of the equation $\sin t = -1/2$ has the form $t = (-1)^{n+1}\frac{\pi}{6} + \pi n$, $n \in \mathbf{Z}$. ▶

Example 1.4.2. What is the value of $\arcsin(\sin 10)$?

◀In Example 1.1.7 it was shown that the point P_{10} lies in the third quadrant. Let $t = \arcsin(\sin 10)$, then $\sin t = \sin 10 < 0$ and $-\pi/2 \leqslant t \leqslant \pi/2$. Consequently, the point P_t lies in the fourth quadrant and has the same ordinate as the point P_{10}; therefore $P_t = P_{\pi-10}$ (Fig. 18), and the equality $t = \pi - 10 + 2\pi k$ holds for some integer k. For the condition $-\pi/2 < t < 0$ to be fulfilled,

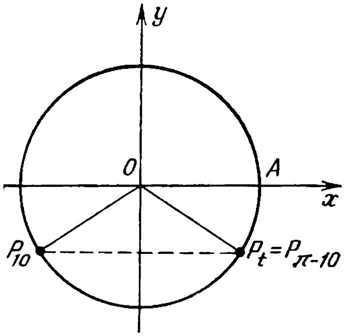

Fig. 18

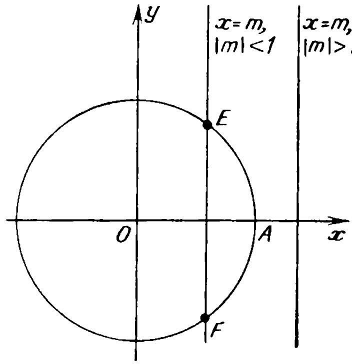

Fig. 19

it is necessary to set $k = 2$. Indeed,

$$-\pi/2 < 3\pi - 10 < 0$$

(these inequalities follow from the estimate $3.1 < \pi < 3.2$). Thus,

$$t = 3\pi - 10 = \arcsin(\sin 10). \blacktriangleright$$

2. Solving the Equation $\cos t = m$. Arc Cosine. To solve the equation $\cos t = m$, it is necessary to find all real numbers t such that the abscissa of the point P_t is equal to m. For this purpose, we draw a straight line $x = m$ and find the points of its intersection with the trigonometric circle. A point of intersection exists if $|m| \leqslant 1$. One of the points of intersection (the point E in Fig. 19) necessarily lies in the upper half-plane, where $y \geqslant 0$, and this point can be written in the form

$$E = P_{t_0}, \quad \text{where} \quad 0 \leqslant t_0 \leqslant \pi.$$

Definition. The *arc cosine* of a number m is a number t_0, lying on the closed interval $[0, \pi]$, such that its cosine is equal to m. The arc cosine of the number m is denoted by $\arccos m$ (or $\cos^{-1} m$).

Obviously, the expression arccos m has sense only for $|m| \leqslant 1$. By definition,

$$\cos(\arccos m) = m, \quad 0 \leqslant \arccos m \leqslant \pi.$$

Note that the lower point of intersection coincides with the point $F = P_{-t_0}$. Therefore the general solution of the equation $\cos t = m$, $m \in [-1, 1]$ has the form

$$t = \pm \arccos m + 2\pi k, \ k \in \mathbf{Z}.$$

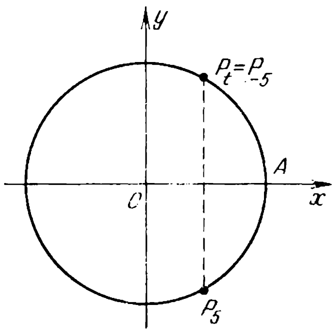

Fig. 20

Note also that the following equality holds:

$$\arccos(-m) = \pi - \arccos m.$$

Example 1.4.3. Find the value of arccos (cos 5).

◀ Let $t = \arccos(\cos 5)$, then $0 \leqslant t \leqslant \pi$, and $\cos t = \cos 5$. The point P_5 lies in the fourth quadrant (Fig. 20) since the inequalities $3\pi/2 < 5 < 2\pi$ hold, therefore the point P_t, lying in the upper half-plane, must coincide with the point P_{-5} which is symmetric to P_5 with respect to the axis of abscissas, that is, $t = -5 + 2\pi k$. The condition $0 \leqslant t \leqslant \pi$ will be fulfilled if we take $k = 1$, therefore $\arccos(\cos 5) = 2\pi - 5$. ▶

Example 1.4.4. Prove that if $0 \leqslant x \leqslant 1$, then $\arcsin x = \arccos \sqrt{1 - x^2}$.

◀ Let $t = \arcsin x$. Then $0 \leqslant t \leqslant \pi/2$ since $\sin t = x \geqslant 0$. Now, from the relationship $\cos^2 t + \sin^2 t = 1$ we get: $|\cos t| = \sqrt{1 - x^2}$. But, bearing in mind that $t \in [0, \pi/2]$, we have: $\cos t = \sqrt{1 - x^2}$, whence $t = \arccos \sqrt{1 - x^2}$. ▶

3. Solving the Equation $\tan t = m$. Arc Tangent. To solve the equation $\tan t = m$, it is necessary to find all real numbers t, such that the line passing through the origin and point P_t intersects the line AB': $x = 1$ at a point Z_t with ordinate equal to m (Fig. 21). The equation of the straight line passing through the origin and P_t is given by the formula $y = mx$. For an arbitrary real number m there are exactly two points of intersection of the line $y = mx$ with the trigonometric circle. One of these points lies in the right-hand half-plane and can be represented in the form $E = P_{t_0}$, where $t_0 \in (-\pi/2, \pi/2)$.

Definition. The *arc tangent* of a number m is a number t_0, lying on the open interval $(-\pi/2, \pi/2)$, such that $\tan t_0 = m$. It is denoted by $\arctan m$ (or $\tan^{-1} m$).

The general solution of the equation $\tan t = m$ (see Fig. 21) is:

$$t = \arctan m + \pi k, \ k \in \mathbf{Z}.$$

For all real values of m the following equalities hold:

$$\tan(\arctan m) = m,$$

$$\arctan(-m) = -\arctan m.$$

Arc cotangent is introduced in a similar way: for any $m \in \mathbf{R}$ the number $t = \operatorname{arccot} m$ is uniquely defined by two conditions: $0 < t < \pi$, $\cot t = m$.

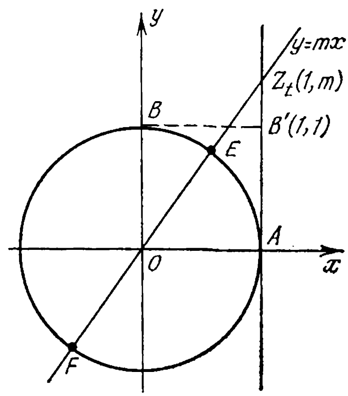

Fig. 21

The general solution of the equation $\cot t = m$ is

$$t = \operatorname{arccot} m + \pi k, \quad k \in \mathbf{Z}.$$

The following identities occur:

$$\cot(\operatorname{arccot} m) = m, \quad \operatorname{arccot}(-m) = \pi - \operatorname{arccot} m.$$

Example 1.4.5. Prove that $\arctan(-2/5) = \operatorname{arccot}(-5/2) - \pi$.

◀ Let $t = \operatorname{arccot}(-5/2)$, then $0 < t < \pi$, $\cot t = -5/2$, and $\tan t = -2/5$. The point P_t lies in the second quadrant, consequently, the point $P_{t-\pi}$ lies in the fourth quadrant, $\tan(t - \pi) = \tan t = -2/5$, and the condition $-\pi/2 < t - \pi < \pi/2$ is satisfied. Consequently, the number $t - \pi$ satisfies both conditions defining arc tangent, i.e.

$$t - \pi = \arctan(-2/5) = \operatorname{arccot}(-5/2) - \pi. \blacktriangleright$$

Example 1.4.6. Prove the identity

$$\sin(\arctan x) = x/\sqrt{1+x^2}.$$

◀ Let $t = \arctan x$. Then $\tan t = x$, and $-\pi/2 < t < \pi/2$. Let us prove that $\sin t = x/\sqrt{1+x^2}$. For this purpose let us note that $\cos t > 0$ since $-\pi/2 < t < \pi/2$. By virtue of (1.10), $1/\cos^2 t = 1 + \tan^2 t = 1 + x^2$ or $\cos t = 1/\sqrt{1+x^2}$, whence $\sin t = \tan t \cos t = x\sqrt{1+x^2}$. ▶

It is clear from the above examples that when solving problems, it is convenient to use more formal definition of inverse trigonometric functions, for instance: $t = \arccos m$ if (1) $\cos t = m$, and (2) $0 \leqslant t \leqslant \pi$. To solve problems on computation involving inverse trigonometric functions, it suffices to remember the above definitions and basic trigonometric formulas.

PROBLEMS

1.1. Locate the points by indicating the quadrants: (a) P_{11}, (b) $P_{10-\sqrt[3]{2}}$, (c) $P_{\sqrt[3]{26}+\sqrt{2}}$.

1.2. Given a regular pentagon inscribed in the trigonometric circle with vertices $A_k = P_{2\pi k/5}$, $k = 0, 1, 2, 3, 4$. Find on which of the arcs joining two neighbouring vertices the following points lie: (a) P_{10}, (b) P_{-11}, (c) P_{12}.

1.3. Given a regular heptagon inscribed in the trigonometric circle with vertices $B_k = P_{2+2\pi k/7}$, $k = 0, 1, 2, 3, 4, 5, 6$. Find on which of the arcs joining the neighbouring vertices the point $P_{\sqrt{29}}$ lies.

1.4. Prove that there is a regular N-gon inscribed in the trigonometric circle such that its vertices include the points $P_{\sqrt{2}+4\pi/11}$, $P_{\sqrt{2}+7\pi/13}$. Find the least possible number N.

1.5. Prove that any two points of the form $P_{\pi x}$, $P_{\pi y}$, where x, y are rational numbers, are vertices of a regular N-gon inscribed in the trigonometric circle.

1.6. Find a necessary and sufficient condition for the points P_α and P_β to be vertices of one and the same regular N-gon inscribed in the trigonometric circle.

1.7. Compare the following pairs of numbers:

(a) $\sin 1$ and $\sin\left(1+\frac{2\pi}{5}\right)$,

(b) $\cos\left(1+\frac{2\pi}{5}\right)$ and $\cos\left(1+\frac{4\pi}{5}\right)$.

1.8. Determine the sign of the indicated number:

(a) $\cos\left(1+\frac{6\pi}{5}\right)$,

(b) $\cos\left(1+\frac{2\pi}{5}\right)\cos\left(1+\frac{4\pi}{5}\right)\cos\left(1+\frac{6\pi}{5}\right)\times$
$\cos\left(1+\frac{8\pi}{5}\right)$.

1.9. Can the sine of an angle be equal to:

(a) $\log a+\frac{1}{\log a}$ $(a>0,\ a\neq 1)$,

(b) $\left(\frac{\sqrt{3}-1}{\sqrt{5}-\sqrt{3}}\right)^{-1}$?

1.10. Determine the sign of the product $\sin 2\cdot\sin 3\cdot\sin 5$.

1.11. Evaluate:

(a) $\sin\frac{1001\pi}{6}$, (b) $\cos\frac{123\pi}{4}$, (c) $\sin(-117\pi/4)$,

(d) $\cos(-205\pi/6)$.

1.12. Determine the sign of the number $\tan 11$.

1.13. Evaluate:

(a) $\tan\frac{1011\pi}{4}$, (b) $\cot\frac{1001\pi}{6}$.

1.14. Prove that for an arbitrary real number $a\in\mathbf{R}$ and an integer $N>1$ the following equalities are valid:

$$\sum_{k=0}^{N-1}\sin\left(a+\frac{2\pi k}{N}\right)=0,\quad \sum_{k=0}^{N-1}\cos\left(a+\frac{2\pi k}{N}\right)=0.$$

1.15. Prove that the function $f(t)=\tan\frac{11t}{34}+\cot\frac{13t}{54}$ is periodic and find its fundamental period.

1.16. Is the function $f(t)=\sin(2x+\cos(\sqrt{2}\,x))$ periodic?

1.17. Prove that the function $f(x)=\cos(x^2)$ is not periodic.

1.18. Prove that the function $\tan(\sqrt{2}x)+\cot(\sqrt{3}x)$ is not periodic.

1.19. Prove that the function $f(t) = \sin 3t + \cos 5t$ is periodic and find its fundamental period.

1.20. Prove that the function $f(x) = \cos(\sqrt[3]{|x|^2})$ is not periodic.

1.21. Find the fundamental period of the function:

(a) $y = \cos \pi x + \sin \frac{\pi x}{3}$,

(b) $y = \sin x + \cos \frac{x}{3} + \tan \frac{x}{10}$.

1.22. Find the fundamental period of the function $y = 15 \sin^2 12x + 12 \sin^2 15x$.

1.23. Prove that the function of the form $f(x) = \cos(ax + \sin(bx))$, where a and b are real nonzero numbers, is periodic if and only if the number a/b is rational.

1.24. Prove that the function of the form $f(x) = \cos(ax) + \tan(bx)$, where a and b are real nonzero numbers, is periodic if and only if the ratio a/b is a rational number.

1.25. Prove that the function $y = \tan 5x + \cot 3x + 4 \sin x \cos 2x$ is odd.

1.26. Prove that the function $y = \cos 4x + \sin^3 \frac{x}{2} \times \tan x + 6x^2$ is even.

1.27. Represent the function $y = \sin(x + 1) \sin^3(2x - 3)$ as a sum of an even and an odd function.

1.28. Represent the function $y = \cos\left(x + \frac{\pi}{8}\right) + \sin\left(2x - \frac{\pi}{12}\right)$ as a sum of an even and an odd function.

1.29. Find all the values of the parameters a and b for which (a) the function $f(t) = a \sin t + b \cos t$ is even, (b) the function $f(t) = a \cos t + b \sin t$ is odd.

In Problems **1.30** to **1.32**, without carrying out computations, determine the sign of the given difference.

1.30. (a) $\sin \frac{2\pi}{9} - \sin \frac{10\pi}{9}$, (b) $\cos 3.13 - \sin 3.13$.

1.31. (a) $\sin 1 - \sin 1.1$, (b) $\sin 2 - \sin 2.1$,
(c) $\sin 131° - \sin 130°$, (d) $\sin 200° - \sin 201°$.

1.32. (a) $\cos 71° - \cos 72°$, (b) $\cos 1 - \cos 0.9$,
(c) $\cos 100° - \cos 99°$, (d) $\cos 3.4 - \cos 3.5$.

1.33. Is the function $\cos(\sin t)$ increasing or decreasing on the closed interval $[-\pi/2, 0]$?

1.34. Is the function $\sin(\cos t)$ increasing or decreasing on the closed interval $[\pi, 3\pi/2]$?

1.35. Prove that the function $\tan(\cos t)$ is decreasing on the closed interval $[0, \pi/2]$.

1.36. Is the function $\cos(\sin(\cos t))$ increasing or decreasing on the closed interval $[\pi/2, \pi]$?

In Problems **1.37** to **1.40**, given the value of one function, find the values of other trigonometric functions.

1.37. (a) $\sin t = 4/5$, $\pi/2 < t < \pi$,
(b) $\sin t = -5/13$, $\pi < t < 3\pi/2$,
(c) $\sin t = -0.6$, $-\pi/2 < t < 0$.

1.38. (a) $\cos t = 7/25$, $0 < t < \pi/2$,
(b) $\cos t = -24/25$, $\pi < t < 3\pi/2$,
(c) $\cos t = 15/17$, $3\pi/2 < t < 2\pi$.

1.39. (a) $\tan t = 3/4$, $0 < t < \pi/2$,
(b) $\tan t = -3/4$, $\pi/2 < t < \pi$.

1.40. (a) $\cot t = 12/5$, $\pi < t < 3\pi/2$,
(b) $\cot t = -5/12$, $3\pi/2 < t < 2\pi$.

1.41. Solve the given equation:
(a) $2\cos^2 t - 5\cos t + 2 = 0$, (b) $6\cos^2 t + \cos t - 1 = 0$.

1.42. Find the roots of the equation $\cos t = -1/2$ belonging to the closed interval $[-2\pi, 6\pi]$.

1.43. Solve the equations:
(a) $\tan t = 0$, (b) $\tan t = 1$, (c) $\tan 2t = \sqrt{3}$,
(d) $\tan 2t = -\sqrt{3}$, (e) $\tan\left(t - \frac{\pi}{4}\right) - 1 = 0$,
(f) $\sqrt{3}\tan\left(t + \frac{\pi}{6}\right) = 1$.

1.44. Compute:
(a) $\arcsin 0 + \arccos 0 + \arctan 0$,
(b) $\arcsin\frac{1}{2} + \arccos\frac{\sqrt{3}}{2} + \arctan\frac{\sqrt{3}}{3}$,
(c) $\arcsin\frac{\sqrt{3}}{2} + \arccos\left(-\frac{\sqrt{3}}{2}\right) - \arctan\left(-\frac{\sqrt{3}}{3}\right)$.

In Problems **1.45** to **1.47**, prove the identities.

1.45. (a) $\tan|\arctan x| = |x|$, (b) $\cos(\arctan x) = 1/\sqrt{1 + x^2}$.

1.46. (a) $\cot |\operatorname{arccot} x| = x$.
(b) $\tan(\operatorname{arccot} x) = 1/x$ if $x \neq 0$,
(c) $\sin(\operatorname{arccot} x) = 1/\sqrt{1+x^2}$,
(d) $\cos(\operatorname{arccot} x) = x/\sqrt{1+x^2}$.

1.47. (a) $\arcsin x = \arctan \dfrac{x}{\sqrt{1-x^2}}$ and $\arccos x = \operatorname{arccot} \dfrac{x}{\sqrt{1-x^2}}$ for $0 \leqslant x < 1$,

(b) $\arcsin x = \operatorname{arccot} \dfrac{\sqrt{1-x^2}}{x}$ for $0 < x \leqslant 1$,

(c)
$$\arctan \frac{1}{x} = \operatorname{arccot} x = \arcsin \frac{1}{\sqrt{1+x^2}} = \arccos \frac{x}{\sqrt{1+x^2}},$$
$$\operatorname{arccot} \frac{1}{x} = \arctan x = \arcsin \frac{x}{\sqrt{1+x^2}} = \arccos \frac{1}{\sqrt{1+x^2}}$$
for $x > 0$.

1.48. Express: (a) $\arcsin \frac{3}{5}$, (b) $\arccos \frac{12}{13}$, (c) $\arctan \frac{5}{12}$, (d) $\arctan \frac{3}{4}$ in terms of values of each of the three other inverse trigonometric functions.

1.49. Express: (a) $\arccos\left(-\frac{1}{3}\right)$, (b) $\arctan\left(-\frac{7}{24}\right)$, (c) $\operatorname{arccot}\left(-\frac{7}{24}\right)$ in terms of values of each of the three other inverse trigonometric functions.

1.50. Find $\sin \alpha$ if $\tan \alpha = 2$ and $\pi < \alpha < 3\pi/2$.

Chapter 2

Identical Transformations of Trigonometric Expressions

2.1. Addition Formulas

There are many trigonometric formulas. Most cause difficulties to school-pupils and those entering college. Note that the two of them, most important formulas, are derived geometrically. These are the fundamental trigonometric identity $\sin^2 t + \cos^2 t = 1$ and the formula for the cosine of the sum (difference) of two numbers which is considered in this section. Note that the basic properties of trigonometric functions from Sec. 1.3 (periodicity, evenness and oddness, monotonicity) are also obtained from geometric considerations. Any of the remaining trigonometric formulas can be easily obtained if the student well knows the relevant definitions and the properties of the fundamental trigonometric functions, as well as the two fundamental trigonometric formulas. For instance, formulas (1.10) and (1.11) from Item 4 of Sec. 1.3 relating $\cos t$ and $\tan t$, and also $\sin t$ and $\cot t$ are not fundamental; they are derived from the fundamental trigonometric identity and the definitions of the functions $\tan t$ and $\cot t$. The possibility of deriving a variety of trigonometric formulas from a few fundamental formulas is a certain convenience, but requires an attentive approach to the logic of proofs. At the same time, such subdivision of trigonometric formulas into fundamental and nonfundamental (derived from the fundamental ones) is conventional. The student should also remember that among the derived formulas there are certain formulas which are used most frequently, e.g. double-argument and half-argument formulas, and also the formulas for transforming a product into a sum. To solve problems successfully, these formulas should be kept in mind so that they can be put to use straight away. A good technique to memorize such formulas consists in following attentively the way they are derived and solving a certain number of problems pertaining to identical transformations of trigonometric expressions.

1. The Cosine of the Sum and Difference of Two Real Numbers. One should not think that there are several basic addition formulas. We are going to derive the formula for the cosine of the sum of two real numbers and

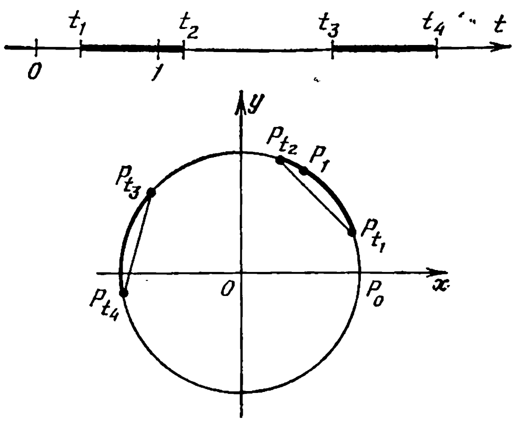

Fig. 22

then show that other addition formulas are derived from it provided that the properties of evenness and oddness of the basic trigonometric functions are taken into consideration.

To prove this formula, we shall need the following note. Under the trigonometric mapping

$$P\colon \mathbf{R} \to S$$

of the real axis **R** onto the trigonometric circle S (see Item 3, Sec. 1.1), line segments of equal length go into arcs of equal size. More precisely, this means the following. Let on the number line be taken four points: t_1, t_2, t_3, t_4 such that the distance from t_1 to t_2 is equal to the distance from t_3 to t_4, that is, such that $|t_1 - t_2| = |t_3 - t_4|$, and let P_{t_1}, P_{t_2}, P_{t_3}, P_{t_4} be points on the coordinate circle corresponding to those points. Then the arcs $P_{t_1}P_{t_2}$ and $P_{t_3}P_{t_4}$ are congruent (Fig. 22). Hence it follows that the chords $P_{t_1}P_{t_2}$ and $P_{t_3}P_{t_4}$ are also congruent: $|P_{t_1}P_{t_2}| = |P_{t_3}P_{t_4}|$.

Theorem 2.1. *For any real numbers t and s the following identity is valid*:

$$\cos(t+s) = \cos t \cos s - \sin t \sin s. \qquad (2.1)$$

Proof. Consider the point P_0 of intersection of the unit circle and the positive direction of the x-axis, $P_0 = (1, 0)$. Then the corresponding points P_t, P_{t+s}, and P_{-s} are images of the point P_0 when the plane is rotated about the origin through angles of t, $t+s$, and $-s$ radians (Fig. 23).

By the definition of the sine and cosine, the coordinates of the points P_t, P_{t+s}, and P_{-s} will be:

$$P_t = (\cos t, \sin t),$$

$$P_{t+s} = (\cos(t+s), \sin(t+s)),$$

$$P_{-s} = (\cos(-s), \sin(-s)) = (\cos s, -\sin s).$$

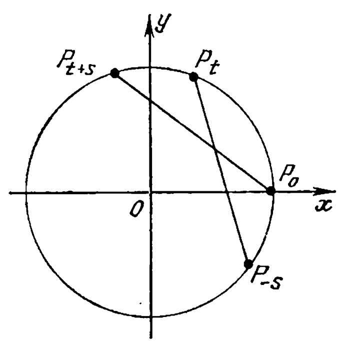

Fig. 23

Here we have also used the basic properties of the evenness of cosine and oddness of sine. We noted before the proof of the theorem that the lengths of the line segments P_0P_{t+s} and $P_{-s}P_t$ are equal, hence we can equate the squares of their lengths. Thus, we get identity (2.1). Knowing the coordinates of the points P_0 and P_{t+s}, we can find the square of the length of the line segment P_0P_{t+s}: $|P_0P_{t+s}|^2 = (1-\cos(t+s))^2 + \sin^2(t+s) = 1 - 2\cos(t+s) + \cos^2(t+s) + \sin^2(t+s)$ or, by virtue of the fundamental trigonometric identity,

$$|P_0P_{t+s}|^2 = 2 - 2\cos(t+s).$$

On the other hand,

$$\begin{aligned}|P_{-s}P_t|^2 &= (\cos s - \cos t)^2 + (-\sin s - \sin t)^2 \\ &= \cos^2 s - 2\cos s \cos t + \cos^2 t \\ &+ \sin^2 s + 2\sin s \cdot \sin t + \sin^2 t \\ &= 2 - 2\cos t \cos s + 2\sin t \sin s.\end{aligned}$$

Here we have used the fundamental trigonometric identity (1.9) once again. Consequently, from the equality $|P_0P_{t+s}|^2 = |P_{-s}P_t|^2$ we get

$$2 - 2\cos(t+s) = 2 - 2\cos t\cos s + 2\sin t\sin s,$$

whence (2.1) follows. ▶

Corollary 1. *For any real numbers t and s we have*

$$\cos(t-s) = \cos t\cos s + \sin t\sin s. \qquad (2.2)$$

Proof. Let us represent the number $t - s$ as $(t + (-s))$ and apply Theorem 2.1:

$$\begin{aligned}\cos(t-s) &= \cos(t+(-s)) \\ &= \cos t\cos(-s) - \sin t\sin(-s).\end{aligned}$$

Taking advantage of the properties of the evenness of cosine and oddness of sine, we get:

$$\cos(t-s) = \cos t\cos s + \sin t\sin s. \ ▶$$

In connection with the proof of identities (2.1) and (2.2), we should like to note that it is necessary to make sure that in deriving a certain identity we do not rely on another identity which, in turn, is obtained from the identity under consideration. For instance, the property of the evenness of cosine is sometimes proved as follows:

$$\cos(-t) = \cos(0 - t) = \cos 0 \cos t + \sin 0 \sin t = \cos t,$$

that is, relying on the addition formula (2.2). When deriving formula (2.2), we use the property of the evenness of the function $\cos t$. Therefore the mentioned proof of the evenness of $\cos t$ may be recognized as correct only provided that the student can justify the addition formula $\cos(t - s)$ without using this property of cosine.

Example 2.1.1. Compute $\cos\frac{\pi}{12} = \cos 15°$.

◀Since $\frac{\pi}{12}=\frac{\pi}{3}-\frac{\pi}{4}$, we get from formula (2.2) for $t=\pi/3$ and $s=\pi/4$:

$$\cos\frac{\pi}{12}=\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}+\sqrt{6}}{4}. \ ▶$$

2. Reduction Formulas.

Corollary 2. *For any real number t we have*

$$\cos\left(\frac{\pi}{2}-t\right)=\sin t, \qquad (2.3)$$

$$\cos\left(\frac{\pi}{2}+t\right)=-\sin t, \qquad (2.4)$$

$$\cos(\pi-t)=-\cos t, \qquad (2.5)$$

$$\cos(\pi+t)=-\cos t. \qquad (2.6)$$

Proof. Let us use formula (2.2):

$$\cos\left(\frac{\pi}{2}-t\right)=\cos\frac{\pi}{2}\cos t+\sin\frac{\pi}{2}\sin t$$
$$=0\cdot\cos t+1\cdot\sin t=\sin t,$$
$$\cos(\pi-t)=\cos\pi\cos t+\sin\pi\sin t$$
$$=(-1)\cdot\cos t-0\cdot\sin t=-\cos t,$$

which yields identities (2.3) and (2.5). We now use (2.1):

$$\cos\left(\frac{\pi}{2}+t\right)=\cos\frac{\pi}{2}\cos t-\sin\frac{\pi}{2}\sin t$$
$$=0\cdot\cos t-1\cdot\sin t=-\sin t,$$
$$\cos(\pi+t)=\cos\pi\cos t-\sin\pi\sin t$$
$$=(-1)\cdot\cos t-0\cdot\sin t=-\cos t.$$

These equalities prove identities (2.4) and (2.6). ▶

Corollary 3. *For any real number t we have*

$$\sin\left(\frac{\pi}{2}-t\right)=\cos t, \qquad (2.7)$$

$$\sin\left(\frac{\pi}{2}+t\right)=\cos t, \qquad (2.8)$$

$$\sin(\pi-t)=\sin t, \qquad (2.9)$$

$$\sin(\pi+t)=-\sin t. \qquad (2.10)$$

Proof. Let us use identity (2.3):

$$\cos t = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - t\right)\right) = \sin\left(\frac{\pi}{2} + t\right),$$

which yields identity (2.7). From identity (2.3) and the property of evenness of cosine there follows:

$$\cos t = \cos(-t) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} + t\right)\right) = \sin\left(\frac{\pi}{2} + t\right),$$

which proves identity (2.8).

Further, by virtue of (2.8) and (2.3), we have:

$$\sin(\pi - t) = \sin\left(\frac{\pi}{2} + \left(\frac{\pi}{2} - t\right)\right) = \cos\left(\frac{\pi}{2} - t\right) = \sin t,$$

and from identities (2.8) and (2.4) we get:

$$\sin(\pi + t) = \sin\left(\frac{\pi}{2} + \left(\frac{\pi}{2} + t\right)\right)$$
$$= \cos\left(\frac{\pi}{2} + t\right) = -\sin t. \quad \blacktriangleright$$

Remark. Using formulas (2.3)-(2.10), we can easily obtain reduction formulas for $\cos\left(\frac{3\pi}{2} \pm t\right)$, $\cos(2\pi - t)$, $\sin\left(\frac{3\pi}{2} \pm t\right)$, $\sin(2\pi - t)$. The reader is invited to do this as an exercise.

Example 2.1.2. Derive the reduction formula for $\cos\left(\frac{3\pi}{2} - t\right)$.

◀ Using formulas (2.6) and (2.3), we have

$$\cos\left(\frac{3\pi}{2} - t\right) = \cos\left(\pi + \left(\frac{\pi}{2} - t\right)\right) = -\cos\left(\frac{\pi}{2} - t\right)$$
$$= -\sin t. \quad \blacktriangleright$$

Corollary 4. *The reduction formulas for tangent and cotangent are*

$$\tan\left(\frac{\pi}{2} - t\right) = \cot t, \quad \tan\left(\frac{\pi}{2} + t\right) = -\cot t, \quad t \neq \pi k,$$

$$\cot\left(\frac{\pi}{2} - t\right) = \tan t, \quad \cot\left(\frac{\pi}{2} + t\right) = -\tan t,$$

$$t \neq \frac{\pi}{2} + \pi k,$$

$$\tan(\pi - t) = -\tan t, \quad t \neq \frac{\pi}{2} + \pi k,$$

$$\cot(\pi - t) = -\cot t, \quad t \neq \pi k \quad (k \in \mathbf{Z}).$$

Proof. All these formulas can be obtained from the definition of the functions $\tan t$ and $\cot t$ applying the appropriate reduction formulas. ▶

Note that the reduction formulas together with the periodicity properties of the basic trigonometric functions make it possible to reduce the computation of the value of a trigonometric function at any point to its computation at a point in the closed interval $[0, \pi/4]$.

To facilitate the memorizing of the reduction formulas, the following mnemonic rule is recommended:

(1) Assuming that $0 < t < \frac{\pi}{2}$, find in which quadrant the point $P_{\frac{\pi k}{2} \pm t}$, $k \in \mathbf{Z}$, lies, determine the sign the given expression has in this quadrant, and put this sign before the obtained result.

(2) When replacing the argument $\pi \pm t$ or $2\pi - t$ by t, the name of the function is retained.

(3) When replacing the argument $\frac{\pi}{2} \pm t$ or $\frac{3\pi}{2} \pm t$ by t, the name of the function should be changed: sine for cosine and cosine for sine, tangent for cotangent and cotangent for tangent.

3. Sine of a Sum (a Difference).

Corollary 5. *For any real numbers t and s the following identities are valid*:

$$\sin (t + s) = \sin t \cos s + \cos t \sin s, \qquad (2.11)$$

$$\sin (t - s) = \sin t \cos s - \cos t \sin s. \qquad (2.12)$$

Proof. Using the reduction formula (2.3), we reduce sine to cosine:

$$\sin (t + s) = \cos \left(\frac{\pi}{2} - (t + s)\right) = \cos \left(\left(\frac{\pi}{2} - t\right) - s\right).$$

We now apply the formula for the cosine of a difference (2.2):

$$\sin (t + s) = \cos \left(\frac{\pi}{2} - t\right) \cos s + \sin \left(\frac{\pi}{2} - t\right) \sin s$$
$$= \sin t \cos s + \cos t \sin s$$

(in the last equality, we have used the reduction formulas (2.3) and (2.7)). To prove formula (2.12), it suffices to represent the difference $t - s$ as $t + (-s)$ and use the proved identity (2.11) and the properties of the evenness of cosine and oddness of sine:

$$\sin(t - s) = \sin(t + (-s)) = \sin t \cos(-s) + \cos t \sin(-s) = \sin t \cos s - \cos t \sin s. \blacktriangleright$$

4. Tangent of a Sum (a Difference).

Corollary 6. *For any real numbers t and s, except for $t = \frac{\pi}{2} + \pi k$, $s = \frac{\pi}{2} + \pi m$, $t + s = \frac{\pi}{2} + \pi n$ $(k, m, n \in \mathbf{Z})$, the following identity holds*:

$$\tan(t + s) = \frac{\tan t + \tan s}{1 - \tan t \tan s}. \qquad (2.13)$$

Remark. The domains of permissible values of the arguments t and s are different for the right-hand and left-hand sides of (2.13). Indeed, the left-hand side is defined for all the values of t and s except $t + s = \frac{\pi}{2} + \pi n$, $n \in \mathbf{Z}$, while the right-hand side only for the values of t and s mentioned in Corollary 6. Thus, for $t = \frac{\pi}{2}$ and $s = \frac{\pi}{4}$ the left-hand side is defined, while the right-hand side is not.

Proof. Let us apply the definition of tangent and formulas (2.11) and (2.1):

$$\tan(t + s) = \frac{\sin(t+s)}{\cos(t+s)} = \frac{\sin t \cos s + \cos t \sin s}{\cos t \cos s - \sin t \sin s}.$$

Since $\cos t \neq 0$ and $\cos s \neq 0$ (by hypothesis), both the numerator and denominator of the fraction can be divided by $\cos t \cos s$, then

$$\tan(t + s) = \frac{\frac{\sin t}{\cos t} + \frac{\sin s}{\cos s}}{1 - \frac{\sin t}{\cos t} \frac{\sin s}{\cos s}} = \frac{\tan t + \tan s}{1 - \tan t \tan s}. \blacktriangleright$$

Corollary 7. *For any real t and s, except for $t = \frac{\pi}{2} + \pi k$, $s = \frac{\pi}{2} + \pi m$, $t - s = \frac{\pi}{2} + \pi n$ $(k, m, n \in \mathbf{Z})$, the* *following identity holds*:

$$\tan(t-s)=\frac{\tan t-\tan s}{1+\tan t\tan s}. \qquad (2.14)$$

Remark. As in (2.13), the domains of permissible values of the arguments of the right-hand and left-hand sides of the identity are different.

Proof. It suffices to replace s by $-s$ in (2.13) and make use of the oddness property of tangent. ▶

Example 2.1.3. Find $\tan\left(\frac{\pi}{4}+t\right)$ if $\tan t=\frac{3}{4}$.

◀ Use formula (2.13), bearing in mind that $\tan\frac{\pi}{4}=1$. We have

$$\tan\left(\frac{\pi}{4}+t\right)=\frac{\tan\frac{\pi}{4}+\tan t}{1-\tan\frac{\pi}{4}\tan t}=\frac{1+\tan t}{1-\tan t}$$

$$=\frac{1+\frac{3}{4}}{1-\frac{3}{4}}=7. \quad ▶$$

Example 2.1.4. Evaluate $\tan\left(\arcsin\frac{3}{5}+\arccos\frac{5}{13}\right)$.

◀ Let $t=\arcsin\frac{3}{5}$, $s=\arccos\frac{5}{13}$. Then, by the definition of inverse trigonometric functions (Sec. 1.4), we have:

$$\sin t = 3/5, \quad 0<t<\pi/2,$$
$$\cos s = 5/13, \quad 0<s<\pi/2.$$

Let us now find $\tan t$ and $\tan s$, noting that $\tan t>0$ and $\tan s>0$:

$$\tan^2 t=\frac{\sin^2 t}{1-\sin^2 t}=\frac{9}{16}=\left(\frac{3}{4}\right)^2,$$
$$\tan^2 s=\frac{1}{\cos^2 s}-1=\left(\frac{13}{5}\right)^2-1=\left(\frac{12}{5}\right)^2$$

(we have used formulas (1.9) and (1.10)). Therefore $\tan t=\frac{3}{4}$, $s=\frac{12}{5}$; and we may use formula (2.13) for the tangent of a sum:

$$\tan(t+s)=\frac{\tan t+\tan s}{1-\tan t\tan s}=\frac{\frac{3}{4}+\frac{12}{5}}{1-\frac{3}{4}\cdot\frac{12}{5}}$$

$$=\frac{15+48}{20-36}=-\frac{63}{16}. \quad \blacktriangleright$$

Example 2.1.5. Evaluate

$$\tan\left(\arccos\left(-\frac{7}{25}\right)+\arcsin\left(-\frac{12}{35}\right)\right).$$

◀ We use the properties of inverse trigonometric functions (see Items 1 and 2 of Sec. 1.4)

$$\arccos\left(-\frac{7}{25}\right)=\pi-\arccos\frac{7}{25},$$

$$\arcsin\left(-\frac{12}{13}\right)=-\arcsin\frac{12}{13},$$

and set $t=\arccos\frac{7}{25}$, $s=\arcsin\frac{12}{13}$ to get

$$\tan\left(\arccos\left(-\frac{7}{25}\right)+\arcsin\left(-\frac{12}{13}\right)\right)$$

$$=\tan(\pi-t-s)=-\tan(t+s),$$

$$\cos t=7/25, \qquad 0<t<\pi/2,$$

$$\sin s=12/13, \qquad 0<s<\pi/2.$$

Proceeding as in the preceding example we have: $\tan t=\frac{24}{7}$ and $\tan s=\frac{12}{5}$,

$$-\tan(t+s)=-\frac{\frac{24}{7}+\frac{12}{5}}{1-\frac{24}{7}\cdot\frac{12}{5}}=\frac{24\cdot 5+12\cdot 7}{24\cdot 12-5\cdot 7}=\frac{204}{253}. \quad \blacktriangleright$$

5. Cotangent of a Sum (a Difference).

Corollary 8. *For any real numbers t and s, except* $t=\pi k$, $s=\pi m$, $t+s=\pi n$ $(k, m, n\in \mathbf{Z})$, *the following identity holds true*:

$$\cot(t+s)=\frac{\cot t\cot s-1}{\cot t+\cot s}. \qquad (2.15)$$

The *proof* follows from the definition of cotangent and formulas (2.1) and (2.11):

$$\cot(t+s) = \frac{\cot(t+s)}{\sin(t+s)} = \frac{\cos t \cos s - \sin t \sin s}{\sin t \cos s + \cos t \sin s}.$$

Since the product $\sin t \sin s$ is not equal to zero since $t \neq \pi k$, $s \neq \pi m$, $(k, m \in \mathbf{Z})$, both the numerator and denominator of the fraction may be divided by $\sin t \sin s$. Then

$$\cot(t+s) = \frac{\frac{\cos t}{\sin t} \cdot \frac{\cos s}{\sin s} - 1}{\frac{\cos t}{\sin t} + \frac{\cos s}{\sin s}} = \frac{\cot t \cot s - 1}{\cot t + \cot s}. \quad \blacktriangleright$$

Corollary 9. *For any real values of t and s, except $t = \pi k$, $s = \pi m$, and $t - s = \pi n$ $(k, m, n \in \mathbf{Z})$, the following identity is valid*:

$$\cot(t-s) = \frac{\cot t \cot s + 1}{\cot t - \cot s}. \tag{2.16}$$

The *proof* follows from identity (2.15) and oddness property of cotangent. ▶

6. Formulas of the Sum and Difference of Like Trigonometric Functions. The formulas to be considered here involve the transformation of the sum and difference of like trigonometric functions (of different arguments) into a product of trigonometric functions. These formulas are widely used when solving trigonometric equations to transform the left-hand side of an equation, whose right-hand side is zero, into a product. This done, the solution of such equations is usually reduced to solving elementary trigonometric equations considered in Chapter 1. All these formulas are corollaries of Theorem 2.1 and are frequently used.

Corollary 10. *For any real numbers t and s the following identities are valid*:

$$\sin t + \sin s = 2 \sin \frac{t+s}{2} \cos \frac{t-s}{2}, \tag{2.17}$$

$$\sin t - \sin s = 2 \sin \frac{t-s}{2} \cos \frac{t+s}{2}. \tag{2.18}$$

The *proof* is based on the formulas of the sine of a sum and a difference. Let us write the number t in the following form: $t=\frac{t+s}{2}+\frac{t-s}{2}$, and the number s in the form $s=\frac{t+s}{2}-\frac{t-s}{2}$, and apply formulas (2.11) and (2.12):

$$\sin t=\sin\frac{t+s}{2}\cos\frac{t-s}{2}+\cos\frac{t+s}{2}\sin\frac{t-s}{2}, \qquad (2.19)$$

$$\sin s=\sin\frac{t+s}{2}\cos\frac{t-s}{2}-\cos\frac{t+s}{2}\sin\frac{t-s}{2}. \qquad (2.20)$$

Adding equalities (2.19) and (2.20) termwise, we get identity (2.17), and, subtracting (2.20) from (2.19), we get identity (2.18). ▶

Corollary 11. *For any real numbers t and s the following identities hold*:

$$\cos t+\cos s=2\cos\frac{t+s}{2}\cos\frac{t-s}{2}, \qquad (2.21)$$

$$\cos t-\cos s=-2\sin\frac{t+s}{2}\sin\frac{t-s}{2}. \qquad (2.22)$$

The *proof* is very much akin to that of the preceding corollary. First, represent the numbers t and s as

$$t=\frac{t+s}{2}+\frac{t-s}{2}, \quad s=\frac{t+s}{2}-\frac{t-s}{2},$$

and then use formulas (2.1) and (2.2):

$$\cos t=\cos\frac{t+s}{2}\cos\frac{t-s}{2}-\sin\frac{t+s}{2}\sin\frac{t-s}{2}, \qquad (2.23)$$

$$\cos s=\cos\frac{t+s}{2}\cos\frac{t-s}{2}+\sin\frac{t+s}{2}\sin\frac{t-s}{2}. \qquad (2.24)$$

Adding equalities (2.23) and (2.24) termwise, we get identity (2.21). Identity (2.22) is obtained by subtracting (2.24) from (2.23) termwise.

Corollary 12. *For any real values of t and s, except* $t=\frac{\pi}{2}+\pi k$, $s=\frac{\pi}{2}+\pi n$ $(k, n\in\mathbf{Z})$, *the following identities are valid*:

$$\tan t+\tan s=\frac{\sin(t+s)}{\cos t\cos s}, \qquad (2.25)$$

$$\tan t-\tan s=\frac{\sin(t-s)}{\cos t\cos s}. \qquad (2.26)$$

Proof. Let us use the definition of tangent:

$$\tan t+\tan s=\frac{\sin t}{\cos t}+\frac{\sin s}{\cos s}$$
$$=\frac{\sin t\cos s+\sin s\cos t}{\cos t\cos s}=\frac{\sin(t+s)}{\cos t\cos s}$$

(we have applied formula (2.11) for the sine of a sum). Similarly (using formula (2.12)), we get:

$$\tan t-\tan s=\frac{\sin t}{\cos t}-\frac{\sin s}{\cos s}$$
$$=\frac{\sin t\cos s-\sin s\cos t}{\cos t\cos s}=\frac{\sin(t-s)}{\cos t\cos s}. \quad ▶$$

7. Formulas for Transforming a Product of Trigonometric Functions into a Sum. These formulas are helpful in many cases, especially in finding derivatives and integrals of functions containing trigonometric expressions and in solving trigonometric inequalities and equations.

Corollary 13. *For any real values of t and s the following identity is valid*:

$$\sin t\cos s=\frac{1}{2}(\sin(t+s)+\sin(t-s)). \qquad (2.27)$$

Proof. Again, we use formulas (2.11) and (2.12):

$$\sin(t+s)=\sin t\cos s+\cos t\sin s,$$
$$\sin(t-s)=\sin t\cos s-\cos t\sin s.$$

Adding these equalities termwise and dividing both sides by 2, we get formula (2.27). ▶

Corollary 14. *For any real numbers t and s the following identities hold true*:

$$\cos t\cos s=\frac{1}{2}(\cos(t+s)+\cos(t-s)), \qquad (2.28)$$

$$\sin t\sin s=\frac{1}{2}(\cos(t-s)-\cos(t+s)). \qquad (2.29)$$

The *proof* of these identities becomes similar to that of the preceding corollary if we apply the formulas for the cosine of a difference and a sum:

$$\cos t\cos s+\sin t\sin s=\cos(t-s), \qquad (2.30)$$
$$\cos t\cos s-\sin t\sin s=\cos(t+s). \qquad (2.31)$$

Adding (2.30) and (2.31) termwise and dividing both sides of the equality by 2, we get identity (2.28). Similarly, identity (2.29) is obtained from the half-difference of equalities (2.30) and (2.31). ▶

8. Transforming the Expression $a \sin t + b \cos t$ by Introducing an Auxiliary Angle.

Theorem 2.2. *For any real numbers a and b such that $a^2 + b^2 \neq 0$ there is a real number φ such that for any real value of t the following identity is valid:*

$$a \sin t + b \cos t = \sqrt{a^2+b^2} \sin (t+\varphi). \qquad (2.32)$$

For φ, we may take any number such that

$$\cos \varphi = a/\sqrt{a^2+b^2}, \qquad (2.33)$$

$$\sin \varphi = b/\sqrt{a^2+b^2}. \qquad (2.34)$$

Proof. First, let us show that there is a number φ which simultaneously satisfies equalities (2.33) and (2.34). We define the number φ depending on the sign of the number b in the following way:

$$\begin{aligned} \varphi &= \arccos \frac{a}{\sqrt{a^2+b^2}} \quad \text{for} \quad b \geqslant 0, \\ \varphi &= -\arccos \frac{a}{\sqrt{a^2+b^2}} \quad \text{for} \quad b < 0. \end{aligned} \qquad (2.35)$$

The quantity $\arccos \frac{a}{\sqrt{a^2+b^2}}$ is defined since $\frac{|a|}{\sqrt{a^2+b^2}} \leqslant 1$ (see Item 2, Sec. 1.4). By definition, we have:

$$\cos \varphi = \cos (|\varphi|) = \cos \left(\arccos \frac{a}{\sqrt{a^2+b^2}} \right) = \frac{a}{\sqrt{a^2+b^2}}.$$

From the fundamental trigonometric identity it follows that

$$\sin^2 \varphi = 1 - \cos^2 \varphi = 1 - \frac{a^2}{a^2+b^2} = \frac{b^2}{a^2+b^2},$$

or $|\sin \varphi| = \frac{|b|}{\sqrt{a^2+b^2}}$. Note that the sign of $\sin \varphi$ coincides with the sign of φ since $|\varphi| \leqslant \pi$. The signs of the numbers φ and b also coincide, therefore

$$\sin \varphi = \frac{b}{\sqrt{a^2+b^2}}.$$

It remains to check that if the number φ satisfies the requirements (2.33) and (2.34), then equality (2.32) is fulfilled. Indeed, we have:

$$a \sin t + b \cos t = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} \sin t + \frac{b}{\sqrt{a^2+b^2}} \cos t\right)$$
$$= \sqrt{a^2+b^2}(\sin t \cos \varphi + \cos t \sin \varphi),$$

and, by virtue of (2.11), we get

$$a \sin t + b \cos t = \sqrt{a^2+b^2} \sin (t+\varphi). \quad \blacktriangleright$$

2.2. Trigonometric Identities for Double, Triple, and Half Arguments

1. Trigonometric Formulas of Double Argument.

Theorem 2.3. *For any real numbers* α *the following identities hold true*:

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha, \tag{2.36}$$
$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha, \tag{2.37}$$
$$\cos 2\alpha = 2 \cos^2 \alpha - 1, \tag{2.38}$$
$$\cos 2\alpha = 1 - 2 \sin^2 \alpha. \tag{2.39}$$

Proof. Applying formula (2.11) for the sine of the sum of two numbers, we get

$$\sin 2\alpha = \sin (\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$$
$$= 2 \sin \alpha \cdot \cos \alpha.$$

It we apply formula (2.1) for the cosine of a sum, then we get

$$\cos 2\alpha = \cos (\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha$$
$$= \cos^2 \alpha - \sin^2 \alpha.$$

Identities (2.38) and (2.39) follow from identity (2.37) we have proved and the fundamental trigonometric identity (1.9):

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$
$$= 2 \cos^2 \alpha - (\cos^2 \alpha + \sin^2 \alpha) = 2 \sin^2 \alpha - 1,$$
$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$
$$= (\cos^2 \alpha + \sin^2 \alpha) - 2 \sin^2 \alpha = 1 - 2 \sin^2 \alpha. \quad \blacktriangleright$$

Corollary 1. *For any real values of* α *the following identities are valid*:

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}, \tag{2.40}$$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}. \tag{2.41}$$

Proof. Formula (2.40) follows directly from identity (2.38). Analogously, formula (2.41) is obtained from identity (2.39). ▶

2. Expressing Trigonometric Functions in Terms of the Tangent of Half Argument (Universal Substitution Formulas). The formulas considered here are of great importance since they make it possible to reduce all basic trigonometric functions to one function, the tangent of half argument.

Corollary 2. *For any real number* α, *except* $\alpha = \pi + 2\pi n$, $n \in \mathbf{Z}$, *the following identities are valid*:

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}, \tag{2.42}$$

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}. \tag{2.43}$$

Remark. The domains of permissible values of the arguments on the right-hand and left-hand sides of (2.42) and (2.43) differ: the left-hand sides are defined for all the values of α, while the right-hand sides only for the α's which are indicated in the corollary.

Proof. By virtue of (2.36) and the fundamental trigonometric identity (1.9), we have:

$$\sin\alpha = \sin\left(2\cdot\frac{\alpha}{2}\right) = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}}.$$

By hypothesis, $\cos\frac{\alpha}{2}$ does not vanish, therefore both the numerator and denominator of the fraction may be divided by $\cos^2 \frac{\alpha}{2}$, whence

$$\sin \alpha = \frac{2 \frac{\sin \alpha}{\cos \alpha}}{1 + \frac{\sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}}} = \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}.$$

We now apply (2.37) and the fundamental trigonometric identity and get

$$\cos \alpha = \cos \left(2 \cdot \frac{\alpha}{2}\right) = \cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}$$

$$= \frac{\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\alpha}{2}} = \frac{1 - \frac{\sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}}}{1 + \frac{\sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}}} = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}. \blacktriangleright$$

Corollary 3. *For any real numbers* α, *except* $\alpha = \frac{\pi}{2} + \pi k$, $\alpha = \pi + 2\pi n$ $(k, n \in \mathbf{Z})$ *the following identity holds:*

$$\tan \alpha = \frac{2 \tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}}. \tag{2.44}$$

Proof. Note that, by hypothesis, $\cos \alpha$ does not vanish, and the condition of Corollary 2 is fulfilled. Consequently, we may use the definition of tangent and then divide termwise identity (2.42) by identity (2.43) to get

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} \div \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}}$$

$$= \frac{2 \tan \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}}. \blacktriangleright$$

Remark. This formula can also be derived from (2.13). Indeed,

$$\tan\alpha = \tan\left(\frac{\alpha}{2}+\frac{\alpha}{2}\right) = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}.$$

Corollary 4. *For any real numbers* α, *except* $\alpha = \pi n$ $(n \in \mathbf{Z})$, *the identity*

$$\cot\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}} \tag{2.45}$$

is fulfilled.

Proof. The values of the number α satisfy the requirements of Corollary 2 and $\sin\alpha$ does not vanish, therefore identity (2.45) follows directly from the definition of cotangent and identities (2.42) and (2.43):

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \div \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

$$= \frac{1=\tan^2\frac{\alpha}{2}}{2\tan\frac{\alpha}{2}}. \quad ▶$$

Example 2.2.1. Evaluate $\sin(2\arctan 5)$.

◀ Let us use formula (2.42). If we set $\alpha = 2\arctan 5$, then $0 < \alpha < \pi$ and $\alpha/2 = \arctan 5$, and, by virtue of (2.42), we have

$$\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2\tan(\arctan 5)}{1+(\tan(\arctan 5))^2} = \frac{2\cdot 5}{26} = \frac{5}{13}. \quad ▶$$

Example 2.2.2. Evaluate $\cos(2\arctan(-7))$.

◀ Let us denote $\alpha = 2\arctan(-7)$, then $\tan\frac{\alpha}{2} = -7$ and $-\pi < \alpha < 0$. Using formula (2.43), we get

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{1-49}{1+49} = -\frac{24}{25}. \quad ▶$$

Example 2.2.3. Evaluate tan (2 arctan 3).

◀Setting $\alpha = 2$ arctan 3, we get $\tan\frac{\alpha}{2} = 3 > 1$ and $\frac{\pi}{2} < \alpha < \pi$. Using formula (2.44), we get

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} = -\frac{2\cdot 3}{8} = -\frac{3}{4}. \blacktriangleright$$

3. Trigonometric Formulas of Half Argument.

Corollary 5. *For any real number α the following identities are valid:*

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}, \tag{2.46}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}, \tag{2.47}$$

where the sign depends on the quadrant in which the point $P_{\alpha/2}$ lies and coincides with the sign of the values contained on the left-hand sides of the equalities (in that quadrant).

Proof. Applying identity (2.40), we get

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2} = \frac{1+\cos\alpha}{2},$$

whence

$$\left|\cos\frac{\alpha}{2}\right| = \sqrt{\frac{1+\cos\alpha}{2}}.$$

To get rid of the modulus sign, the expression $\cos\frac{\alpha}{2}$ should be given the sign corresponding to the quadrant $\frac{\alpha}{2}$ lies in; whence follows formula (2.46). Similarly (2.41) yields equality (2.47). ▶ .

Corollary 6. *For any real number α, except $\alpha = \pi(2n + 1)$ $(n \in \mathbf{Z})$, the following identities are valid*

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \tag{2.48}$$

(*the sign before the radical depends on the quadrant the point* $P_{\alpha/2}$ *lies in*),

$$\tan\frac{\alpha}{2}=\frac{\sin\alpha}{1+\cos\alpha}. \tag{2.49}$$

Proof. Identity (2.48) is obtained if we take into consideration that $\cos\alpha\neq 0$ and divide identity (2.47) by (2.46) termwise. Further, by virtue of (2.36) and (2.40), we have:

$$\tan\frac{\alpha}{2}=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}=\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}=\frac{\sin\alpha}{1+\cos\alpha}. \ \blacktriangleright$$

Corollary 7. *For any real number* α, *except* $\alpha=\pi n$ $(n\in\mathbf{Z})$, *the following identity is valid*:

$$\tan\frac{\alpha}{2}=\frac{1-\cos\alpha}{\sin\alpha}. \tag{2.50}$$

Proof. In this case the condition $\sin\frac{\alpha}{2}\neq 0$ is fulfilled, therefore from (2.36) and (2.41) there follows

$$\tan\frac{\alpha}{2}=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}=\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$=2\,\frac{1-\cos(2(\alpha/2))}{\sin(2(\alpha/2))}=\frac{1-\cos\alpha}{\sin\alpha}. \ \blacktriangleright$$

Corollary 8. *For any real value of* α, *except* $\alpha=2\pi n$, $n\in\mathbf{Z}$, *the following identities hold true*:

$$\cot\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} \tag{2.51}$$

(*the sign before the radical depends on the quadrant the point* $P_{\alpha/2}$ *lies in*)

$$\cot\frac{\alpha}{2}=\frac{\sin\alpha}{1-\cos\alpha}. \tag{2.52}$$

Proof. Since by hypothesis $\sin\frac{\alpha}{2}\neq 0$, formula (2.51) is obtained after termwise division of (2.46) by (2.47).

From identities (2.36) and (2.41) it also follows that

$$\cot\frac{\alpha}{2}=\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}=\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}}=\frac{\sin\alpha}{1-\cos\alpha}. \blacktriangleright$$

Corollary 9. *For any real number α, except $\alpha=\pi n$, $n\in\mathbf{Z}$, the following identity is valid:*

$$\cot\alpha=\frac{1+\cos\alpha}{\sin\alpha}.$$

Proof. By virtue of (2.36) and (2.40), we have:

$$\cot\frac{\alpha}{2}=\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}=\frac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=\frac{1+\cos\alpha}{\sin\alpha}. \blacktriangleright$$

Example 2.2.4. Evaluate $\sin\frac{\pi}{8}$, $\cos\frac{\pi}{8}$, $\tan\frac{\pi}{8}$.

◀Let us first apply identities (2.46) and (2.47) with $\alpha=\frac{\pi}{4}$ and take the radical to be positive since $\frac{\alpha}{2}$ belongs to the first quadrant. We have:

$$\cos\frac{\pi}{8}=\sqrt{\frac{1+\cos\frac{\pi}{4}}{2}}=\frac{\sqrt{2+\sqrt{2}}}{2},$$

$$\sin\frac{\pi}{8}=\sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}=\frac{\sqrt{2-\sqrt{2}}}{2}.$$

Further,

$$\tan\frac{\pi}{8}=\frac{\sin\frac{\pi}{8}}{\cos\frac{\pi}{8}}=\frac{\sqrt{2-\sqrt{2}}}{2+\sqrt{2}}$$

$$=\sqrt{\frac{(2-\sqrt{2})^2}{(2+\sqrt{2})(2-\sqrt{2})}}=\sqrt{2}-1. \blacktriangleright$$

Example 2.2.5. Evaluate $\cos\left(\frac{1}{2}\arccos\left(-\frac{1}{10}\right)\right)$.

◀Let $\alpha=\arccos\left(-\frac{1}{10}\right)$, then $\cos\alpha=-\frac{1}{10}$, $0<\alpha<\pi$ and $\frac{\alpha}{2}$ lies in the first quadrant. Now, using formula (2.46) with a positive radical, we get

$$\cos\left(\frac{1}{2}\arccos\left(-\frac{1}{10}\right)\right)=\sqrt{\frac{1+\cos\alpha}{2}}$$
$$=\sqrt{\frac{9}{20}}=\frac{3\sqrt{5}}{10}.$$

4. Trigonometric Formulas of Triple Argument.
Theorem 2.4. *For any real numbers* α *the following identities are valid*:

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha, \qquad (2.53)$$
$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha. \qquad (2.54)$$

Proof. From (2.11), (2.36), (2.39) and the fundamental trigonometric identity it follows that

$$\begin{aligned}\sin 3\alpha &= \sin(\alpha+2\alpha)=\sin\alpha\cos 2\alpha+\cos\alpha\sin 2\alpha\\ &=\sin\alpha\,(1-2\sin^2\alpha)+\cos\alpha\,(2\sin\alpha\cos\alpha)\\ &=\sin\alpha-2\sin^3\alpha+2\sin\alpha\cos^2\alpha\\ &=\sin\alpha-2\sin^3\alpha+2\sin\alpha\,(1-\sin^2\alpha)\\ &=3\sin\alpha-4\sin^3\alpha.\end{aligned}$$

Further, by virtue of (2.1), (2.36), (2.38), and the fundamental trigonometric identity, we have

$$\begin{aligned}\cos 3\alpha &= \cos(\alpha+2\alpha)=\cos\alpha\cos 2\alpha-\sin\alpha\sin 2\alpha\\ &=\cos\alpha\,(2\cos^2\alpha-1)-\sin\alpha\,(2\sin\alpha\cos\alpha)\\ &=2\cos^3\alpha-\cos\alpha-2\sin^2\alpha\cos\alpha\\ &=2\cos^3\alpha-\cos\alpha-2\,(1-\cos^2\alpha)\cos\alpha\\ &=4\cos^3\alpha-3\cos\alpha.\ \blacktriangleright\end{aligned}$$

Example 2.2.6. Evaluate $\cos\left(3\arccos\left(-\frac{1}{3}\right)\right)$.

◀ Let $\alpha=\arccos\left(-\frac{1}{3}\right)$, then $\cos\alpha=-\frac{1}{3}$. We make use of identity (2.54) to get

$$\begin{aligned}\cos 3\alpha &= 4\cos^3\alpha-3\cos\alpha\\ &=4\left(-\frac{1}{27}\right)-3\left(-\frac{1}{3}\right)=1-\frac{4}{27}=\frac{23}{27}.\ \blacktriangleright\end{aligned}$$

Example 2.2.7. Evaluate $\sin\left(3 \arcsin \frac{1}{5}\right)$.

◀Let $\alpha = \arcsin \frac{1}{5}$, then $\sin \alpha = \frac{1}{5}$. We use identity (2.53); whence there follows:

$$\sin 3\alpha = 3\cdot\frac{1}{5} - 4\cdot\frac{1}{125} = \frac{71}{125}. \text{ ▶}$$

2.3. Solution of Problems Involving Trigonometric Transformations

1. Evaluation of Trigonometric Expressions. A number of problems require the value of a trigonometric function for values of the argument that have not been tabulated nor can be reduced to tabular values by reduction or periodicity formulas, that is, they do not have the form $\pi k/6$ or $\pi k/4$, $k \in \mathbf{Z}$. Sometimes, the value of the argument can be expressed in terms of such tabular values and then apply the addition formulas for half or multiple arguments.

Example 2.3.1. Evaluate $\tan\frac{\pi}{12}$.

◀Since $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$, we may apply formula (2.14) for the tangent of a difference:

$$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1-\frac{1}{\sqrt{3}}}{1+1\cdot\frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = 2-\sqrt{3}. \text{ ▶}$$

Example 2.3.2. Evaluate $\cot\frac{5\pi}{12}$.

◀Noting that $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$, we apply formula (2.15) for the cotangent of a sum:

$$\cot\frac{5\pi}{12} = \cot\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \frac{\cot\frac{\pi}{4}\cot\frac{\pi}{6} - 1}{\cot\frac{\pi}{4} + \cot\frac{\pi}{6}}$$

$$= \frac{1\cdot\sqrt{3}-1}{1+\sqrt{3}} = \frac{(\sqrt{3}-1)^2}{3-1} = 2-\sqrt{3}. \text{ ▶}$$

In the cases when the argument is expressed in terms of inverse trigonometric functions, we have to transform the given expression so as to take advantage of the definition of inverse trigonometric functions.

Example 2.3.3. Evaluate

$$A = \tan\left(\frac{3\pi}{4} - \frac{1}{4}\arcsin\left(-\frac{4}{5}\right)\right).$$

◀ Using first identity (2.50) for the tangent of half argument and then the corresponding reduction formula, we get

$$A = \frac{1-\cos\left(\frac{3\pi}{2} - \frac{1}{2}\arcsin\left(-\frac{4}{5}\right)\right)}{\sin\left(\frac{3\pi}{2} - \frac{1}{2}\arcsin\left(-\frac{4}{5}\right)\right)}$$

$$= \frac{1+\sin\left(\frac{1}{2}\arcsin\left(-\frac{4}{5}\right)\right)}{-\cos\left(\frac{1}{2}\arcsin\left(-\frac{4}{5}\right)\right)}.$$

We set $\alpha = \arcsin\left(-\frac{4}{5}\right)$, then $\sin\alpha = -\frac{4}{5}$, $-\frac{\pi}{2} < \alpha < 0$, $\cos^2\alpha = 1 - \sin^2\alpha = 1 - \left(-\frac{4}{5}\right)^2 = \frac{9}{25}$. Consequently, $\cos\alpha = \frac{3}{5}$, and, by virtue of (2.47) and (2.46) for the sine and cosine of half argument, we get

$$\sin\frac{\alpha}{2} = -\sqrt{\frac{1-\cos\alpha}{2}} = -\sqrt{\frac{1-\frac{3}{5}}{2}} = -\frac{1}{\sqrt{5}},$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+\frac{3}{5}}{2}} = \frac{2}{\sqrt{5}}.$$

The signs before the radicals have been determined from the condition $-\frac{\pi}{2} < \alpha < 0$. Finally,

$$A = \left(1 - \frac{1}{\sqrt{5}}\right) \div \left(-\frac{2}{\sqrt{5}}\right) = \frac{1-\sqrt{5}}{2}.$$ ▶

Example 2.3.4. Evaluate

$$\arcsin\frac{3}{5}+\arcsin\frac{12}{13}.$$

◀Let $\alpha=\arcsin\frac{3}{5}$, $\beta=\arcsin\frac{12}{13}$. Then $\sin\alpha=\frac{3}{5}$, $0<\alpha<\frac{\pi}{2}$, $\sin\beta=\frac{12}{13}$, $0<\beta<\frac{\pi}{2}$ and $0<\alpha+\beta<\pi$, i.e. the number $\alpha+\beta$ lies in the domain of values of arc cosine. Since the points P_α and P_β lie in the first quadrant,

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=4/5,$$

$$\cos\beta=\sqrt{1-\sin^2\beta}=5/13,$$

and we may apply formula (2.1) for the cosine of a sum:

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$=\frac{4}{5}\cdot\frac{5}{13}-\frac{3}{5}\cdot\frac{12}{13}=-\frac{16}{65},$$

whence it follows that $\alpha+\beta=\arccos\left(-\frac{16}{65}\right)$. ▶

When solving such problems, a common error consists in that the magnitude of the argument $\alpha+\beta$ is not aken into account. They reason like this: by the formula or the sine of the sum of numbers, we may write

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$=\frac{3}{5}\cdot\frac{5}{13}+\frac{4}{5}\cdot\frac{12}{13}=\frac{63}{65},$$

tand then conclude incorrectly that $\alpha+\beta=\arcsin\frac{63}{65}$, although the number $\alpha+\beta$ does not belong to the domain of values of arc sine since $\alpha+\beta>\frac{\pi}{2}$.

Example 2.3.5. Evaluate $\arctan 4-\arctan 5$.

◀ Let $\alpha=\arctan 4$, $\beta=\arctan 5$, then $\tan\alpha=4$, $0<\alpha<\frac{\pi}{2}$, $\tan\beta=5$, $0<\beta<\frac{\pi}{2}$ and $-\frac{\pi}{2}<\alpha-\beta<\frac{\pi}{2}$. Let us use formula (2.14):

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\frac{4-5}{1+4\cdot5}=-\frac{1}{21}.$$

Consequently, $\alpha-\beta = \arctan\left(-\frac{1}{21}\right)$ since the number $\alpha-\beta$ lies in the domain of values of arc tangent. ▶

Example 2.3.6. Evaluate $\sin 10°\cdot\sin 30°\cdot\sin 50°\cdot\sin 70°$.

◀Let us make use of the reduction formula (2.3) and multiply the given expression by cos 10°; we get

$$\cos 10° \sin 10° \sin 30° \cos 40° \cos 20°.$$

Now, we may apply formula (2.36) for the sine of a double angle for three times, namely:

$$\cos 10° \sin 10° \sin 30° \cos 20° \cos 40°$$
$$= \frac{1}{2}(2\cos 10° \sin 10°)\frac{1}{2}\cos 20° \cos 40°$$
$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}(2\sin 20° \cos 20°)\cos 40°$$
$$= \frac{1}{8}\cdot\frac{1}{2}(2\sin 40° \cos 40°) = \frac{1}{16}\sin 80°.$$

Since the initial expression was multiplied by $\cos 10° = \sin 80°$, it is now obvious that it is equal to $\frac{1}{16}$. ▶

Example 2.3.7. Prove the equality

$$\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°} = 4.$$

◀Multiply both sides of the equality by the number $\sin 10° \cos 10°$ which is not zero. This transformation is reversible, and the equality being proved now has the form

$$\cos 10° - \sqrt{3}\sin 10° = 4\sin 10° \cos 10°,$$

or

$$2\left(\frac{1}{2}\cos 10° - \frac{\sqrt{3}}{2}\sin 10°\right) = 2\sin 20°,$$
$$2\sin(30° - 10°) = 2\sin 20°.$$

We have used formulas (2.36) and (2.12) and have shown that both sides of the equality are equal to the same number $2\sin 20°$. ▶

The examples considered above show that in computational problems it is often convenient first to carry out simplifications with the aid of known trigonometric formulas. For instance, it is useful to single out the expression of the form $\sin^2\alpha+\cos^2\alpha$ which is equal to 1 by virtue of the fundamental trigonometric identity.

Example 2.3.8. Evaluate

$$\sin^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}.$$

◀ A long solution would be first to compute the values $\sin\frac{\pi}{8}$, $\cos\frac{3\pi}{8}$, $\sin\frac{5\pi}{8}$, and $\cos\frac{7\pi}{8}$ with the aid of half-argument formulas and then to raise the terms to the fourth power and to add them together. However, we first use reduction formulas (2.3), (2.8), and (2.5) according to which

$$\cos\frac{3\pi}{8}=\cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right)=\sin\frac{\pi}{8},$$

$$\sin\frac{5\pi}{8}=\sin\left(\frac{\pi}{2}+\frac{\pi}{8}\right)=\cos\frac{\pi}{8}=-\cos\frac{7\pi}{8}.$$

Consequently, the given expression can be rewritten as follows:

$$2\left(\sin^4\frac{\pi}{8}+\cos^4\frac{\pi}{8}\right)$$

$$=2\left(\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}\right)^2-2\cdot 2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}$$

$$=2\cdot 1-\sin^2\frac{\pi}{4}=2-\frac{1}{2}=\frac{3}{2}.$$

Here, we have used the fundamental trigonometric identity, formula (2.36) for the sine of a double angle, and also the tabular value $\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$. ▶

Example 2.3.9. It is known that $\cos\varphi+\sin\varphi=a$, where φ and a are real numbers. Find $\sin^3\varphi+\cos^3\varphi$.

◀ $\sin^3\varphi+\cos^3\varphi$

$$=(\sin\varphi+\cos\varphi)(\sin^2\varphi-\sin\varphi\cos\varphi+\cos^2\varphi)$$

$$=a\left(\frac{3}{2}(\sin^2\varphi+\cos^2\varphi)\right.$$

$$\left.-\frac{1}{2}(\sin^2\varphi+2\sin\varphi\cos\varphi+\cos^2\varphi)\right)$$

$$= a\left(\frac{3}{2}-\frac{1}{2}(\sin\varphi+\cos\varphi)^2\right) = a\left(\frac{3}{2}-\frac{a^2}{2}\right)$$
$$= \frac{3}{2}a - \frac{1}{2}a^3. \blacktriangleright$$

Example 2.3.10. Evaluate

$$A = \frac{\cos 70^\circ \cos 10^\circ + \cos 80^\circ \cos 20^\circ}{\cos 68^\circ \cos 8^\circ + \cos 82^\circ \cos 22^\circ}.$$

◀ First use the reduction formulas and then formula (2.2) for the cosine of the difference between two numbers:

$$A = \frac{\cos 70^\circ \cos 10^\circ + \cos(90^\circ - 10^\circ)\cos(90^\circ - 70^\circ)}{\cos 68^\circ \cos 8^\circ + \cos(90^\circ - 8^\circ)\cos(90^\circ - 68^\circ)}$$
$$= \frac{\cos 70^\circ \cos 10^\circ + \sin 10^\circ \sin 70^\circ}{\cos 68^\circ \cos 8^\circ + \sin 8^\circ \sin 68^\circ} = \frac{\cos(70^\circ - 10^\circ)}{\cos(68^\circ - 8^\circ)}$$
$$= \frac{\cos 60^\circ}{\cos 60^\circ} = 1. \blacktriangleright$$

Example 2.3.11. Find $\sin\frac{\alpha+\beta}{2}$ and $\cos\frac{\alpha+\beta}{2}$, if $\sin\alpha+\sin\beta = -\frac{21}{65}$, $\cos\alpha+\cos\beta = -\frac{27}{65}$, and $\pi < \alpha - \beta < 3\pi$.

◀ It follows from the two equalities that

$$(\cos\alpha+\cos\beta)^2 + (\sin\alpha+\sin\beta)^2 = \frac{(27)^2+(21)^2}{(65)^2},$$

whence we get:

$$(\cos^2\alpha + \sin^2\alpha) + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + (\cos^2\beta + \sin^2\beta) = 18/65,$$

$$1 + 2\cos(\alpha-\beta) + 1 = 18/65,$$
$$1 + \cos(\alpha-\beta) = 9/65,$$
$$2\cos^2\frac{\alpha-\beta}{2} = 9/65.$$

Consequently, by virtue of the inequalities $\frac{\pi}{2} < \frac{\alpha-\beta}{2} < \frac{3\pi}{2}$

$$\cos\frac{\alpha-\beta}{2} = -\frac{3}{\sqrt{130}}.$$

Transform the sum of two sines into a product using formula (2.17):

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$=-\frac{3\sqrt{2}}{\sqrt{65}}\sin\frac{\alpha+\beta}{2}.$$

Consequently,

$$\sin\frac{\alpha+\beta}{2}=\left(-\frac{21}{65}\right)\div\left(-\frac{3\sqrt{2}}{\sqrt{65}}\right)=\frac{7}{\sqrt{130}}.$$

Similarly, by virtue of the identity (2.21), we get

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$=-\frac{3\sqrt{2}}{\sqrt{65}}\cos\frac{\alpha+\beta}{2}.$$

Therefore

$$\cos\frac{\alpha+\beta}{2}=\left(-\frac{27}{65}\right)\div\left(-\frac{3\sqrt{2}}{\sqrt{65}}\right)=\frac{9}{\sqrt{130}}.$$ ▶

An interesting method of solving problems on computing the values of trigonometric expressions consists in the following: we first try to find and then apply an algebraic condition which is satisfied by the given expression. Here is an instructive example: $\sin\frac{2\pi}{5}=\sin 72°=\cos 18°$.

Example 2.3.12. Prove that

$$\cos 18°=\sqrt{\frac{5+\sqrt{5}}{8}}.$$

◀Consider the continued identity based on the formula for the sine of a double argument:

$$\cos 18°\sin 18°\cos 36°=\frac{1}{2}(2\sin 18°\cos 18°)\cos 36°$$
$$=\frac{1}{2}\sin(2\cdot 18°)\cos 36°=\frac{1}{2}\sin 36°\cos 36°$$
$$=\frac{1}{4}(2\sin 36°\cos 36°)=\frac{1}{4}\sin 72°$$
$$=\frac{1}{4}\cos(90°-72°)=\frac{1}{4}\cos 18°,$$

which yields the equality

$$\sin 18° \cos 36° = \frac{1}{4},$$

since $\cos 18° \neq 0$. Using formula (2.39), we write $\cos 36° = 1 - 2\sin^2 18°$ and, consequently, get the relationship

$$\sin 18° (1 - 2\sin^2 18°) = \frac{1}{4}$$

or

$$8\sin^3 18° - 4\sin 18° + 1 = 0,$$

that is, an algebraic equation which is satisfied by the number sin 18°. Setting $\sin 18° = x$, we can solve the cubic equation thus obtained

$$8x^3 - 4x + 1 = 0$$

by factorizing its left-hand side by grouping the terms in the following way:

$$8x^3 - 4x + 1 = 2x(4x^2 - 1) - (2x - 1)$$
$$= (2x - 1)(4x^2 + 2x - 1).$$

Consequently, our equation takes the form: $(2x - 1) \times (4x^2 + 2x - 1) = 0$, its roots being $x_1 = \frac{1}{2}$, $x_2 = \frac{-1-\sqrt{5}}{4}$, $x_3 = \frac{-1+\sqrt{5}}{4}$. Since $0 < 18° < \sin 30° = \frac{1}{2}$ (by Theorem 1.4), then $x_1 \neq \sin 18°$, $x_2 \neq \sin 18°$, consequently, $\sin 18° = \frac{-1+\sqrt{5}}{4}$. Hence (since $\cos 18° > 0$)

$$\cos 18° = \sqrt{1 - \sin^2 18°} = \sqrt{1 - \frac{(\sqrt{5}-1)^2}{16}}$$
$$= \sqrt{1 - \frac{3-\sqrt{5}}{8}} = \sqrt{\frac{5+\sqrt{5}}{8}}. \blacktriangleright$$

2. Simplifying Trigonometric Expressions and Proving Trigonometric Identities. The usual method of proving a trigonometric identity consists in that one of its sides is transformed with the aid of various trigonometric and algebraic operations and also with the aid of the relationships given in the hypothesis so as to get finally the expression which represents the other side of the identity. We can make sure that the left-hand and right-hand sides coincide, transforming them separately so as to get equal expressions. It is only reguired that all the transformations carried out be reversible in the domain of permissible values of the arguments of the given equality (that is, for all values of the arguments for which all the expressions involved in the given identity make sense). This means that not only from each equality obtained during the process of transformations there follows a consequent equality, but also vice versa, the preceding equality itself must follow from the consequent one. This method of reasoning, of course, seems to be rather general and is used for solving equations and systems as well as for proving and solving inequalities.

Example 2.3.13. Prove that if

$$\frac{\sin(x-\alpha)}{\sin(x-\beta)}=\frac{a}{b}, \quad \frac{\cos(x-\alpha)}{\cos(x-\beta)}=\frac{A}{B}$$

and $aB + bA \neq 0$, then

$$\cos(\alpha-\beta)=\frac{aA+bB}{aB+bA}.$$

◀ Using identities (2.11), (2.36), (2.17), we get

$$\frac{aA+bB}{aB+bA}=\frac{\frac{a}{b}\cdot\frac{A}{B}+1}{\frac{a}{b}+\frac{A}{B}}=\frac{\frac{\sin(x-\alpha)}{\sin(x-\beta)}\cdot\frac{\cos(x-\alpha)}{\cos(x-\beta)}+1}{\frac{\sin(x-\alpha)}{\sin(x-\beta)}+\frac{\cos(x-\alpha)}{\cos(x-\beta)}}$$

$$=\frac{\sin(x-\alpha)\cos(x-\alpha)+\sin(x-\beta)\cos(x-\beta)}{\sin(x-\alpha)\cos(x-\beta)+\cos(x-\alpha)\sin(x-\beta)}$$

$$=\frac{\frac{1}{2}(\sin 2(x-\alpha)+\sin 2(x-\beta))}{\sin((x-\alpha)+(x-\beta))}$$

$$=\frac{\sin(2x-(\alpha+\beta))\cos(\alpha-\beta)}{\sin(2x-(\alpha+\beta))}=\cos(\alpha-\beta).$$

The last transformation consisted in dividing both the numerator and denominator of the fraction by the number $\sin(2x-(\alpha+\beta))$. This transformation is reversible since $\sin(2x - (\alpha + \beta)) = \frac{(aB + bA)\sin(x-\beta)\cos(x-\beta)}{bB} \neq 0$ for the values of the arguments x, α, β considered in the problem. ▶

Example 2.3.14. Simplify the expression

$$\sin^3 2\alpha \cos 6\alpha + \cos^3 2\alpha \sin 6\alpha.$$

◀Applying formulas (2.53) and (2.54) for the sine and cosine of a triple argument and also formula (2.11), we get

$$\begin{aligned}
&\sin^3 2\alpha \cos 6\alpha + \cos^3 2\alpha \sin 6\alpha \\
&= \left(\frac{3}{4}\sin 2\alpha - \frac{1}{4}\sin 6\alpha\right)\cos 6\alpha \\
&\quad + \left(\frac{3}{4}\cos 2\alpha + \frac{1}{4}\cos 6\alpha\right)\sin 6\alpha \\
&= \frac{3}{4}\sin 2\alpha\cos 6\alpha - \frac{1}{4}\sin 6\alpha\cos 6\alpha \\
&\quad + \frac{3}{4}\cos 2\alpha\sin 6\alpha + \frac{1}{4}\cos 6\alpha\sin 6\alpha \\
&= \frac{3}{4}(\sin 2\alpha\cos 6\alpha + \cos 2\alpha\sin 6\alpha) \\
&= \frac{3}{4}\sin(2\alpha + 6\alpha) = \frac{3}{4}\sin 8\alpha.
\end{aligned}$$

All the transformations carried out are reversible, and any real value of α is permissible. ▶

Example 2.3.15. Prove that the expression

$$2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x)$$

is independent of x.

◀Apply the fundamental trigonometric identity:

$$\begin{aligned}
&2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) \\
&= 2((\sin^2 x)^3 + (\cos^2 x)^3) - 3\sin^4 x - 3\cos^4 x \\
&= 2(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x) \\
&\quad - 3\sin^4 x - 3\cos^4 x \\
&= 2(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x) \\
&\quad - 3\sin^4 x - 3\cos^4 x \\
&= -(\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x) \\
&= -(\sin^2 x + \cos^2 x)^2 = -1.
\end{aligned}$$ ▶

One should remember that in problems on transforming a trigonometric expression it is always *supposed*, although frequently not stipulated explicitly, that the given expression must be transformed in the domain of its definition, that is, only for the values of the arguments for which the suggested expression has sense.

Example 2.3.16. Prove the identity

$$\frac{3+4\cos 4\alpha+\cos 8\alpha}{3-4\cos 4\alpha+\cos 8\alpha}=\cot^4 2\alpha.$$

◀ The right-hand side of the identity is defined for numbers α such that $\sin 2\alpha \neq 0$, but it is not yet clear whether the domain of the right-hand side coincides with the domain of the left-hand side, which is defined by the condition $3-4\cos 4\alpha+\cos 8\alpha \neq 0$. It is not convenient to begin the solution by determining the domain of the left-hand side. One has first to carry out formal transformations using identities (2.40) and (2.41):

$$\frac{3+4\cos 4\alpha+\cos 8\alpha}{3-4\cos 4\alpha+\cos 8\alpha}=\frac{2+4\cos 4\alpha+2\cos^2 4\alpha}{2-4\cos 4\alpha+2\cos^2 4\alpha}$$

$$=\frac{2(1+2\cos 4\alpha+\cos^2 4\alpha)}{2(1-2\cos 4\alpha+\cos^2 4\alpha)}=\frac{2(1+\cos 4\alpha)^2}{2(1-\cos 4\alpha)^2}$$

$$=\frac{2(2\cos^2 2\alpha)^2}{2(2\sin^2 2\alpha)^2}=\cot^4 2\alpha.$$

As a result of these transformations, it has been cleared up, in passing, that the denominator of the left-hand side is equal to $2(2\sin^2 2\alpha)^2$, and the domain of definition of the left-hand side coincides with that of the right-hand side. In the given domain of definition $\sin 2\alpha \neq 0$, that is, $\alpha \neq \frac{\pi k}{2}$, $k \in \mathbf{Z}$, and all the transformations carried out are reversible. ▶

When proving identities involving inverse trigonometric functions, their domains of definition should be treated attentively.

Example 2.3.17. Prove that if $x \in [0, 1]$, $y \in [0, 1]$, then the following identity holds true:

$$\arcsin x+\arcsin y=\arccos\left(\sqrt{1-x^2}\sqrt{1-y^2}-xy\right).$$

Let $\alpha=\arcsin x$, $\beta=\arcsin y$, then $\sin\alpha=x\geqslant 0$, $0\leqslant\alpha\leqslant\pi/2$, $\sin\beta=y\geqslant 0$, $0\leqslant\beta\leqslant\pi/2$, $0\leqslant\alpha+\beta\leqslant$

π, and the number $\alpha + \beta$ lies in the range of arc cosine. Further note that

$$\cos\alpha = \sqrt{1-x^2}, \quad \cos\beta = \sqrt{1-y^2},$$

since the points P_x and P_y lie in the first quadrant. Now, we may apply formula (2.1) for the cosine of a sum to get

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$= \sqrt{1-x^2}\sqrt{1-y^2} - xy.$$

Hence it follows that

$$\alpha+\beta = \arccos\left(\sqrt{1-x^2}\sqrt{1-y^2} - xy\right),$$

since $\alpha + \beta \in [0, \pi]$. ▶

Note that the identity being proved ceased to be valid if the condition $x \in [0, 1]$ and $y \in [0, 1]$ are not fulfilled. For instance, for $x = -1/2$, $y = -1/2$ the left-hand side is negative, while the right-hand side is positive.

3. Transforming Sums and Products of Trigonometric Expressions.

Example 2.3.18. Find the product

$$P = \cos\alpha\cos 2\alpha\cos 4\alpha \ldots \cos 2^n\alpha$$

for $\alpha \neq \pi k$, $k \in \mathbf{Z}$.

◀Multiply the given product by the number $\sin\alpha$ which is nonzero by hypothesis ($\alpha \neq \pi k$, $k \in \mathbf{Z}$). This transformation is reversible, and we get

$$P\sin\alpha = (\sin\alpha\cos\alpha)\cos 2\alpha \ldots \cos 2^n\alpha$$
$$= \frac{1}{2}\sin 2\alpha\cos 2\alpha\cos 4\alpha \ldots \cos 2^n\alpha$$
$$= \frac{1}{4}(2\sin 2\alpha\cos 2\alpha)\cos 4\alpha \ldots \cos 2^n\alpha$$
$$= \frac{1}{2^3}\sin 8\alpha \ldots \cos 2^n\alpha = \ldots$$
$$= \frac{1}{2^n}\sin 2^n\alpha\cos 2^n\alpha = \frac{1}{2^{n+1}}\sin 2^{n+1}\alpha.$$

Consequently,

$$P = \frac{1}{2^{n+1}}\,\frac{\sin 2^{n+1}\alpha}{\sin\alpha}. \ ▶$$

Example 2.3.19. Compute the sum

$$S = \sin\alpha + \sin 2\alpha + \ldots + \sin n\alpha.$$

◀ If $\alpha = \pi k$ $(k \in \mathbf{Z})$, then $S = \sin\pi k + \sin 2\pi k + \ldots + \sin n\pi k = 0$. Let $\alpha \neq \pi k$ $(k \in \mathbf{Z})$. We multiply the sum S by $\sin\frac{\alpha}{2} \neq 0$, and then use formula (2.29) for the product of sines. We get

$$S\sin\frac{\alpha}{2} = (\sin\alpha + \sin 2\alpha + \ldots + \sin n\alpha)\sin\frac{\alpha}{2}$$

$$= \sin\alpha\sin\frac{\alpha}{2} + \sin 2\alpha\sin\frac{\alpha}{2} + \ldots + \sin n\alpha\sin\frac{\alpha}{2}$$

$$= \frac{1}{2}\left(\cos\frac{\alpha}{2} - \cos\frac{3\alpha}{2}\right) + \frac{1}{2}\left(\cos\frac{3\alpha}{2} - \cos\frac{5\alpha}{2}\right)$$

$$+ \ldots + \frac{1}{2}\left(\cos\frac{(2n-3)\alpha}{2} - \cos\frac{(2n-1)\alpha}{2}\right)$$

$$+ \frac{1}{2}\left(\cos\frac{(2n-1)\alpha}{2} - \cos\frac{(2n+2)\alpha}{2}\right)$$

$$= \frac{1}{2}\left(\cos\frac{\alpha}{2} - \cos\frac{(2n+1)\alpha}{2}\right) = \sin\frac{(n+1)\alpha}{2}\sin\frac{n\alpha}{2}$$

(we have also applied formula (2.22)), whence

$$S = \frac{\sin\frac{(n+1)\alpha}{2}\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}. \blacktriangleright$$

Example 2.3.20. Compute the sum

$$S = \cos\alpha + \cos 2\alpha + \ldots + \cos n\alpha.$$

◀ If $\alpha = 2\pi k$, $k \in \mathbf{Z}$, then

$$S = \cos 2\pi k + \cos 4\pi k + \ldots + \cos n\pi k$$
$$= 1 + \ldots + 1 = n.$$

Let $\alpha \neq 2\pi k$, $k \in \mathbf{Z}$, then $\sin\frac{\alpha}{2} \neq 0$ and, using the formulas (2.27) and (2.18), we get

$$S\sin\frac{\alpha}{2}$$

$$= \sin\frac{\alpha}{2}\cos\alpha + \sin\frac{\alpha}{2}\cos 2\alpha + \ldots + \sin\frac{\alpha}{2}\cos n\alpha$$

$$=\frac{1}{2}\left(\sin\frac{3\alpha}{2}-\sin\frac{\alpha}{2}\right)+\frac{1}{2}\left(\sin\frac{5}{2}\alpha-\sin\frac{3\alpha}{2}\right)$$
$$+\ldots+\frac{1}{2}\left(\sin\frac{(2n+1)\alpha}{2}-\sin\frac{(2n-1)\alpha}{2}\right)$$
$$=\frac{1}{2}\left(\sin\frac{(2n+1)\alpha}{2}-\sin\frac{\alpha}{2}\right)=\sin\frac{n\alpha}{2}\cos\frac{(n+1)\alpha}{2}.$$

Consequently,

$$S=\frac{\sin\frac{n\alpha}{2}\cos\frac{(n+1)\alpha}{2}}{\sin\frac{\alpha}{2}}. \blacktriangleright$$

Example 2.3.21. Prove the identity

$$\frac{1}{2}\tan\frac{\alpha}{2}+\frac{1}{4}\tan\frac{\alpha}{4}+\ldots+\frac{1}{2^n}\tan\frac{\alpha}{2^n}$$
$$=\frac{1}{2^n}\cot\frac{\alpha}{2^n}-\cot\alpha.$$

◀ First, we prove a useful auxiliary formula:

$$\tan x-\cot\alpha=\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}$$
$$=-\frac{\cos^2 x-\sin^2 x}{\sin x\cdot\cos x}=-2\frac{\cos 2x}{\sin 2x}$$
$$=-2\cot 2x.$$

Then we subtract the number $\frac{1}{2^n}\cot\frac{\alpha}{2^n}$ from the left-hand side of the identity and apply n times the formula just derived:

$$\frac{1}{2}\tan\frac{\alpha}{2}+\frac{1}{4}\tan\frac{\alpha}{4}+\ldots+\frac{1}{2^{n-1}}\tan\frac{\alpha}{2^{n-1}}$$
$$+\frac{1}{2^n}\tan\frac{\alpha}{2^n}-\frac{1}{2^n}\cot\frac{\alpha}{2^n}$$
$$=\frac{1}{2}\tan\frac{\alpha}{2}+\frac{1}{4}\tan\frac{\alpha}{4}+\ldots+\frac{1}{2^{n-1}}\tan\frac{\alpha}{2^{n-1}}$$
$$+\frac{1}{2^n}\left(\tan\frac{\alpha}{2^n}-\cot\frac{\alpha}{2^n}\right)=\frac{1}{2}\tan\frac{\alpha}{2}+\frac{1}{4}\tan\frac{\alpha}{4}$$
$$+\ldots+\frac{1}{2^{n-1}}\tan\frac{\alpha}{2^{n-1}}-\frac{1}{2^{n-1}}\cot\frac{\alpha}{2^{n-1}}$$
$$=\frac{1}{2}\tan\frac{\alpha}{2}+\frac{1}{4}\tan\frac{\alpha}{4}+\ldots-\frac{1}{2^{n-2}}\cot\frac{\alpha}{2^{n-2}}$$

$$= \ldots = \frac{1}{2}\tan\frac{\alpha}{2} + \frac{1}{4}\tan\frac{\alpha}{4} - \frac{1}{4}\cot\frac{\alpha}{4}$$

$$= \frac{1}{2}\tan\frac{\alpha}{2} + \frac{1}{4}\left(\tan\frac{\alpha}{4} - \cot\frac{\alpha}{4}\right)$$

$$= \frac{1}{2}\tan\frac{\alpha}{2} - \frac{1}{2}\cot\frac{\alpha}{2}$$

$$= \frac{1}{2}\left(\tan\frac{\alpha}{2} - \cot\frac{\alpha}{2}\right) = -\cot\alpha.$$

The given identity follows from this continued equality. Note that the domains of definition of both sides of the given identity coincide with the set of real numbers α such that $\sin\alpha \neq 0$, that is, $\alpha \neq \pi k$, $k \in \mathbf{Z}$.

PROBLEMS

In Problems **2.1** to **2.17**, prove the given identities:

2.1. $\dfrac{\tan 2t + \cot 3s}{\cot 2t + \tan 3s} = \dfrac{\tan 2t}{\tan 3s}$.

2.2. $\cot^2 t - \cot^2 s = \dfrac{\cos^2 t - \cos^2 s}{\sin^2 t \sin^2 s}$.

2.3. $\dfrac{\sin 4\alpha}{1+\cos 4\alpha} \cdot \dfrac{\cos 2\alpha}{1+\cos 2\alpha} = \cot\left(\dfrac{3}{2}\pi - \alpha\right)$.

2.4. $(\cos\alpha - \cos\beta)^2 - (\sin\alpha - \sin\beta)^2$

$$= -4\sin^2\frac{\alpha-\beta}{2}\cos(\alpha+\beta).$$

2.5. $\dfrac{\sin^4 t + \cos^4 t - 1}{\sin^6 t + \cos^6 t - 1} = \dfrac{2}{3}$.

2.6. $\cot t - \tan t - 2\tan 2t = 4\cot 4t$.

2.7. $\tan 6t - \tan 4t - \tan 2t = \tan 6t \tan 4t \tan 2t$.

2.8. $\tan 3t = \dfrac{3\tan t - \tan^3 t}{1 - 3\tan^2 t}$.

2.9. $\sin^4 t = \dfrac{1}{8}\left(\cot 4t - 4\cos 2t + \dfrac{6}{2}\right)$.

2.10. $\cos 4t = 8\cos^4 t - 8\cos^2 t + 1$.

2.11. $\dfrac{3 - 4\cos 2t + \cos 4t}{3 + 4\cos 2t + \cos 4t} = \tan^4 t$.

2.12. $\cos t + \cos(t+s) + \cos(t+2s)$.

$$+\ldots+\cos(t+ns) = \frac{\cos\left(t+\frac{ns}{2}\right)\sin\left(\frac{n+1}{2}s\right)}{\sin\frac{s}{2}}.$$

2.13. $\sin t + \sin(t+s) + \sin(t+2s)$

$$+\ldots+\sin(t+ns) = \frac{\sin\left(t+\frac{ns}{2}\right)\sin\left(\frac{n+1}{2}s\right)}{\sin\frac{s}{2}}.$$

2.14. $8\cos^4 t + 4\cos^3 t - 8\cos^2 t - 3\cos t + 1$

$$= 2\cos\frac{7}{2}t\cos\frac{t}{2}.$$

2.15. $\sin(t+s+u) = \sin t\cos s\cos u$

$$+\cos t\sin s\cos u + \cos t\cos s\sin u$$

$$-\sin t\sin s\sin u.$$

2.16. $\cos(t+s+u) = \cos t\cos s\cos u$

$$-\sin t\sin s\cos u - \sin t\cos s\sin u$$

$$-\cos t\sin s\sin u.$$

2.17. $\dfrac{\tan^2 t - \tan^2 s}{1-\tan^2 t\tan^2 s} = \tan(t+s)\tan(t-s)$.

In Problems **2.18** to **2.23**, simplify the indicated expressions.

2.18. $\sin^2\left(\frac{t}{2}+2s\right) - \sin^2\left(\frac{t}{2}-2s\right)$.

2.19. $\cos^2(t+2s) + \sin^2(t-2s) - 1$.

2.20. $\cos^6\left(t-\frac{\pi}{2}\right) + \sin^6\left(t-\frac{3\pi}{2}\right)$

$$-\frac{3}{4}\left(\sin^2\left(t+\frac{\pi}{2}\right) - \cos^2\left(t+\frac{3\pi}{2}\right)\right)^2.$$

2.21. $\sin^2\left(\frac{3\pi}{4}-2t\right) - \sin^2\left(\frac{7\pi}{6}-2t\right)$

$$-\sin\frac{13\pi}{12}\cos\left(\frac{11\pi}{12}-4t\right).$$

2.22. $\sin^3 2t \cos 6t + \cos^3 2t \sin 6t$.

2.23. $4(\sin^4 t + \cos^4 t) - 4(\sin^6 t + \cos^6 t) - 1$.

In Problems **2.24** to **2.28**, transform the given expressions into a product.

2.24. $\sin 6t - 2\sqrt{3}\cos^2 3t + \sqrt{3}$.

2.25. $\tan^4 t - 4\tan^2 t + 3$.

2.26. $\dfrac{\sin^2(t+s)-\sin^2 t-\sin^2 s}{\sin^2(t+s)-\cos^2 t-\cos^2 s}$

2.27. $\sin^2(2t - s) - \sin^2 2t - \sin^2 s$.

2.28. $\cos 22t + 3\cos 18t + 3\cos 14t + \cos 10t$.

In Problems **2.29** to **2.32**, check the indicated equalities.

2.29. $\cot 70° + 4\cos 70° = \sqrt{3}$.

2.30. $\sin^2\left(\arctan 3 - \operatorname{arccot}\left(-\frac{1}{2}\right)\right) = \frac{1}{2}$.

2.31. $\sin\left(2\arctan\frac{1}{2}\right) + \tan\left(\frac{1}{2}\arcsin\frac{15}{17}\right) = \frac{7}{5}$.

2.32. $\sin\left(2\arctan\frac{1}{2}\right) - \tan\left(\frac{1}{2}\arcsin\frac{15}{17}\right) = \frac{1}{5}$.

In Problems **2.33.** to **2.36**, compute the given expressions.

2.33. $\arccos(\cos(2\arctan(\sqrt{2}-1)))$.

2.34. $\sin^2\left(\arctan\frac{1}{2} - \arctan\left(-\frac{1}{3}\right)\right)$.

2.35. $\cos\left(\frac{1}{2}\arccos\frac{3}{5} - 2\arctan(-2)\right)$.

2.36. $\dfrac{\sin 22° \cos 8° + \cos 158° \cos 98°}{\sin 23° \cos 7° + \cos 157° \cos 97°}$.

2.37. Find $\tan\frac{t}{2}$ if $\sin t + \cos t = 1/5$.

2.38. Knowing that α, β, and γ are internal angles of a triangle, prove the equality

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

Chapter 3

Trigonometric Equations and Systems of Equations

3.1. General

Written entry examinations include, as a rule, problems on solving trigonometric equations. This is partly explained by the fact that there is no general method which would be applicable for solving any trigonometric equations and in each concrete case the search for a solution require, a certain of ease in carrying out identical transformations and the knack of finding and applying the proper trigonometric formula. In most cases, the transformations used to solve such problems are generally aimed at the reduction of a given equation to several simple equations to be solved in a regular way, as it was described in Sec. 1.4.

It is of importance to note that the form of notation of the roots of trigonometric equations often depends on the method applied in solving a given equation. To prove the fact that two different notations of the answer are equivalent is sometimes an interesting problem in itself, although the examination requires to solve the given equation using only one method, rather than to transform the answer into other notations.

When solving trigonometric equations and systems of equations the student has frequently to deal with rather complicated expressions composed of trigonometric functions. Competition problems often involve expressions that, along with trigonometric functions, contain other types of functions (inverse trigonometric, exponential, logarithmic, rational, fractional, etc.). The student must remember that in most cases such expressions are not defined for all values of the variables in the given expression. Frequently, identical transformations result in some simplifications, however, the equation (or system) obtained may have another domain of permissible values of the unknowns. This can lead to "superfluous" or "extraneous" roots if the domain of permissible values is broadened as a result of a certain transformation or even to a loss of roots if as a result of some transformation the domain of permissible values is narrowed. A good method to avoid such troubles is to watch for the invertibility of the identical transformations being carried out on the domain under considertion. Whenever the transformations are not invertible a check is required. Namely, first, find and check the possible values of the roots which might have disappeared as a result of the reduction of the domain of permissible values, and, second, make sure that there are no extraneous roots which do not belong to the domain of permissible values of the initial expression.

Example 3.1.1. Solve the equation

$$\sin x + 7\cos x + 7 = 0.$$

◀One of the methods for solving this equation (though not the best) consists in using the universal substitution formulas (2.42) and (2.43):

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}, \quad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}.$$

As a result, we get the following equation:

$$\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{7\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}} + 7 = 0,$$

which is equivalent to $\tan\frac{x}{2} = -7$, and, consequently, to $x = 2\pi k - 2\arctan 7$, $k \in \mathbf{Z}$. However, the above reasoning is erroneous, since the universal substitution formulas are applicable only to the x's for which $\tan\frac{x}{2}$ is defined, that is, $x \neq \pi + 2\pi k$, $k \in \mathbf{Z}$. Therefore, as a result of the transformations, the domain of definition of the function on the left-hand side of the equation is reduced and in order to get a correct answer, it is necessary to check whether there are roots among the numbers which are left outside the domain of definition of the expression obtained, that is, to check whether or not the numbers $x = \pi + 2\pi k$, $k \in \mathbf{Z}$, are also roots of this expression. For these values of x we get $\sin x = 0$, $\cos x = -1$, and the equation turns into an identity (we mean the original equation), therefore the correct answer is:

$$x = -2 \arctan 7 + 2\pi k, \quad x = \pi + 2\pi k, \quad k \in \mathbf{Z}.$$

This problem can be solved in a shorter way, by using only invertible transformations. For this purpose, it is necessary to use the method of introducing an auxiliary angle (Theorem 2.2). We divide both sides of the equation by the number $\sqrt{1^2 + 7^2} = \sqrt{50}$, transpose the number $7/\sqrt{50}$ to the right-hand side, and then rewrite the equation as follows

$$\sin (x + \varphi) = -7/\sqrt{50},$$

where $\varphi = \arcsin \dfrac{7}{\sqrt{50}}$. Thus, the original equation has been reduced to a simple trigonometric equation of the form $\sin t = a$ considered in Sec. 1.4, and the general solution is written in the following form:

$$x + \varphi = (-1)^n \arcsin (-7)/\sqrt{50}) + \pi n, \quad n \in \mathbf{Z},$$

or

$$x = -\arcsin \frac{7}{\sqrt{50}} + (-1)^{n+1} \arcsin \frac{7}{\sqrt{50}} + \pi n, \quad n \in \mathbf{Z}.$$

Thus, for an even $n = 2k$, $k \in \mathbf{Z}$, we get a series of solutions

$$x = -2 \arcsin \frac{7}{\sqrt{50}} + 2\pi k,$$

and for an odd $n = 2k + 1$ another series of solutions

$$x = \pi + 2\pi k, \quad k \in \mathbf{Z}.$$

The equivalence of the two notations of the answer now follows from the equality $\sin (\arctan x) = x/\sqrt{1 + x^2}$ (see Example 1.4.6) or $\arctan x = \arcsin \dfrac{x}{\sqrt{1+x^2}}$, whence $\arcsin \dfrac{7}{\sqrt{50}} = \arctan 7$. ▶

Example 3.1.2. Among the roots of the equation

$$\frac{\cos 3\pi x}{1+\sqrt{3}\tan \pi x}=0$$

find the one which has the least distance from the number $\sqrt{8}$ on the number line.

◀The function $\tan \pi x$ is defined if $\pi x \neq \frac{\pi}{2}+\pi k$, i.e. $x \neq \frac{1}{2}+k$, $k \in \mathbf{Z}$.

In order to find the domain of definition of the function on the left-hand side of the equation, it is necessary, in addition, to bear in mind that the denominator must not vanish, that is, $1+\sqrt{3}\tan \pi x \neq 0$, or $\tan \pi x \neq -1/\sqrt{3}$, which is equivalent to the condition $\pi x \neq -\frac{\pi}{6}+\pi k$, that is, $x \neq -\frac{1}{6}+k$, $k \in \mathbf{Z}$. Thus, the domain of the function $\frac{\cos 3\pi x}{1+\sqrt{3}\tan \pi x}$ consists of real numbers x such that $x \neq \frac{1}{2}+k$ and $x \neq -\frac{1}{6}+k$ for $k \in \mathbf{Z}$. In the given domain of permissible values the equation is equivalent to the following:

$$\cos 3\pi x = 0$$

whose solution has the form:

$$3\pi x=\frac{\pi}{2}+\pi n, \quad \text{i.e} \quad x=\frac{1}{6}+\frac{n}{3}, \quad n \in \mathbf{Z}.$$

Let us now find out which of the obtained values of x do not belong to the domain of definition. We have:

$$\frac{1}{6}+\frac{n}{3}=\frac{1}{2}+k \quad \text{for} \quad n=1+3k,$$

$$\frac{1}{6}+\frac{n}{3}=-\frac{1}{6}+k \text{ for } \quad n=-1+3k.$$

Note that $n=3k \pm 1$, $k \in \mathbf{Z}$, is the set of all the integers not divisible by 3. Consequently, the formula for x describing the solutions of the given equation must contain only n divisible by 3. Setting $n=3k$, we get $x=\frac{1}{6}+k$, $k \in \mathbf{Z}$. Further, from these solutions we choose the one which satisfies the additional condition of the problem (with the least distance from the number $\sqrt{8} = 2\sqrt{2}$). Since $1.41 < \sqrt{2} < 1.42$, we have $2.82 < 2\sqrt{2} < 2.84$, and it suffices to consider the two numbers

$$x = \frac{1}{6} + 2 \quad \text{and} \quad x = \frac{1}{6} + 3,$$

which are the closest to the number $\sqrt{8}$. The problem has been reduced to comparing the two numbers

$$\left|2\frac{1}{6} - 2\sqrt{2}\right| = 2\sqrt{2} - 2\frac{1}{6}$$

and

$$\left|3\frac{1}{6} - 2\sqrt{2}\right| = 3\frac{1}{6} - 2\sqrt{2}.$$

The following inequality holds true:

$$3\frac{1}{6} - 2\sqrt{2} < 2\sqrt{2} - 2\frac{1}{6}.$$

Indeed, it is equivalent to the inequality $4\sqrt{2} > 5\frac{1}{3}$ which is proved by squaring both members since both of them are positive. Therefore the number $x = 3\frac{1}{6}$ is the sought-for solution of the given equation. ▶

If in this example the domain of permissible values is treated inaccurately, for instance, one forgets about the condition $1 + \sqrt{3} \tan \pi x \neq 0$ or that $\tan \pi x$ must be defined, then a wrong answer is obtained.

Even these two examples show the importance of taking into account the domain of permissible values when solving trigonometric equations. However, it is not always convenient to write out the domain of permissible values in an explicit form, that is, to indicate explicitly all the values of the unknown belonging to this domain. It suffices to write the conditions wherefrom this domain can be found. Thus, in Example 3.1.2, these conditions were $\cos \pi x \neq 0$ (i.e. $\tan \pi x$ is defined) and $1 + \sqrt{3} \tan \pi x \neq 0$. Sometimes, one succeeds in solving such conditions with respect to x in a simple way (as in

Example 3.1.2), but in most cases this is a cumbersome problem in itself. However, when checking individual values of the unknown which may turn out to be roots, one may also successfully use an implicit representation of the domain of permissible values, checking whether the indicated conditions are met for the values under consideration.

Example 3.1.3. Solve the equation

$$2\sin\left(3x+\frac{\pi}{4}\right)=\sqrt{1+8\sin 2x\cos^2 2x}\,. \qquad (3.1)$$

◀The domain of permissible values of the given equation is specified implicitly by the conditions: $1 + 8\sin 2x\cos^2 2x \geqslant 0$, $\sin\left(3x+\frac{\pi}{4}\right)\geqslant 0$. If we square both sides of the equation, then, on the given domain, the original equation (3.1) is equivalent to the following equation:

$$4\sin^2\left(3x+\frac{\pi}{4}\right)=1+8\sin 2x\cos^2 2x. \qquad (3.2)$$

However, if one does not take into account the domain of permissible values, then, although the roots of the original equation (3.1) are also the roots of equation (3.2), but all the roots of (3.2) will not necessarily be the roots of (3.1). Therefore on finding all the roots of (3.2) we have to choose those which will be the roots of the original equation. Applying formula (2.41), we get

$$\sin^2\left(3x+\frac{\pi}{4}\right)=\frac{1-\cos\left(6x+\frac{\pi}{2}\right)}{2}=\frac{1}{2}(1+\sin 6x),$$

using formulas (2.36), (2.27), we have

$$8\sin 2x\cos^2 2x = 4\cos 2x\,(2\sin 2x\cos 2x)$$
$$= 4\cos 2x\sin 4x = 2\,(\sin 6x + \sin 2x).$$

Therefore equation (3.2) may be rewritten as follows

$$2 + 2\sin 6x = 1 + 2\sin 6x + 2\sin 2x,$$

or

$$\sin 2x=\frac{1}{2}. \qquad (3.3)$$

For a further investigation, the solutions of this equation should be conveniently written in the form of two series of solutions (but not united, as usual, into one; see Sec. 1.4):

$$x=\frac{\pi}{12}+\pi n, \quad x=\frac{5\pi}{12}+\pi n, \quad n\in \mathbf{Z}.$$

Since equation (3.3) is equivalent to equation (3.2), we have to check whether all of its solutions are the solutions of the original equation.

Substituting the found values of x into the right-hand side of the original equation, we get the number 2, that is, the condition $1+8\sin 2x\cos^2 2x\geqslant 0$ has been fulfilled. For $x=\frac{\pi}{12}+\pi n$, $n\in \mathbf{Z}$, the left-hand side of the original equation is equal to

$$2\sin\left(3x+\frac{\pi}{4}\right)=2\sin\left(\frac{\pi}{2}+2\pi n\right)=2\cos\pi n,$$

If n is an even number, then $2\cos\pi n=2$, and if n is odd, then $2\cos\pi n=-2$. Hence, from the first series the solutions of the original equation are only the numbers

$$x=\frac{\pi}{12}+2\pi k, \quad k\in \mathbf{Z}.$$

For $x=\frac{5\pi}{12}+\pi n$, $n\in \mathbf{Z}$, the left-hand side of the original equation is equal to

$$2\sin\left(3x+\frac{\pi}{4}\right)=2\sin\left(\frac{3\pi}{2}+3\pi n\right)=-2\cos\pi n.$$

If n is an even number, then $-2\cos\pi n=-2$, and if n is odd, then $-2\cos\pi n=2$. Consequently, from the second series, the following numbers are the solutions of the original equation:

$$x=\frac{5\pi}{12}+(2k+1)\pi, \quad k\in \mathbf{Z}.$$

Answer: $x=\frac{\pi}{12}+2\pi k$, $x=\frac{5\pi}{12}+(2k+1)\pi$, $k\in \mathbf{Z}$. ▶

When solving this problem, most errors occur owing to incorrect understanding of the symbol $\sqrt{\ }$. As in algebra, in trigonometry this radical sign means an *arithmetic* square root whose value is always nonnegative. This note is as essential as the requirement that a nonnegative expression stand under the radical sign of an arithmetic root. If for some values of the arguments these conditions are not fulfilled, then the equality under consideration has no sense.

3.2. Principal Methods of Solving Trigonometric Equations

1. Solving Trigonometric Equations by Reducing Them to Algebraic Ones. This widely used method consists in transforming the original equation to the form

$$F(f(t)) = 0, \tag{3.4}$$

where $F(x)$ is a polynomial and $f(t)$ is a trigonometric function; in other words, it is required, using trigonometric identities, to express all the trigonometric functions in the equation being considered in terms of one trigonometric function.

If $x_1, x_2, \ldots, x_m$ are roots of the polynomial F, that is,

$$F(x_1) = 0, \ F(x_2) = 0, \ \ldots, \ F(x_m) = 0,$$

then the transformed equation (3.4) decomposes into m simple equations

$$f(t) = x_1, \ f(t) = x_2, \ \ldots, \ f(t) = x_m.$$

For instance, if the original equation has the form

$$G(\sin t, \cos t) = 0,$$

where $G(x, y)$ is a polynomial of two variables x and y, then the given equation can be reduced to an algebraic equation with the aid of the universal substitution formulas by getting rid of the denominators during the process of transformation. As it was stressed in Sec. 3.1, such a reduction requires control over the invertibility of all the transformations carried out, and in case of violation of invertibility a check is required.

Example 3.2.1. Solve the equation

$$\cos 2t - 5 \sin t - 3 = 0.$$

◀By formula (2.39), we have $1 - 2\sin^2 t - 5 \sin t - 3 = 0$, or

$$2 \sin^2 t + 5 \sin t + 2 = 0.$$

We set $x = \sin t$; then the original equation takes the form of an algebraic equation:

$$2x^2 + 5x + 2 = 0.$$

Solving this equation we get $x_1 = -1/2$, $x_2 = -2$. All the transformations carried out are invertible, therefore the original equation is decomposed into two simple equations:

$$\sin t = -\frac{1}{2} \quad \text{and} \quad \sin t = 2.$$

The second equation has no solutions since $|\sin t| \leqslant 1$, therefore we take $\sin t = -1/2$, that is,

$$t = (-1)^{n+1} \frac{\pi}{6} + \pi n, \quad n \in \mathbf{Z}. \quad ▶$$

Example 3.2.2. Solve the equation

$$\tan x + \tan\left(\frac{\pi}{4} + x\right) = -2.$$

◀By formula (2.13) for the tangent of the sum of two angles, we have:

$$\tan\left(\frac{\pi}{4} + x\right) = \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x} = \frac{1 + \tan x}{1 - \tan x}.$$

Hence, $\tan x + \frac{1+\tan x}{1-\tan x} = -2$. Setting $y = \tan x$, we get an algebraic equation:

$$y + \frac{1+y}{1-y} = -2,$$

or

$$y(1 - y) + 1 + y = -2(1 - y), \quad y = \pm\sqrt{3},$$

consequently, $\tan x = \pm\sqrt{3}$, that is,

$$x = \pm\frac{\pi}{3} + \pi n, \quad n \in \mathbf{Z}.$$

Both series of solutions belong to the domain of permissible values of the original equation which was not reduced under transformation.

Example 3.2.3. Solve the equation

$$(1 - \tan x)(1 + \sin 2x) = 1 + \tan x - \cos 2x.$$

◀Note that the numbers $x = \frac{\pi}{2} + \pi k$, $k \in \mathbf{Z}$, are not solutions of the given equation, therefore we may consider the given equation on a smaller domain of permissible values specified by the condition $x \neq \frac{\pi}{2} + \pi k$, $k \in \mathbf{Z}$, and use the universal substitution formulas (2.42) and (2.43) which are reversible transformations in the given domain:

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}, \quad \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}.$$

We set $y = \tan x$, then the given equation is reduced to an algebraic one:

$$(1-y)\left(1 + \frac{2y}{1+y^2}\right) = 1 + y - \frac{1-y^2}{1+y^2}.$$

Since $1 + y^2 \neq 0$, this equation is equivalent to

$$(1 - y)(1 + y^2 + 2y) = (1 + y)(1 + y^2) - 1 + y^2,$$

whence, by successive invertible transformations, we get:

$$(1 - y)(1 + y)^2 = (1 + y)(1 + y^2) + (y - 1)(y + 1),$$
$$(1 - y)(1 + y)^2 = (1 + y)(1 + y^2 + y - 1),$$
$$(1 - y)(1 + y)^2 = (1 + y)^2 y,$$
$$(1 + y)^2(1 - 2y) = 0.$$

The roots of the obtained equation are: $y_1 = -1$ and $y_2 = 1/2$. Consequently, the original equation is broken into two simple equations: $\tan x = -1$ and $\tan x = 1/2$ in the sense that the set of solutions of the original equation is a union of the sets of solutions of the obtained equations, and we get

$$x = -\frac{\pi}{4} + \pi n, \quad x = \arctan\frac{1}{2} + \pi n, \quad n \in \mathbf{Z}. \blacktriangleright$$

Example 3.2.4. Solve the equation

$$2 \sin 4x + 16 \sin^3 x \cos x + 3 \cos 2x - 5 = 0.$$

◀Note that, by virtue of formulas (2.36) and (2.38),

$$2 \sin 4x = 8 \sin x \cos x \cos 2x$$
$$= 8 \sin x \cos x - 16 \sin^3 x \cos x,$$

and the equation takes the form

$$8 \sin x \cos x + 3 \cos 2x - 5 = 0,$$

or

$$4 \sin 2x + 3 \cos 2x = 5. \tag{3.5}$$

Let us make use of the universal substitution formula

$$\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}, \quad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$

and designate $y = \tan x$. Then the equation is transformed into an algebraic one:

$$\frac{8y}{1+y^2} + \frac{3-3y^2}{1+y^2} = 5,$$

or $8y + 3 - 3y^2 = 5 + 5y^2$, whence

$$y^2 - y + \frac{1}{4} = 0.$$

Consequently, $y = 1/2$ or $\tan x = 1/2$, whence $x = \arctan\frac{1}{2} + \pi n$, $n \in \mathbf{Z}$. It remains to check that no roots are lost during the process of solution. Indeed, only those x's might be lost for which $\tan x$ has no sense, that is, $x = \frac{\pi}{2} + \pi k$, $k \in \mathbf{Z}$. Substituting these values into the left-hand side of (3.5), which is equivalent to the original one, we get

$$4 \sin (\pi + 2\pi k) + 3 \cos (\pi + 2\pi k) = -3.$$

Consequently, besides $x = \arctan\frac{1}{2} + \pi n$, $n \in \mathbf{Z}$, the equation has no other roots. ▶

2. Other Methods of Reducing Trigonometric Equations to Several Simple Equations. Basically, we mean here application of the formulas for transforming the sum or difference of trigonometric functions into a product (see Sec. 2.1, Item 6). However, it is often necessary first to carry out additional identical transformations. In particular, the left-hand sides of formulas (2.17), (2.18), (2.21), (2.22), (2.25), (2.26) contain the basic trigonometric functions (in the first power), therefore to use them, it is, for instance, useful to apply formulas (2.40), (2.41) which reduce the power of trigonometric functions in the given expression.

Example 3.2.5. Solve the equation

$$\sin^2 x + \cos^2 3x = 1.$$

◀Applying identities (2.40) and (2.41), we get

$$\frac{1}{2} - \frac{1}{2}\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 6x = 1,$$

or $\frac{1}{2}(\cos 6x - \cos 2x) = 0$, whence, by virtue of identity (2.22) we get

$$-\sin 4x \sin 2x = 0.$$

The original equation has been broken into two equations:

$$\sin 4x = 0 \quad \text{and} \quad \sin 2x = 0.$$

Note that the solutions of the equation $\sin 2x = 0$ ($x = \pi k/2$, $k \in \mathbf{Z}$) are solutions of the equation $\sin 4x = 0$ (since $\sin 4(\pi k/2) = \sin 2\pi k = 0$, $k \in \mathbf{Z}$), therefore it suffices to find the roots of the equation $\sin 4x = 0$. Consequently, $4x = \pi n$ or:

$$x = \pi n/4, \quad n \in \mathbf{Z}.$$ ▶

Example 3.2.6. Solve the equation

$$\sin x + \sin 2x + 2\sin x \sin 2x = 2\cos x + \cos 2x.$$

◀By virtue of identity (2.29) for the product of sines, we have

$$\sin x + \sin 2x + \cos x - \cos 3x = 2\cos x + \cos 2x,$$

or

$$\sin x + \sin 2x - \cos x - \cos 2x - \cos 3x = 0,$$

$$\sin x + \sin 2x - (\cos x + \cos 3x) - \cos 2x = 0.$$

Applying formulas (2.21) and (2.36), we get

$$\sin x + 2 \sin x \cos x - 2 \cos 2x \cos x - \cos 2x = 0,$$

or

$$\sin x (1 + 2 \cos x) - \cos 2x (1 + 2 \cos x) = 0,$$

$$(\sin x - \cos 2x)(1 + 2 \cos x) = 0.$$

Thus, the original equation has been decomposed into two equations:

(1) $1 + 2 \cos x = 0$ or $\cos x = -1/2$,

for which $x = \pm\frac{2\pi}{3} + 2\pi n$, $n \in \mathbf{Z}$,

(2) $\sin x - \cos 2x = 0$.

By formula (2.39), we transform this equation to the form

$$\sin x - 1 + 2 \sin^2 x = 0$$

and set $y = \sin x$:

$$2y^2 + y - 1 = 0.$$

The quadratic equation thus obtained has two roots: $y_1 = -1$, $y_2 = 1/2$. In the first case

$$\sin x = -1, \quad \text{or} \quad x = -\frac{\pi}{2} + 2\pi n, \quad n \in \mathbf{Z}.$$

In the second case

$$\sin x = 1/2, \quad \text{that is,} \quad x = (-1)^n \frac{\pi}{6} + \pi n, \quad n \in \mathbf{Z}.$$

Thus, the solutions of the original equation are written in the form of three series

$$x = \pm \frac{2\pi}{3} + 2\pi n, \qquad x = -\frac{\pi}{2} + 2\pi n,$$

$$x = (-1)^n \frac{\pi}{6} + \pi n, \quad n \in \mathbf{Z}. \ ▶$$

Example 3.2.7. Solve the equation

$$\tan 2x + \frac{1}{\sin x} = \cot x + \frac{1}{\sin 5x}.$$

◀Transform the given equation:

$$\frac{\sin 2x}{\cos 2x} - \frac{\cos x}{\sin x} + \frac{1}{\sin x} - \frac{1}{\sin 5x} = 0,$$

or

$$\frac{\sin 2x \sin x - \cos 2x \cos x}{\sin x \cos 2x} + \frac{\sin 5x - \sin x}{\sin x \sin 5x} = 0.$$

Applying identities (2.1) and (2.18), we get

$$\frac{-\cos 3x}{\sin x \cos 2x} + \frac{2 \cos 3x \sin 2x}{\sin x \sin 5x} = 0,$$

or

$$-\frac{\cos 3x (\sin 5x - 2 \sin 2x \cos 2x)}{\sin x \cos 2x \sin 5x} = 0,$$

$$-\frac{\cos 3x (\sin 5x - \sin 4x)}{\sin x \cos 2x \sin 5x} = 0,$$

and, again by formula (2.18),

$$\frac{\cos 3x \cdot 2 \sin \frac{x}{2} \cos \frac{9x}{2}}{\sin x \cos 2x \sin 5x} = 0.$$

Note that if $\sin x = 0$ (i.e. $x = \pi n$, $n \in \mathbf{Z}$), then also $\sin 5x = 0$, and the equality $\sin \frac{x}{2} = 0$ means that $\sin x = 0$ ($\sin 2\pi n = 0$, $n \in \mathbf{Z}$). Consequently, the domain of permissible values of the given equation can be specified by two conditions:

$$\cos 2x \neq 0 \quad \text{and} \quad \sin 5x \neq 0,$$

and on this domain the original equation is decomposed into two equations:

(1) $\cos 3x = 0$,

(2) $\cos \frac{9}{2} x = 0$.

Let us solve equation (1). We have $3x = \frac{\pi}{2} + \pi n$ ($n \in \mathbf{Z}$), or $x = \frac{\pi}{6} + \frac{\pi n}{3}$, and we have to check whether the constraints specifying the domain of permissible values are met. For $n = 3k$, $k \in \mathbf{Z}$, the expression $\cos\left(2\left(\frac{\pi}{6} + \frac{\pi n}{3}\right)\right) = \cos\frac{\pi(2n+1)}{3}$ takes on values equal to 1/2, for $n = 3k + 1$, $k \in \mathbf{Z}$, values equal to $-1/2$, and for $n = 3k + 2$, $k \in \mathbf{Z}$, values equal to -1, that is, $\cos 2x \neq 0$ for all these values of x. Further, for $n = 6k$ or $n = 6k + 2$, $k \in \mathbf{Z}$, the expression $\sin\left(5\left(\frac{\pi}{6} + \frac{\pi n}{3}\right)\right) = \sin\left(\frac{5\pi}{6} + \frac{5\pi n}{3}\right)$ takes on values equal to 1/2, for $n = 6k + 1$, $k \in \mathbf{Z}$, values equal to 1, for $n = 6k + 3$ or $n = 6k + 5$, $k \in \mathbf{Z}$, values equal to $-1/2$, and for $n = 6k + 4$, $k \in \mathbf{Z}$, values equal to -1, that is, $\sin 5x \neq 0$ for the indicated values of x, and all of them belong to the domain of permissible values.

Consider now equation (2). We have:

$$\frac{9}{2}x = \frac{\pi}{2} + \pi n, \quad \text{or} \quad x = \frac{\pi}{9} + \frac{2\pi n}{9}, \quad n \in \mathbf{Z},$$

and check whether the constraints $\cos 2x \neq 0$ and $\sin 5x \neq 0$ are met. Note that the expression $\cos\left(2\left(\frac{\pi}{9} + \frac{2\pi n}{9}\right)\right)$ takes on one of the following nine values: $\cos\frac{2\pi}{9}$, $-\frac{1}{2}$, $\cos\frac{10\pi}{9}$, $\cos\frac{14\pi}{9}$, 1, $\cos\frac{22\pi}{9}$, $\cos\frac{26\pi}{9}$, $\cos\frac{10\pi}{3}$, $\cos\frac{34\pi}{3}$, none of them being zero. Similarly, we check to see that the expression $\sin\left(5\left(\frac{\pi}{9} + \frac{2\pi n}{9}\right)\right)$ does not vanish either for any integral values of n. Thus, we have found all the solutions of the original equation: $x = \frac{\pi}{6} + \frac{\pi n}{3}$, $x = \frac{\pi}{9} + \frac{2\pi n}{9}$, $n \in \mathbf{Z}$. ▶

Example 3.2.8. Solve the equation

$$5\cos 3x + 3\cos x = 3\sin 4x.$$

◀Let us first note that if we apply twice formula (2.36) for the sine of a double angle, we shall get the identity $\sin 4x = 2\sin 2x\cos 2x = 4\sin x\cos x\cos 2x$. Using this identity and (2.54), we rewrite the given equation in the form

$$5(4\cos^3 x - 3\cos x) + 3\cos x = 12\sin x\cos x\cos 2x,$$

or

$$\cos x\,(20\cos^2 x - 15 + 3 - 12\sin x\,(1 - 2\sin^2 x)) = 0,$$

$$\cos x\,(20\,(1 - \sin^2 x) - 12 - 12\sin x\,(1 - 2\sin^2 x)) = 0,$$

$$\cos x\,(20 - 20\sin^2 x - 12 - 12\sin x + 24\sin^3 x) = 0,$$

$$\cos x\,(6\sin^3 x - 5\sin^2 x - 3\sin x + 2) = 0,$$

$$\cos x\,(6\sin^2 x\,(\sin x - 1) + \sin x\,(\sin x - 1) - 2\,(\sin x - 1)) = 0,$$

$$\cos x\,(\sin x - 1)\,(6\sin^2 x + \sin x - 2) = 0.$$

Thus, the given equation decomposes into the following three equations:

(1) $\cos x = 0, \quad x = \frac{\pi}{2} + \pi n, \quad n \in \mathbf{Z},$

(2) $\sin x = 1, \quad x = \frac{\pi}{2} + 2\pi n, \quad n \in \mathbf{Z}$

(we see that the solutions of equation (2) are at the same time solutions of equation (1)),

(3) $6\sin^2 x + \sin x - 2 = 0.$

Setting $y = \sin x$ we get an algebraic equation

$$6y^2 + y - 2 = 0,$$

whose roots are $y_1 = -2/3$ and $y_2 = 1/2$, and it remains to consider two cases:

(a) $\sin x = -\frac{2}{3}, \; x = (-1)^{n+1} \arcsin \frac{2}{3} + \pi n, \quad n \in \mathbf{Z},$

(b) $\sin x = \frac{1}{2}, \; x = (-1)^n \frac{\pi}{6} + \pi n, \quad n \in \mathbf{Z}.$

Thus, all the solutions of the original equation are described by the formulas

$$x = \frac{\pi}{2} + \pi n, \quad x = (-1)^{n+1} \arcsin \frac{2}{3} + \pi n,$$

$$x = (-1)^n \frac{\pi}{6} + \pi n, \quad n \in \mathbf{Z}. \blacktriangleright$$

Example 3.2.9. Solve the equation

$$\cot 2x + 3\tan 3x = 2\tan x + \frac{2}{\sin 4x}.$$

◀ Let us represent the given equation in the form

$$(\cot 2x + \tan x) + 3(\tan 3x - \tan x) = \frac{2}{\sin 4x}$$

and use the following identity:

$$\cot 2x + \tan x = \frac{\cos 2x}{\sin 2x} + \frac{\sin x}{\cos x}$$

$$= \frac{\cos 2x \cos x + \sin 2x \sin x}{\sin 2x \cos x} = \frac{\cos x}{\sin 2x \cos x} = \frac{1}{\sin 2x},$$

whose domain of permissible values is specified by the condition $\sin 2x \neq 0$, since in this case also $\cos x \neq 0$. Applying this equality and formula (2.26) for the difference of tangents, we get

$$\frac{1}{\sin 2x} + 3\frac{\sin 2x}{\cos 3x \cos x} = \frac{2}{\sin 4x}.$$

The domain of permissible values for the given equation can be specified by the following two conditions: $\sin 4x \neq 0$, that is, $x \neq \pi n/4$, $n \in \mathbf{Z}$, and $\cos 3x \neq 0$, that is, $x \neq \frac{\pi}{6} + \frac{\pi n}{3}$, $n \in \mathbf{Z}$. We transform the obtained equation on the given domain:

$$3\frac{\sin 2x}{\cos 3x \cos x} = \frac{2}{\sin 4x} - \frac{1}{\sin 2x},$$

$$3\frac{\sin 2x}{\cos 3x \cos x} = \frac{1}{\sin 2x \cos 2x} - \frac{1}{\sin 2x},$$

$$3\frac{\sin 2x}{\cos 3x \cos x} = \frac{1 - \cos 2x}{\sin 2x \cos 2x},$$

$$\frac{6\sin x \cos x}{\cos 3x \cos x} = \frac{2\sin^2 x}{2\sin x \cos x \cos 2x}.$$

Since $\sin x \neq 0$ and $\cos x \neq 0$, we have

$$\frac{6\cos x}{\cos 3x} = \frac{1}{\cos 2x}.$$

Thus, in the domain of permissible values the original equation is equivalent to

$$6\cos x \cos 2x = \cos 3x,$$

and, by virtue of identity (2.54) for the cosine of a triple angle, we have

$$6 \cos x \cos 2x = 4 \cos^3 x - 3 \cos x,$$

whence

$$6 \cos 2x = 4 \cos^2 x - 3 \,(\cos x \neq 0),$$
$$6 \cos 2x = 2 (1 + \cos 2x) - 3,$$
$$6 \cos 2x = 2 \cos 2x - 1,$$
$$4 \cos 2x = -1, \ \cos 2x = -1/4,$$

whence $x = \pm \frac{1}{2} \arccos\left(-\frac{1}{4}\right) + \pi n, \ n \in \mathbf{Z}$.

It is easy to check that the found values of x satisfy the conditions specifying the domain of permissible values in which all the transformations carried out were invertible. ▶

Example 3.2.10. Solve the equation

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cos 4x = 0.$$

◀Note that $x = 2\pi n, \ n \in \mathbf{Z}$, are not solutions of the given equation, therefore we may assume that $x \neq 2\pi n$. Then $\sin \frac{x}{2} \neq 0$, and the following equalities hold:

$$\cos x + \cos 2x + \cos 3x + \cos 4x$$

$$= \frac{1}{2 \sin \frac{x}{2}} \left(2 \cos x \sin \frac{x}{2} + 2 \cos 2x \sin \frac{x}{2}\right.$$

$$\left. + 2 \cos 3x \sin \frac{x}{2} + 2 \cos 4x \sin \frac{x}{2}\right)$$

$$= \frac{1}{2 \sin \frac{x}{2}} \left(\sin \frac{3}{2} x - \sin \frac{x}{2} + \sin \frac{5}{2} x - \sin \frac{3}{2} x\right.$$

$$\left. + \sin \frac{7}{2} x - \sin \frac{5}{2} x + \sin \frac{9}{2} x - \sin \frac{7}{2} x\right)$$

$$= \frac{1}{2 \sin \frac{x}{2}} \left(\sin \frac{9}{2} x - \sin \frac{x}{2}\right) = \frac{\sin \frac{9}{2} x}{2 \sin \frac{x}{2}} - \frac{1}{2}.$$

Consequently, the original equation can be transformed to the form

$$\frac{\sin \frac{9}{2} x}{2 \sin \frac{x}{2}}=0 \quad \text{or} \quad \sin \frac{9}{2} x=0,$$

whence $x=\frac{2\pi k}{9}$, $k \in \mathbf{Z}$, provided $x \neq 2\pi n$ which is equivalent to $\frac{2\pi k}{9} \neq 2\pi n$ or $k \neq 9n$ $(n \in \mathbf{Z})$. ▶

3. Solving Trigonometric Equations Using the Properties of Trigonometric Functions. Frequently, we have to deal with equations of the form $f(t) = g(t)$, where f and g are some functions containing trigonometric expressions such that enable us to investigate the domains of values $E(f)$ and $E(g)$ and to prove that these domains either do not intersect or have few points in common. In such cases, the solutions of the equation $f(t) = g(t)$ should be sought for among such t's which satisfy (simpler) equations $f(t) = a$, $g(t) = a$, where a is a real number such that $a \in E(f)$ and $a \in E(g)$, that is, $a \in E(f) \cap E(g)$.

Example 3.2.11. Solve the equation

$$\sin^2 4x + \cos^2 x = 2 \sin 4x \cos^4 x.$$

◀Let us write the given equation in the form

$$\sin^2 4x - 2 \sin 4x \cos^4 x = -\cos^2 x.$$

Adding $\cos^8 x$ to both sides of the equation, we get

$$\sin^2 4x - 2 \sin 4x \cos^4 x + \cos^8 x = \cos^8 x - \cos^2 x$$

or

$$(\sin 4x - \cos^4 x)^2 = -\cos^2 x (1 - \cos^6 x).$$

The left-hand side of the equation is nonnegative, while the right-hand side is nonpositive ($\cos^2 x \geqslant 0$, $1 - \cos^6 x \geqslant 0$), consequently, the equality will be valid only when the following conditions are fulfilled simultaneously:

$$\begin{cases} -\cos^2 x\,(1-\cos^6 x)=0, \\ (\sin 4x-\cos^4 x)^2=0. \end{cases}$$

The first equation decomposes in to two:

(1) $\cos^2 x = 0$ or $\cos x = 0$,

whence $x=\frac{\pi}{2}+\pi n$, $n\in \mathbf{Z}$. The obtained values also satisfy the second equation since

$$\sin\left(4\left(\frac{\pi}{2}+\pi n\right)\right)=\sin(2\pi+4\pi n)=0.$$

(2) $1-\cos^6 x = 0$ or $\cos x = \pm 1$,

whence $x = \pi n$, $n \in \mathbf{Z}$. Substituting these values of x into the second equation, we get $(\sin 4\pi n - \cos^4 \pi n)^2 = 0$ or $(0-1)^2 = 0$ which is wrong.

Thus, the solution of the original equation consists of the numbers $x=\frac{\pi}{2}+\pi n$, $n\in \mathbf{Z}$. ▶

Example 3.2.12. Solve the equation

$$\sin^6 x + \cos^6 x = p,$$

where p is an arbitrary real number.

◀Note that

$$\begin{aligned} &\sin^6 x+\cos^6 x \\ &\quad=(\sin^2 x+\cos^2 x)(\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x) \\ &\quad=\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x \\ &\quad=(\sin^4 x+2\sin^2 x\cos^2 x+\cos^4 x)-3\sin^2 x\cos^2 x \\ &\quad=(\sin^2 x+\cos^2 x)^2-\frac{3}{4}(2\sin x\cos x)^2 \\ &\quad=1-\frac{3}{4}\sin^2 2x=1-\frac{3}{8}(1-\cos 4x) \\ &\quad=\frac{3}{8}\cos 4x+\frac{5}{8}, \end{aligned}$$

and the given equation takes the form

$$\frac{3}{8}\cos 4x+\frac{5}{8}=p \quad \text{or} \quad \cos 4x=\frac{8p-5}{3}.$$

The equation has the solution $x = \pm\frac{1}{4}\arccos\frac{8p-5}{3}+\frac{\pi n}{2}$, $n\in\mathbf{Z}$, for $-1\leqslant(8p-5)/3\leqslant 1$ or $1/4\leqslant p\leqslant 1$. ▶

Example 3.2.13. Solve the equation

$$\left(\cos\frac{x}{4}-2\sin x\right)\sin x+\left(1+\sin\frac{x}{4}-2\cos x\right)\cos x=0.$$

◀Remove the parentheses and then use the fundamental trigonometric identity and formula (2.11) for the sine of the sum of two numbers. We get

$$\cos\frac{x}{4}\sin x-2\sin^2 x+\cos x+\sin\frac{x}{4}\cos x-2\cos^2 x=0,$$

that is,

$$\sin\left(x+\frac{x}{4}\right)+\cos x-2\left(\sin^2 x+\cos^2 x\right)=0,$$

or

$$\sin\frac{5x}{4}+\cos x=2.$$

Note that the sum in the left-hand side of the obtained equation will equal 2 only if $\sin\frac{5x}{4}=1$ and $\cos x=1$ simultaneously, that is, our equation is equivalent to the system of equations:

$$\begin{cases}\sin\frac{5x}{4}=1,\\ \cos x=1,\end{cases}$$

whence

$$\begin{cases}\frac{5x}{4}=\frac{\pi}{2}+2\pi n, & n\in\mathbf{Z},\\ x=2\pi k, & k\in\mathbf{Z},\end{cases}$$

and the equality $2\pi k=\frac{2\pi}{5}+\frac{8\pi}{5}n$ must hold, whence $k=\frac{1+4n}{5}$. Since $k\in\mathbf{Z}$, we have $n=5m+1$, $m\in\mathbf{Z}$ (since for the remaining integral n's, that is, $n=5m$,

$n = 5m + 2$, $n = 5m + 3$, $n = 5m + 4$, it is obvious that $k \notin \mathbf{Z}$), and then $x = 2\pi + 8\pi m$, $m \in \mathbf{Z}$, that is, $x = 2\pi(4m + 1)$, $m \in \mathbf{Z}$.

3.3. Solving Trigonometric Equations and Systems of Equations in Several Unknowns

The presence of two or more unknowns involve certain difficulties in solving trigonometric equations and systems. The solution of such an equation or system is defined as a set of values of the variables which turn the given equation or each of the equations of a system into a true numerical equality. To solve a given equation or system is to find all such sets. Therefore, answering a problem of this type by giving the values taken on by each unknown is senseless. One of the difficulties encountered in solving such problems is also that the set of solutions for these equations and systems, is, as a rule, infinite. Therefore, to write the answer in a correct way and to choose desired solutions, one has to consider different cases, to check the validity of auxiliary inequalities, etc. In some cases, when solving systems of equations we can eliminate one of the unknowns rather easily by expressing it in terms of other unknowns from one of the equations of the system. Another widely used method is to try to reduce a trigonometric system to a system of algebraic equations involving some trigonometric functions as new unknowns. As in solving trigonometric equations in one unknown, we can try to carry out identical transformations to decompose one or more of the equations to the simple equations of the type $\sin(x + 2y) = -1$, $\tan(x - y) = \sqrt{3}$, and so forth.

Example 3.3.1. Solve the system of equations

$$\begin{cases} \sqrt{\sin x}\cos y = 0, \\ 2\sin^2 x - \cos 2y - 2 = 0. \end{cases}$$

◀It follows from the first equation that $\sin x \geqslant 0$, two cases are possible here: if $\sin x = 0$, then the equation turns into an identity, and if $\sin x > 0$, then the equation implies that $\cos y = 0$. Consequently, the system is equivalent to the collection of two systems:

$$\begin{cases} \sin x = 0, \\ 2\sin^2 x - \cos 2y - 2 = 0 \end{cases}$$

and

$$\begin{cases} \cos y = 0, \\ 2\sin^2 x - \cos 2y - 2 = 0, \quad \sin x > 0. \end{cases}$$

The first system has no solutions ($\cos 2y + 2 \neq 0$), while the second is equivalent to the system of two simple equations

$$\begin{cases} \cos y = 0, \\ \sin x = \sqrt{2}/2. \end{cases}$$

Consequently, the set of all solutions of the original system consists of pairs of numbers (x, y) of the kind

$$\left((-1)^k \frac{\pi}{4} + \pi k, \ \frac{\pi}{2} + \pi l\right), \quad k, l \in \mathbf{Z}. \ ▶$$

Example 3.3.2. Solve the equation

$$\frac{3+2\cos(x-y)}{2} = \sqrt{3+2x-x^2}\cos^2\frac{x-y}{2} + \frac{\sin^2(x-y)}{2}.$$

◀ Using formula (2.40) for reducing the power of cosine, we get

$$\frac{3+2\cos(x-y)}{2} = \sqrt{3+2x-x^2}\,\frac{1+\cos(x-y)}{2} + \frac{\sin^2(x-y)}{2},$$

or

$$1 - \sin^2(x-y) + (2 - \sqrt{3+2x-x^2})\cos(x-y) + 2 - \sqrt{3+2x-x^2} = 0,$$

$$\cos^2(x-y) + (2 - \sqrt{3+2x-x^2})\cos(x-y) + 2 - \sqrt{3+2x-x^2} = 0. \qquad (3.6)$$

Let us set $t = \cos(x-y)$ and $a = 2 - \sqrt{3+2x-x^2}$, then equation (3.6) can be rewritten as follows:

$$t^2 + at + a = 0.$$

The given equality can be regarded as a quadratic equation with respect to t which has solutions only if its discriminant is nonnegative, hence,

$$a^2 - 4a \geqslant 0 \quad \text{or} \quad a(a-4) \geqslant 0,$$

whence $a \leqslant 0$ or $a \geqslant 4$. However, according to the introduced notation, $a = 2 - \sqrt{3 + 2x - x^2} = 2 - \sqrt{4 - (x-1)^2}$, and therefore $0 = 2 - \sqrt{4} \leqslant a \leqslant 2$. Consequently, $a = 0$, whence $2 - \sqrt{4 - (x-1)^2} = 0$ or $x = 1$. Then equation (3.6) takes the form $\cos^2(1 - y) = 0$ or $\cos(y - 1) = 0$, whence

$$y - 1 = \frac{\pi}{2} + \pi n, \quad n \in \mathbf{Z}.$$

Thus, all the solutions of the original equation are pairs of numbers (x, y) of the form $\left(1, 1 + \frac{\pi}{2} + \pi n\right)$, $n \in \mathbf{Z}$. ▶

Example 3.3.3. Solve the system of equations

$$\begin{cases} \cot x + \sin 2y = \sin 2x, \\ 2 \sin y \sin(x + y) = \cos x. \end{cases}$$

◀ Using formula (2.29), we can represent the second equation as follows:

$$\cos x - \cos(x + 2y) = \cos x.$$

Consequently, $\cos(x + 2y) = 0$, whence $x + 2y = \frac{\pi}{2} + \pi k$, $x = \frac{\pi}{2} - 2y + \pi k$, $k \in \mathbf{Z}$. Note that $\cot x = \cot\left(\frac{\pi}{2} - 2y\right) = \tan 2y$ and $\sin 2x = \sin(\pi - 4y) = \sin 4y$. Therefore the substitution of x into the first equation yields:

$$\tan 2y + \sin 2y = \sin 4y,$$

or

$$\sin 2y\,(1 + \cos 2y - 2\cos^2 2y) = 0,$$

$$\sin 2y\,(\cos 2y - 1)\left(\cos 2y + \frac{1}{2}\right) = 0.$$

The last equation decomposes into three equations:

$$\sin 2y = 0, \ \cos 2y = 1 \ \text{and} \ \cos 2y = -1/2.$$

From the equation $\sin 2y = 0$ we get $y = \frac{\pi n}{2}$, $n \in \mathbf{Z}$, and therefore $x = \frac{\pi}{2} + \pi l$, $l \in \mathbf{Z}$. From the equation $\cos 2y = 1$ we find $2y = 2\pi n$, $y = \pi n$, $n \in \mathbf{Z}$; hence, $x = \frac{\pi}{2} - 2\pi n + \pi k = \frac{\pi}{2} + \pi l$, $l \in \mathbf{Z}$. We see that the set of solutions of the second equation belongs to the set of solutions of the first equation. Finally, from the equation $\cos 2y = -\frac{1}{2}$ it follows that $2y = \pm\frac{2\pi}{3} + 2\pi n$, and therefore $y = \pm\frac{\pi}{3} + \pi n$, $n \in \mathbf{Z}$; hence, $x = \frac{\pi}{2} \pm \frac{2\pi}{3} - 2\pi n + \pi k = \frac{\pi}{2} \pm \frac{2\pi}{3} + \pi l$, $l \in \mathbf{Z}$. The final answer:

$$\left(\frac{\pi}{2} + \pi l, \frac{\pi n}{2}\right), \left(\frac{\pi}{2} \pm \frac{2\pi}{3} + \pi l, \pm\frac{\pi}{3} + \pi n\right),$$

$n, l \in \mathbf{Z}$. ▶

Example 3.3.4. Solve the system of equations

$$\begin{cases} |x| + |y| = 3, \\ \sin \frac{\pi x^2}{2} = 1. \end{cases}$$

◀From the second equation it follows directly that $\frac{\pi x^2}{2} = \frac{\pi}{2} + 2\pi k$, that is, $x^2 = 1 + 4k$, $k \in \mathbf{Z}$, whence $x_1 = -\sqrt{4k+1}$, $x_2 = \sqrt{4k+1}$. By virtue of the first equation of the system, we have $|x| \leqslant 3$. Consequently, k may take on only the values 0, 1, 2. Thus, there are six values of x, namely: ± 1, $\pm\sqrt{5}$, ± 3. If $x = \pm 1$, then $|y| = 2$, that is, $y = \pm 2$; if $x = \pm\sqrt{5}$, then $|y| = 3 - \sqrt{5}$, that is, $y = \pm(3 - \sqrt{5})$; finally, if $x = \pm 3$, then $|y| = 0$, that is, $y = 0$.

Answer: $(\pm 1, \pm 2)$, $(\pm 5, \pm(3 - \sqrt{5}))$, $(\pm 3, 0)$, all combinations of sign being possible, that is, the system has ten solutions. ▶

Example 3.3.5. Find out for what values of a the system of equations

$$\begin{cases} \sin x \cos 2y = (a^2 - 1)^2 + 1, \\ \cos x \sin 2y = a + 1 \end{cases}$$

has a solution. Find all the solutions.

◀Since the left-hand sides of the equations do not, obviously, exceed 1, the given system may have a solution only for the a's such that

$$\begin{cases} (a^2-1)^2+1 \leqslant 1, \\ |a+1| \leqslant 1. \end{cases}$$

Solving the first inequality, we get $a = \pm 1$, the second inequality being satisfied for $a = -1$.

Thus, the original system of equations has a solution only for $a = -1$ and, consequently, takes the form

$$\begin{cases} \sin x \cos 2y = 1, \\ \cos x \sin 2y = 0. \end{cases}$$

Adding and subtracting the equations of the system termwise, we get the system

$$\begin{cases} \sin x \cos 2y + \cos x \sin 2y = 1, \\ \sin x \cos 2y - \cos x \sin 2y = 1, \end{cases}$$

which is equivalent to the given system, whence (by formulas (2.11) and (2.12))

$$\begin{cases} \sin(x+2y) = 1, \\ \sin(x-2y) = 1. \end{cases}$$

Obviously, the last system yields an algebraic system

$$\begin{cases} x+2y = \frac{\pi}{2} + 2\pi k, & k \in \mathbf{Z}, \\ x-2y = \frac{\pi}{2} + 2\pi n, & n \in \mathbf{Z}. \end{cases}$$

Solving this system, we get

$$x = \frac{\pi}{2} + (k+n)\pi, \quad y = \frac{\pi(k-n)}{2}.$$

Thus, $\left(\frac{\pi}{2} + (k+n)\pi, \ \frac{\pi}{2}(k-n)\right)$, $k, n \in \mathbf{Z}$, are all the solutions of the original system. ▶

Example 3.3.6. Find all the solutions of the system

$$\begin{cases} |\sin x| \sin y = -1/4, \\ \cos(x+y) + \cos(x-y) = 3/2, \end{cases}$$

such that $0 < x < 2\pi$, $\pi < y < 2\pi$.

◀Consider the case when $\sin x \geqslant 0$, then, by hypothesis, $0 < x \leqslant \pi$, and

$$|\sin x| \sin y = \sin x \sin y = \frac{1}{2} (\cos (x - y) - \cos (x + y)).$$

Then the system takes the form

$$\begin{cases} \cos (x - y) - \cos (x + y) = -1/2, \\ \cos (x + y) + \cos (x - y) = 3/2, \end{cases}$$

whence

$$\begin{cases} \cos (x - y) = 1/2, \\ \cos (x + y) = 1. \end{cases}$$

By subtracting and adding the inequalities $0 < x \leqslant \pi$ and $\pi < y < 2\pi$ we get $-2\pi < x - y < 0$ and $\pi < x + y < 3\pi$. Thus, we have two systems of equations

$$\begin{cases} x - y = -\frac{\pi}{3}, \\ x + y = 2\pi, \end{cases} \quad \begin{cases} x - y = -\frac{5\pi}{3}, \\ x + y = 2\pi, \end{cases}$$

whose solutions are $(5\pi/6, 7\pi/6)$ and $(\pi/6, 11\pi/6)$, respectively.

Similarly, if $\sin x < 0$ we get

$$\begin{cases} \cos (x - y) - \cos (x + y) = 1/2, \\ \cos (x + y) + \cos (x - y) = 3/2 \end{cases}$$

and $\pi < x < 2\pi$, whence

$$\begin{cases} \cos (x - y) = 1, \\ \cos (x + y) = 1/2, \end{cases} \quad -\pi < x - y < \pi, \quad 2\pi < x + y < 4\pi,$$

whence

$$\begin{cases} x - y = 0, \\ x + y = 7\pi/3, \end{cases} \quad \text{and} \quad \begin{cases} x - y = 0, \\ x + y = 11\pi/3. \end{cases}$$

The solutions of these systems are $(7\pi/6, 7\pi/6)$ and $(11\pi/6, 11\pi/6)$, respectively. Finally, the original system has the following solutions:

$$(5\pi/6, 7\pi/6), (\pi/6, 11\pi/6),$$
$$(7\pi/6, 7\pi/6), (11\pi/6, 11\pi/6). \blacktriangleright$$

Example 3.3.7. Find all the values of a for which the indicated system of equations has a solution:

$$\begin{cases} \left|12\sqrt{\cos\frac{\pi y}{2}}-5\right|-\left|12\sqrt{\cos\frac{\pi y}{2}}-7\right| \\ +\left|24\sqrt{\cos\frac{\pi y}{2}}+13\right|=11-\sqrt{\sin\frac{\pi(x-2y-1)}{3}}, \\ 2(x^2+(y-a)^2)-1=2\sqrt{x^2+(y-a)^2-\frac{3}{4}}. \end{cases}$$

◀The left-hand side of the first equation has the form $F\left(\sqrt{\cos\frac{\pi y}{2}}\right)$, where $F(z)=|12z-5|-|12z-7|+|24z+13|$. Bearing in mind that $z=\sqrt{\cos\frac{\pi y}{2}}\geqslant 0$, we test the function $F(z)$ for $z\geqslant 0$ using the method of intervals. For this purpose, it is necessary to mark on the number line all the points at which the expressions inside the modulus sign change sign and to consider our function separately on each of the intervals thus obtained, thus getting rid of the modulus sign. Then, we get:

(1) $0\leqslant z\leqslant 5/12$, $F(x)=(5-12z)+(12z-7)+(24z+13)=11+24z$,

(2) $5/12\leqslant z\leqslant 7/12$, $F(z)=(12z-5)+(12z-7)+(24z+13)=48z+1$,

(3) $z\geqslant 7/12$, $F(z)=(12z-5)-(12z-7)+(24z+13)=24z+15$.

Thus, $F(z)$ is an increasing function for $z\geqslant 0$, its least value being $F(0)=11$. The right-hand side of the first equation takes on values $\leqslant 11$ $\left(\text{since } \sqrt{\sin\frac{\pi(x-2y-1)}{4}}\geqslant 0\right)$. Consequently, the first equation of the original system is equivalent to the system

$$\begin{cases} \cos\frac{\pi y}{2}=0, \\ \sin\frac{\pi(x-2y-1)}{3}=0, \end{cases}$$

or

$$\begin{cases} y=2t+1, \\ x-2y-1=3v, \end{cases} \quad t, v\in \mathbf{Z}. \tag{3.7}$$

Consider now the second equation of the original system. We substitute $u=\sqrt{x^2+(y-a)^2-\frac{3}{4}}$, and get $2\left(u^2+\frac{3}{4}\right)-1=2u$ or $u^2-u+\frac{1}{4}=0$, whence $u=1/2$, that is, $x^2+(y-a)^2-\frac{3}{4}=\frac{1}{4}$, or

$$x^2+(y-a)^2=1. \qquad (3.8)$$

Since x is an integer, only the cases $x = \pm 1$, $x = 0$ are possible. Let us consider them separately.

Case (a) $x = -1$. From system (3.7), it follows that

$$2y = -3v - 2, \quad y = 2t + 1, \quad v, t \in \mathbf{Z}.$$

Rewriting these equalities in the form $2(y + 1) = -3v$, $y + 1 = 2t + 2$, we conclude that $y + 1$ is divisible by 6. In addition, from (3.8) for $x = -1$ we have $(y - a)^2 = 0$, hence $y = a = 6k - 1$, $k \in \mathbf{Z}$.

Case (b) $x = 1$. Here again from (3.8) and (3.7): $y = a$, $2y = -3v$, $y = 2t + 1$ for some integers v and t, whence $2(a - 3) = -3(v + 2)$, $a - 3 = 2(t - 1)$. Hence, $a - 3$ is divisible by 6. In this case $a = 6k + 3$, $k \in \mathbf{Z}$.

Case (c) $x = 0$. Here, according to (3.8), $(y - a)^2 = 1$, $y = a \pm 1$, and system (3.7) takes the form

$$2y = -3v - 1, \quad y = 2t + 1, \text{ or}$$

$$2(y - 1) = -3(v + 1), \quad y - 1 = 2t, \quad v, t \in \mathbf{Z}.$$

Consequently, $y - 1$ is divisible by 6, and since $a = y \pm 1$, we have $a = 6k$ or $a = 6k + 2$, $k \in \mathbf{Z}$. Putting together all the results, we get the final answer: the system has a solution if a takes on the values $6k - 1$, $6k$, $6k + 2$, $6k + 3$, $k \in \mathbf{Z}$. ▶

Example 3.3.8. Find all the solutions of the equation

$$\sqrt{2-|y|}\,(5\sin^2 x - 6\sin x\cos x - 9\cos^2 x + 3\sqrt[3]{33}) = \arcsin^2 x + \arccos^2 x - \frac{5}{4}\pi^2.$$

◀ Let us prove that the left-hand side of the equation is always nonnegative, while the right-hand side is nonpositive. Indeed, $\sqrt{2-|y|} \geqslant 0$. Now, we transform the expression in the parentheses:

$5\sin^2 x - 6\sin x\cos x - 9\cos^2 x + 3\sqrt[3]{33}$

$= 6\sin^2 x - (\sin^2 x + 6\sin x\cos x + 9\cos^2 x) + 3\sqrt[3]{33}$
$= 6\sin^2 x - (\sin x + 3\cos x)^2 + 3\sqrt[3]{33}$
$= 6\sin^2 x + 3\sqrt[3]{33} - 10\sin^2(x+\varphi)$,

where $\varphi = \arcsin 3/\sqrt[3]{10}$ by Theorem 2.2. We then note that $3\sqrt[3]{33} > 10$ (this is checked by cubing the positive right-hand and left-hand sides) and also that $10\sin^2(x+\varphi) \leqslant 10$, therefore the expression in the parentheses is always positive, that is, the left-hand side of the equation is nonnegative and vanishes only in the case $\sqrt{2-|y|} = 0$, that is, $y = \pm 2$.

As to the right-hand side, we use the identity

$$\arcsin x + \arccos x = \pi/2.$$

(If we rewrite this identity in the form $\arccos x = \frac{\pi}{2} - \arcsin x$, then its validity follows from the fact that $\cos\left(\frac{\pi}{2} - \arcsin x\right) = \sin(\arcsin x) = x$ and that $\frac{\pi}{2} - \arcsin x \in [0, \pi]$, that is, belongs to the range of values of arc cosine since $x \in [-\pi/2, \pi/2]$.) Let us set $t = \arcsin x$. Then the right-hand side of the equation takes the form $t^2 + \left(\frac{\pi}{2} - t\right)^2 - \frac{5}{4}\pi^2 = 2t^2 - \pi t - \pi^2$, where $t \in [-\pi/2, \pi/2]$. The greatest value of this quadratic function is attained at the end point $t = -\pi/2$ of the closed interval $[-\pi/2, \pi/2]$ and equals zero: $2(-\pi/2)^2 - \pi(-\pi/2) - \pi^2 = 0$. Thus, the right-hand side of the original equation is nonpositive and vanishes for $\arcsin x = -\pi/2$, that is, for $x = -1$.

Consequently, the equation has two solutions: $(-1, -2)$ and $(-1, 2)$. ▶

PROBLEMS

In Problems **3.1** to **3.36**, solve equations.

3.1. $\sin\left(\frac{\pi}{2} + 2x\right)\cot 3x + \sin(\pi + 2x) - \sqrt{2}\cos 5x = 0.$

3.2. $\sin x\cos 2x + \cos x\cos 4x = \sin\left(\frac{\pi}{4} + 2x\right) \times \sin\left(\frac{\pi}{4} - 3x\right).$

3.3. $\sin 2x = \cos^4 \frac{x}{2} - \sin^4 \frac{x}{2}$.

3.4. $(1 + \cos 4x) \sin 2x = \cos^2 2x$.

3.5. $\sin^2 2z + \sin^2 3z + \sin^2 4z + \sin^2 5z = 2$.

3.6. $\sin 2x \sin 6x - \cos 2x \cos 6x = \sqrt{2} \sin 3x \cos 8x$.

3.7. $\sin 3x \cos 3x = \sin 2x$.

3.8. $\cos 2x - 5 \sin x - 3 = 0$.

3.9. $3 \sin 2x + 2 \cos 2x = 3$.

3.10. $\cot\left(\frac{3\pi}{2} - x\right) + \cot^2 x + \frac{1 + \cos 2x}{\sin^2 x} = 0$.

3.11. $6 \sin^2 x + \sin x \cos x - \cos^2 x = 2$.

3.12. $\cos 7x + \sin 8x = \cos 3x - \sin 2x$.

3.13. $\sin^2 x - 2 \sin x \cos x = 3 \cos^2 x$.

3.14. $\cos 5x + \cos 7x = \cos(\pi + 6x)$.

3.15. $4 \sin x \cos\left(\frac{\pi}{2} - x\right) + 4 \sin(\pi + x) \cos x$

$$+ 2 \sin\left(\frac{3\pi}{2} - x\right) \cos(\pi + x) = 1.$$

3.16. $\sin x - \sin 2x + \sin 5x + \sin 8x = 0$.

3.17. $2 \sin z - \cos z = 2/5$.

3.18. $\cos\left(\frac{\pi}{2} + 5x\right) + \sin x = 2 \cos 3x$.

3.19. $(1 + \sin x) \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \frac{1}{\cos x} - \cos x$.

3.20. $\cos x - \sqrt{3} \sin x = \cos 3x$.

3.21. $\sin 2z + 5(\sin z + \cos z) + 1 = 0$.

3.22. $\sin^3 2t + \cos^3 2t + \frac{1}{2} \sin 4t = 1$.

3.23. $\tan z \tan 2z = \tan z + \tan 2z$.

3.24. $\frac{\sin^3 x + \cos^3 x}{2 \cos x - \sin x} = \cos 2x$.

3.25. $\frac{\cot 4t}{\sin^2 t} + \frac{\cot t}{\sin^2 4t} = 0$.

3.26. $\tan^4 x = 36 \cos^2 2x$.

3.27. $\cot x - \tan x - 2 \tan 2x - 4 \tan 4x + 8 = 0$.

3.28. $4 \sin^3 x \cos 3x + 4 \cos^3 x \sin 3x = 3 \sin 2x$.

3.29. $2 \cos z \sin^3\left(\frac{3\pi}{2} - z\right) - 5 \sin^2 z \cos^2 z$

$$+ \sin z \cos^3\left(\frac{3\pi}{2} + z\right) = \cos 2z.$$

3.30. $\sin 2x \sin 6x \cos 4x + \frac{1}{4} \cos 12x = 0.$

3.31. $\cos^{58} x + \sin^{40} x = 1.$

3.32. $\log_{\cos x} \sin x = 1.$

3.33. $\cot (\sin x) = 1.$

3.34. $\tan 5x - 2 \tan 3x = \tan 5x \tan^2 3x.$

3.35. $\cot x - 1 = \frac{\cos 2x}{\tan x + 1}.$

3.36. $\cos^2 3x + \frac{1}{4}\cos^2 x = \cos 3x \cos^4 x.$

3.37. Find all the solutions of the equation

$$1 + \cos x + \cos 2x + \sin x + \sin 2x + \sin 3x,$$

which satisfy the condition $\frac{\pi}{2} < \left| 3x - \frac{\pi}{2} \right| \leqslant \pi.$

3.38. Find all the solutions of the equation

$$2 - \sqrt{3} \cos 2x + \sin 2x = 4 \cos^2 3x,$$

which satisfy the inequality $\cos \left(2x - \frac{\pi}{4}\right) > 0.$

3.39. Solve the equation

$$\sqrt{\frac{1 - 4\cos^2 4x}{8 \cos \left(2x - \frac{2\pi}{3}\right)}} = \cos \left(2x - \frac{\pi}{6}\right).$$

3.40. Find all the solutions of the equation

$$2 + \cos \frac{3}{2} x + \sqrt{3} \sin \frac{3}{2} x = 4 \sin^2 \frac{x}{4},$$

satisfying the condition $\sin \left(\frac{x}{2} + \frac{\pi}{4}\right) > 0.$

3.41. Show that the equation

$$\cot 2x + \cot 3x + \frac{1}{\sin x \sin 2x \sin 3x} = 0$$

has no roots.

3.42. Find all the solutions of the equation

$$4 \left(3 \sqrt{4x - x^2} \sin^2 \left(\frac{x+y}{2}\right) + 2 \cos (x + y)\right) = 13 + 4 \cos^2 (x + y)$$

in two unknowns x and y.

3.43. Find all the solutions of the equation

$$\sqrt{x^2-4}\,(3\sin^2 x+10\sin x\cos x+11\cos^2 x-2\sqrt[3]{301}) = 5\pi^2-4\arcsin^2 y-4\arccos^2 y.$$

In Problems **3.44** to **3.50**, solve the given systems of equations.

3.44. $\begin{cases} \cos x\sqrt{\cos 2x}=0, \\ 2\sin^2 x-\cos\left(2y-\frac{\pi}{3}\right)=0. \end{cases}$

3.45. $\begin{cases} x+y=\pi/6, \\ 5(\sin 2x+\sin 2y)=2(1+\cos^2(x-y)). \end{cases}$

3.46. $\begin{cases} x-y=5\pi/3, \\ \sin x=2\sin y. \end{cases}$

3.47. $\begin{cases} \sin x\cos y=1/4, \\ \sin y\cos x=3/4. \end{cases}$

3.48. $\begin{cases} x-y=-1/3, \\ \cos^2 \pi x-\sin^2 \pi x=1/2. \end{cases}$

3.49. $\begin{cases} x+y=\pi/4, \\ \tan x\tan y=1/6. \end{cases}$

3.50. $\begin{cases} \sqrt{2}\sin x=\sin y, \\ \sqrt{2}\cos x=\sqrt{3}\cos y. \end{cases}$

3.51. Find all the values of a for which the system of equations

$$\begin{cases} \left|6\sqrt{\cos\frac{\pi y}{4}}-5\right|-\left|1-6\sqrt{\cos\frac{\pi y}{4}}\right| \\ \quad+\left|12\sqrt{\cos\frac{\pi y}{4}}+1\right|=5+\sin^2\frac{\pi(y-2x)}{12}, \\ 10-9(x^2+(y-a)^2=3\sqrt{x^2+(y-a)^2-\frac{8}{9}} \end{cases}$$

has a solution.

Chapter 4

Investigating Trigonometric Functions

4.1. Graphs of Basic Trigonometric Functions

To construct graphs of functions, one must know how to use all basic properties of these functions, such as periodicity, evenness or oddness, increase or decrease of a function on an interval, as well as the arrangement of points of extremum. However, constructing the graph of a function cannot be a substitute for a rigorous proof of the properties of the function. Nevertheless, a graph illustrates vividly the properties of a function. At an examination the student is usually asked to construct graphs of functions composed of elementary functions, and most often such a graph is constructed by translating or changing the graphs of elementary functions.

This section deals with the construction of graphs of basic trigonometric functions using the properties of these functions which were considered in Chapter 1. Using the formulas introduced in Chapter 2, it is possible to construct similar graphs for many other trigonometric functions.

First of all, let us recall that the graph of a function f with domain $D(f)$ is defined as a set of points on the coordinate plane with coordinates (x, y) such that $y = f(x)$. This definition should be always referred to when proving the properties of graphs of functions and when considering operations with graphs.

1. Properties and Graph of the Function $f(x) = \sin x$.

(1) The domain of definition $D(f) = \mathbf{R}$, the range of values $E(f) = [-1, 1]$.

(2) $\sin x$ is a periodic function. Any number of the form $2\pi k$, $k \in \mathbf{Z}$, is a period of this function, 2π being its fundamental period (see Sec. 1.3, Item 1). Consequently, when constructing the graph we can first confine ourselves to the construction of it for the closed interval

$[-\pi, \pi]$ of length 2π, and then translate this section by $2\pi k$, $k \in \mathbf{Z}$, along the x-axis. This is because all points of the form $(x + 2\pi k, \sin x) = (x + 2\pi k, \sin (x + 2\pi k))$ have the same values as the value of the point $(x, \sin x)$ on the graph.

(3) $\sin x$ is an odd function, therefore its graph is symmetric with respect to the origin. Indeed, for any point $(x, \sin x)$ of the graph the point $(-x, -\sin x) = (-x, \sin (-x))$, which is obtained by the application of central symmetry to the point $(x, \sin x)$, also lies on this graph (see Sec. 1.3, Item 2). Consequently, to construct the graph of the function on $[-\pi, \pi]$, it suffices to construct it on $[0, \pi]$, and then to map it central-symmetrically with respect to the origin.

(4) On the interval $[0, \pi]$ the graph has two points $(0, 0)$ and $(\pi, 0)$ in common with the x-axis. In general, the equality $\sin x = 0$ is equivalent to that $x = \pi k$, $k \in \mathbf{Z}$.

(5) The function $\sin x$ increases on the interval $[0, \pi/2]$ and decreases on $[\pi/2, \pi]$, this means that if $0 \leqslant x_1 < x_2 \leqslant \pi/2$, then $\sin x_1 < \sin x_2$, and if $\pi/2 \leqslant x_1 < x_2 \leqslant \pi$, then $\sin x_1 > \sin x_2$ (see Sec. 1.3, Item 3). Hence it follows that $(\pi/2, 1)$ is a point of maximum of the function $\sin x$.

The graph of the function $\sin x$ is constructed now in several steps. Firstly, we construct the graph on the interval $[0, \pi]$. This can be done by compiling a table of values of the function $\sin x$ for some points of the interval $[0, \pi]$, for instance,

(x)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

and, on plotting on the coordinate plane points of the form $(x, \sin x)$, where x are numbers from the table, we join these points with a smooth line. By mapping central-symmetrically the constructed section of the graph with respect to the point 0 and then applying a series of translations along the x-axis by $2\pi k$, $k \in \mathbf{Z}$, we get the graph of the function $\sin x$ which is called the *sinusoid* or *sine curve* (Fig. 24).

Another method of construction of the graph which does not require the computation of individual values of the function $\sin x$ consists in using the trigonometric circle. For this purpose, we mark the interval $[0, \pi/2]$

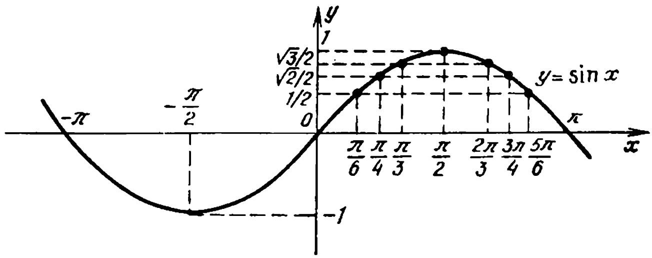

Fig. 24

on the x-axis and on the same drawing construct a circle of unit radius centred on the extension of the x-axis. To construct the graph of the function $\sin x$, we shall "unwind" the circle on the number line marking the ordinates of points P_t corresponding to real numbers t. To construct sufficiently many points belonging to the graph, we may bisect the interval $[0, \pi/2]$ and the subintervals thus obtained, as well as the corresponding arcs on the trigonometric circle. Note that after marking the point $x = \pi/2$ on the x-axis, all further constructions are carried out with the aid of a pair of compasses and a ruler (Fig. 25).

Note that the function $\sin x$ increases from -1 to 1 on any interval of the form $\left[-\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k\right]$, $k \in \mathbf{Z}$, and decreases from 1 to -1 on any interval of the form $\left[\frac{\pi}{2} + 2\pi k, \frac{3\pi}{2} + 2\pi k\right]$, $k \in \mathbf{Z}$. The maximal value $\sin x = 1$ is attained at points $x = \frac{\pi}{2} + 2\pi k$, $k \in \mathbf{Z}$, while the minimal value $\sin x = -1$ at points $x = -\frac{\pi}{2} + 2\pi k$, $k \in \mathbf{Z}$.

2. Properties and Graph of the Function $f(x) = \cos x$. The graph of $\cos x$ is best of all constructed by using the identity $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ (the reduction formula (2.28)). It follows from this identity that the graph of the function $\cos x$ is obtained from the graph of the function $\sin x$ by translating the latter leftwards along the

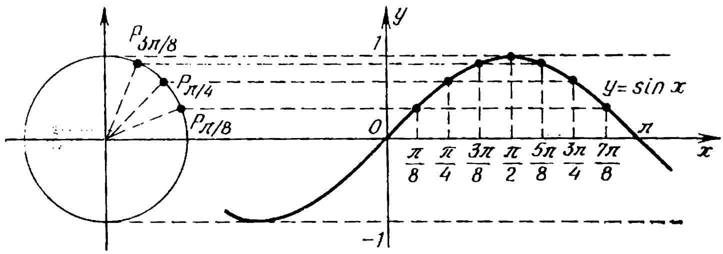

Fig. 25

x-axis by $\pi/2$. Indeed, for every point $(x, \sin x)$ of the graph of $\sin x$ the point $\left(x - \frac{\pi}{2}, \sin x\right)$ lies on the graph of $\cos x$, since $\cos\left(x - \frac{\pi}{2}\right) = \sin\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) = \sin x$. The converse is also true: for any point $(x, \cos x)$ of the graph of $\cos x$, the point $\left(x + \frac{\pi}{2}, \cos x\right)$ lies on the graph of $\sin x$ since $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.

Practically, it is more convenient first to construct the sine curve $y = \sin x$, and then to translate the y-axis to the right by $\pi/2$ (see Fig. 26, where the old y-axis is drawn in a dashed line and the new axis in a continuous line).

Consider the properties of the function $\cos x = f(x)$.

(1) $D(f) = \mathbf{R}$, $E(f) = [-1, 1]$.

(2) $\cos x$ is a periodic function. Any number of the form $2\pi k$ is a period of the function ($k \in \mathbf{Z}$), 2π being its fundamental period (see Sec. 1.3, Item 1).

(3) $\cos x$ is an even function, and its graph is symmetric about the axis of ordinates: if a point $(x, \cos x)$ lies on the graph of $\cos x$, then the point $(-x, \cos x) = (-x, \cos(-x))$ also lies on this graph.

(4) $\cos x = 0$ for $x = \frac{\pi}{2} + \pi k, \quad k \in \mathbf{Z}$.

(5) $\cos x$ decreases from 1 to -1 on any interval of the form $[2\pi k, \pi + 2\pi k]$, $k \in \mathbf{Z}$, and increases from -1 to 1 on any interval of the form $[-\pi + 2\pi k, 2\pi k]$. For $x = \pi + 2\pi k$, $k \in \mathbf{Z}$, the function $\cos x$ takes on the minimal

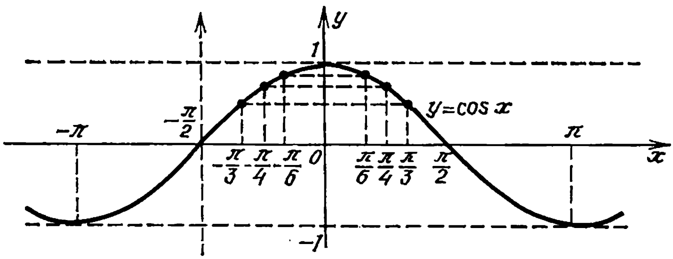

Fig. 26

value -1, while for $x = 2\pi k$, $k \in \mathbf{Z}$, the maximal value 1.

Example 4.1.1. Find the least and greatest values of the function $f(x) = \sin(\cos(\sin x))$ on the closed interval $[\pi/2, \pi]$.

◀Let $\pi/2 \leqslant x_1 < x_2 \leqslant \pi$, then $0 \leqslant \sin x_2 < \sin x_1 \leqslant 1$, and the points $P_{\sin x_1}$, $P_{\sin x_2}$ lie in the first quadrant since $1 < \pi/2$. Since the function $\cos x$ decreases on the interval $[0, \pi/2]$, we have

$$0 < \cos(\sin x_1) < \cos(\sin x_2) \leqslant 1.$$

But the points $P_{\cos(\sin x_1)}$ and $P_{\cos(\sin x_2)}$ also lie in the first quadrant and the function $\sin x$ increases on the interval $[0, \pi/2]$, therefore

$$0 < \sin(\cos(\sin x_1)) < \sin(\cos(\sin x_2)) < 1,$$

that is, the function $f(x) = \sin(\cos(\sin x))$ is increasing on the interval $[\pi/2, \pi]$, consequently, the minimal value of $f(x)$ on this interval is equal to $f(\pi/2) = \sin(\cos 1)$, while the maximal value to $f(\pi) = \sin(\cos 0) = \sin 1$.▶

Example 4.1.2. Compare the numbers $\sin(\cos 1)$ and $\cos(\sin 1)$.

◀According to the reduction formula (2.7), $\cos(\sin 1) = \sin\left(\frac{\pi}{2} - \sin 1\right)$, and it suffices to compare the numbers $\cos 1$ and $\frac{\pi}{2} - \sin 1$ from the interval $[0, \pi/2]$ (by virtue of the monotonicity of the function $\sin x$ on this interval). Note that

$$\cos 1 + \sin 1 = \sqrt{2}\sin\left(1 + \frac{\pi}{4}\right),$$

consequently, the inequality

$$\cos 1 + \sin 1 \leqslant \sqrt{2} < \pi/2$$

is true, whence, since the function $\sin x$ increases on the interval $[0, \pi/2]$, we have

$$\sin(\cos 1) < \sin\left(\frac{\pi}{2} - \sin 1\right) = \cos(\sin 1).$$ ▶

3. Properties and Graph of the Function $f(x) = \tan x$.

(1) The domian of definition is the set of real numbers except for the numbers of the form $\frac{\pi}{2} + \pi k$, $k \in \mathbf{Z}$, the range of values $E(f) = \mathbf{R}$.

(2) $\tan x$ is a periodic function. Any number of the form πk, $k \in \mathbf{Z}$, may be a period of $\tan x$, π being its fundamental period (see Sec. 1.3, Item 1).

(3) $\tan x$ is an odd function, and, consequently, its graph is symmetric with respect to the origin (see Sec. 1.3, Item 2).

(4) $\tan x = 0$ for $x = \pi k$, $k \in \mathbf{Z}$.

(5) On any interval of the form $\left(-\frac{\pi}{2} + \pi k, \frac{\pi}{2} + \pi k\right)$, $k \in \mathbf{Z}$, the function $\tan x$ increases from $-\infty$ to $+\infty$.

The graph $y = \tan x$ (tangent curve) can also be constructed using a table of values, for instance,

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

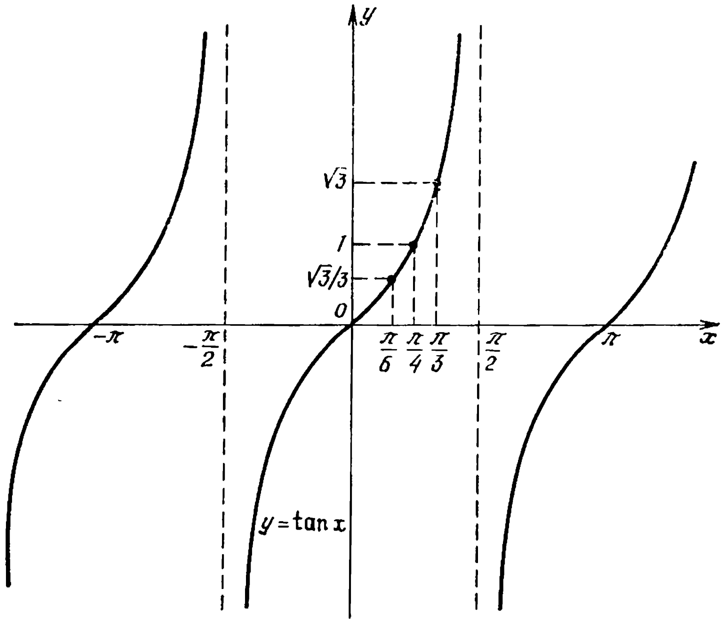

Fig. 27

which was compiled only for the values $x \in [0, \pi/2)$ by virtue of Properties (2) and (3) (Fig. 27).

Another method of constructing the graph (with the aid of the trigonometric circle and the line of tangents) makes it possible to plot arbitrarily many points for the tangent curve without computing individual values of the function $\tan x$ (Fig. 28).

Example 4.1.3. Find the greatest and least values of the function $\tan(\cos x)$ on the interval $[\pi/2, \pi]$.

◀ Let $\pi/2 \leqslant x_1 < x_2 \leqslant \pi$, then $-\frac{\pi}{2} < -1 \leqslant \cos x_2 < \cos x_1 \leqslant 0$, since the function $\cos x$ decreases from 0 to -1 on the interval $[\pi/2, \pi]$. Therefore the points $P_{\cos x_1}$ and $P_{\cos x_2}$ lie in the fourth quadrant, and

$$-\tan 1 \leqslant \tan(\cos x_2) < \tan(\cos x_1) \leqslant 0,$$

since $\tan x$ increases in the fourth quadrant. Consequently, the function $f(x) = \tan(\cos x)$ decreases on the interval

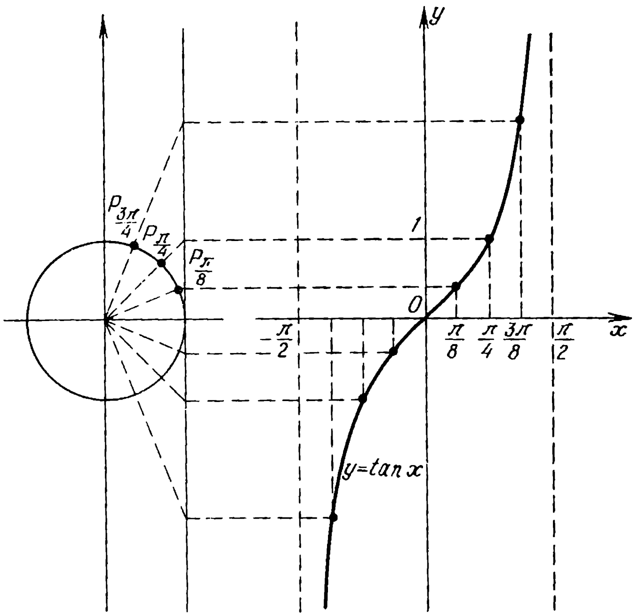

Fig. 28

$[\pi/2, \pi]$, its greatest value is $f(\pi/2) = \tan(0) = 0$, the least value being $f(\pi) = \tan(-1) = -\tan 1$. ▶

4. The Graph of Harmonic Oscillations. *Harmonic oscillations* are defined as rectilinear motions of a point governed by the rule $s = A \sin(\omega t + \alpha)$, where $A > 0$, $\omega > 0$, and t denotes the time coordinate. Consider the method of constructing graphs of such oscillations.

Example 4.1.4. Construct the graph of the function $f(x) = \sin 4x$.

◀ We first represent the graph of the function $\sin x$. If the point $M(a, b)$ lies on this graph, that is, $b = \sin a$, then the point $N(a/4, b)$ lies on the graph of the function $f(x) = \sin 4x$ since $b = \sin\left(4 \cdot \frac{a}{4}\right)$. Thus, if we take a point on the graph of $\sin x$, then a point having the same ordinate and whose abscissa equals one-fourth of the abscissa of that point will lie on the graph of the function $\sin 4x$. Consequently, the graph of $\sin 4x$ is obtained from the graph of $\sin x$ by contracting the latter along the axis of abscissas four-fold (Fig. 29).

To construct the graph of the function $y = \sin 4x$, it suffices to do the following. From the equation $\sin 4x = 0$ we find the points of intersection of the graph with the

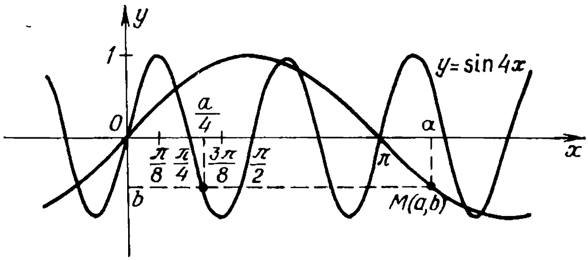

Fig. 29

x-axis: $x = \pi k/4$, $k \in \mathbf{Z}$. Further, $\sin 4x = 1$ for $x = \frac{\pi}{8} + \frac{\pi k}{2}$, $k \in \mathbf{Z}$, and $\sin 4x = -1$ for $x = -\frac{\pi}{8} + \frac{\pi k}{2}$, $k \in \mathbf{Z}$. In addition, the fundamental period of the function $f(x) = \sin 4x$ is equal to $\pi/2$:

$$f\left(x \pm \frac{\pi}{2}\right) = \sin\left(4\left(x \pm \frac{\pi}{2}\right)\right) = \sin(4x \pm 2\pi) = f(x)$$

(see Sec. 1.3, Item 1). Now we can construct the graph: mark the points $(0, 0)$, $(\pi/4, 0)$, $(\pi/2, 0)$ and also $(\pi/8, 1)$, $(3\pi/8, -1)$, join them with a curve resembling the sine curve (on the interval $[0, \pi/2]$), and then apply translations along the axis of abscissas, the origin going into the points of the form $(\pi k/2, 0)$. ▶

In the same way, we can construct the graph of any function of the form $\sin \omega x$, where $\omega > 0$. To this effect, we have to draw the graph of the function $\sin x$ and then to contract it ω times along the axis of abscissas, that is, to replace each point $M(a, b)$ of this graph by the point $N(a/\omega, b)$. The following is of importance: if $0 < \omega < 1$, then, instead of contracting, the graph is stretched $1/\omega$ times: the points will move away from the axis of ordinates.

It is not difficult now to construct the graph of any function of the form $A \sin(\omega x + \alpha)$, $A > 0$. Using the equality $A \sin(\omega x + \alpha) = A \sin\left(\omega\left(x + \frac{\alpha}{\omega}\right)\right)$, we are able to construct a graph in several steps.

(1) Construct the graph of the function $\sin \omega x$ using the method described above.

(2) The graph of the function $\sin\left(\omega\left(x + \frac{\alpha}{\omega}\right)\right)$ is obtained from the graph of the function $\sin \omega x$ by translating the latter along the axis of abscissas by $-\alpha/\omega$. Indeed, for every point $(x, \sin \omega x)$ of the graph of $\sin \omega x$ the point $\left(x - \frac{\alpha}{\omega}, \sin \omega x\right)$ lies on the graph of the function $\sin\left(\omega\left(x + \frac{\alpha}{\omega}\right)\right)$, since $\sin\left(\omega\left(x - \frac{\alpha}{\omega} + \frac{\alpha}{\omega}\right)\right) = \sin \omega x$. Usually, the graph of the function $\sin \omega x$ is constructed in a simpler way, the line $x = \alpha/\omega$ being taken for the new axis of ordinates.

(3) To obtain the graph of the function $\sin\left(\omega\left(x + \frac{\alpha}{\omega}\right)\right)$ from the graph of the function $A \sin\left(\omega\left(x + \frac{\alpha}{\omega}\right)\right)$ it remains to multiply the ordinate of each point of the graph by A, that is, to stretch this graph A times along the axis of ordinates (if $0 < A < 1$, then, instead of being stretched, the graph is contracted $1/A$ times).

In practice, we may proceed as follows: to find the points of intersection of the graph of the function $A \sin(\omega x + \alpha)$ with the axis of abscissas by solving the equation $\sin(\omega x + \alpha) = 0$, and, from the equations $\sin(\omega x + \alpha) = 1$ and $\sin(\omega x + \alpha) = -1$, to find the points of extremum of the function $A \sin(\omega x + \alpha)$ (at the points of maximum the function takes on the value A, at the points of minimum the value $-A$); then join the found points to obtain a curve of sinusoid type. Note that the function $y = A \sin(\omega x + \alpha)$ is periodic with the fundamental period $2\pi/\omega$ since $A \sin\left(\omega\left(x + \frac{2\pi}{\omega}\right) + \alpha\right) = A \sin(\omega x + \alpha + 2\pi) = A \sin(\omega x + \alpha)$.

Example 4.1.5. Construct the graph of the function $4 \sin\left(3x - \frac{\pi}{2}\right)$.

◀The fundamental period of the function is $2\pi/3$. Find the points of intersection of this graph with the axis of abscissas: $\sin\left(3x - \frac{\pi}{2}\right) = 0$, that is, $3x - \frac{\pi}{2} = \pi k$, $k \in \mathbf{Z}$, or $x = \frac{\pi}{6} + \frac{\pi k}{3}$, $k \in \mathbf{Z}$. Further, solving the equation $\sin\left(3x - \frac{\pi}{2}\right) = 1$, we get $3x - \frac{\pi}{2} = \frac{\pi}{2} + 2\pi k$,

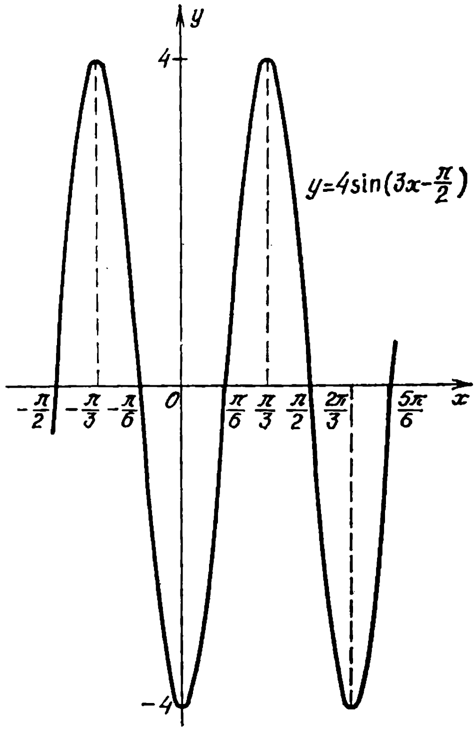

Fig. 30

that is, $x = \frac{\pi}{3} + \frac{2\pi k}{3}$, $k \in \mathbf{Z}$. At these points the function takes on the maximal value equal to 4 equation $\sin\left(3x - \frac{\pi}{2}\right) = -1$ we find $3x -$ $2\pi k$, that is, $x = \frac{2\pi k}{3}$, $k \in \mathbf{Z}$. At these p tion $4\sin\left(3x-\frac{\pi}{2}\right)$ takes on the minimal value equal to -4. Mark the found points on the coordinate plane and join them with a smooth line to get the required graph (Fig. 30). ▶

Example 4.1.6. Graph the function $\sin 2x+\sqrt{3}\cos 2x$.

◀The most convenient technique here is to reduce this

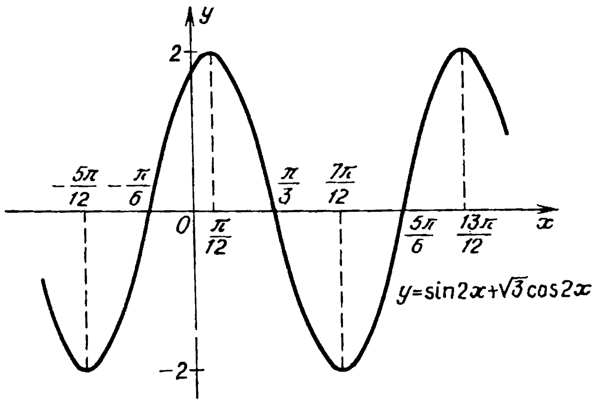

Fig. 31

function to the function $A\sin(\omega x+\alpha)$ using the method of introducing an auxiliary angle (see Theorem 2.2):

$$\sin 2x+\sqrt{3}\cos 2x=2\left(\frac{1}{2}\sin 2x+\frac{\sqrt{3}}{2}\cos 2x\right)$$
$$=2\sin\left(2x+\frac{\pi}{3}\right).$$

The fundamental period of this function is π.

(1) The points of intersection of the graph with the axis of abscissas:

$$\sin\left(2x+\frac{\pi}{3}\right)=0, \quad \text{that is,} \quad x=-\frac{\pi}{6}+\frac{\pi k}{2}, \quad k\in\mathbf{Z}.$$

(2) The points of maximum of the function:

$$\sin\left(2x+\frac{\pi}{3}\right)=1, \quad \text{that is,} \quad x=\frac{\pi}{12}+\pi k, \quad k\in\mathbf{Z},$$

therefore the points of the form $\left(\frac{\pi}{12}+\pi k,\ 2\right)$, $k\in\mathbf{Z}$, lie on the graph of the function.

(3) The points of minimum of the function:

$$\sin\left(2x+\frac{\pi}{3}\right)=-1, \quad \text{that is,} \quad x=-\frac{5\pi}{12}+\pi k, \ k\in \mathbf{Z},$$

therefore the points $\left(-\frac{5\pi}{12}+\pi k, \ -2\right)$, $k\in\mathbf{Z}$, lie on the graph.

The graph of the function $y=\sin 2x+\sqrt{3}\cos 2x$ is shown in Fig. 31. ▶

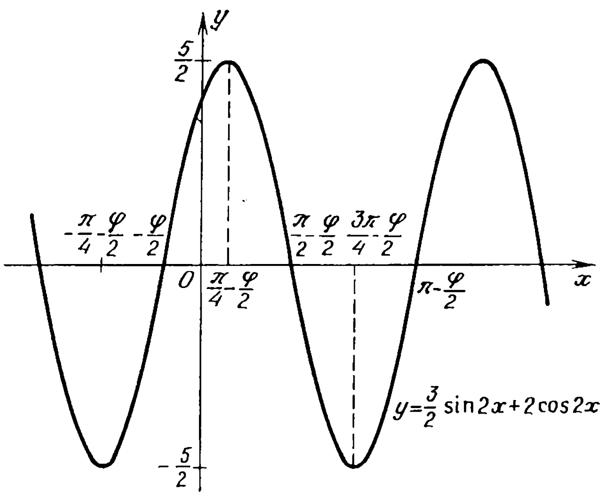

Fig. 32

Example 4.1.7. Graph the function

$$f(x)=\frac{3}{2}\sin 2x+2\cos 2x.$$

◀Using Theorem 2.2, we get $f(x)$

where $\varphi=\arcsin\frac{4}{5}$, $0<\varphi<\pi/2$.

qualities are valid: $\sqrt{2}/2<4/5<$

the function $\sin x$ increases o

we get $\pi/4<\arcsin\frac{4}{5}<\pi/3$

Note that the fundament

equals π.

$s\ x \mid < \varepsilon.$

(1) The points of intersection of the graph with the axis of abscissas:

$$\sin(2x+\varphi)=0, \quad \text{that is,} \quad x=-\frac{\varphi}{2}+\frac{\pi k}{2}, \quad k\in \mathbf{Z}.$$

(2) The points of maximum of the function:

$$\sin(2x+\varphi)=1, \quad \text{that is,} \quad x=\frac{\pi}{4}-\frac{\varphi}{2}+\pi k, \quad k\in \mathbf{Z}.$$

(3) The points of minimum of the function:

$$\sin(2x-\varphi)=-1, \text{ that is, } x=-\frac{\pi}{4}-\frac{\varphi}{2}+\pi k, \quad k\in \mathbf{Z}.$$

With the aid of these points we construct the required graph as in the preceding example (Fig. 32). ▶

4.2. Computing Limits

The theory of limits underlies the important notions of the continuity and differentiability of a function and the finding of derivatives and integrals. We confine ourselves to solving problems on finding limits at certain points of functions represented by trigonometric expressions. To solve these problems, one should know well the definition of the limit of a function at a point $a \in \mathbf{R}$, basic properties of limits (the limit of a sum, product, and ratio, as well as Theorem 4.2 on the first remarkable limit.

Definition. Let a function $f(x)$ be defined on the set $D(f) \subseteq \mathbf{R}$, and let a point a be such that any of its neighbourhoods contains infinitely many points of $D(f)$ (an *accumulation* or *limit point* of the set $D(f)$). Then the number b is said to be the *limit of the function* $f(x)$ *at the point* a if for any positive number ε there is a positive number δ, dependent on ε, such that for any point $x \in D(f)$ satisfying the condition $0 < |x-a| < \delta$ there holds the inequality $|f(x)-b| < \varepsilon$. Written:

$$b=\lim_{x\to a} f(x).$$

Definition. A function $f(x)$ is said to be *continuous at a point* $a \in D(f)$ if $\lim\limits_{x\to a} f(x) = f(a)$.

A function is *continuous on a set* $X \subseteq D(f)$ if it is continuous at each point of this set. The sum, difference, and product of two functions continuous on one and the same set are also continuous on this set. If the denominator of a fraction does not vanish on a set, then the quotient of two functions continuous on this set is also continuous.

The following statement plays a key role in testing trigonometric functions for continuity and computing various limits:

For all real numbers x, satisfying the condition $0 < |x| < \pi/2$, there hold the inequalities

$$|\sin x| < |x| < |\tan x|. \qquad (4.1)$$

The proof will be given later on (see Example 5.1.1), and now we are going to deduce the continuity of basic trigonometric functions.

Theorem 4.1. *The functions* $\sin x$, $\cos x$, $\tan x$, $\cot x$ *are continuous in the domains of their definition.*

Proof. Let us prove, for instance, that cosine is continuous throughout the number line, that is, show that for any $x_0 \in \mathbf{R}$

$$\lim_{x \to x_0} \cos x = \cos x_0.$$

Indeed,

$$\begin{aligned} |\cos x - \cos x_0| &= \left|2 \sin \frac{x_0 - x}{2} \sin \frac{x_0 + x}{2}\right| \\ &= 2\left|\sin \frac{x_0 - x}{2}\right| \left|\sin \frac{x_0 + x}{2}\right| \\ &\leqslant 2\left|\sin \frac{x_0 - x}{2}\right| \\ &\leqslant 2\left|\frac{x_0 - x}{2}\right| = |x - x_0|. \end{aligned}$$

Therefore, setting $\delta = \varepsilon$, for any $\varepsilon > 0$ for $0 < |x - x_0| < \delta$ we have

$$|\cos x - \cos x_0| \leqslant |x - x_0| < \varepsilon.$$

The continuity of sine is proved in similar fashion and that of tangent and cotangent follows from the property of continuity of the quotient of two continuous functions. ▶

Example 4.2.1. Find out whether the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if} \quad x \neq 0, \quad x \in \mathbf{R}. \\ 0 & \text{if} \quad x = 0, \end{cases}$$

is continuous on **R**.

◀ For $x \neq 0$ the given function has the form of the product $f_1(x) \cdot f_2(x)$ of two functions $f_1(x) = x$ and $f_2(x) = \sin \frac{1}{x}$. The function $f_1(x)$ is continuous and the function $f_2(x) = \sin \frac{1}{x}$ can be represented in the form of the composition

$$f_2(x) = f_4(f_3(x))$$

of the functions $f_3(x) = \frac{1}{x}$, $x \neq 0$, and $f_4(y) = \sin y$. By Theorem 4.1 the function $f_4(y)$ is continuous on **R** and the function $f_3(x)$ is continuous for $x \neq 0$ by virtue of the remark on the continuity of a fraction. Consequently, the function $f_2(x)$ is continuous on the set $\{x \in \mathbf{R}\colon\ x \neq 0\}$; it remains only to check whether it is continuous for $x = 0$, that is, whether the equality

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0 = f(0)$$

is fulfilled.

For $\varepsilon > 0$ we set $\delta = \varepsilon$, then for $0 < |x| < \delta$ we have

$$\left| x \sin \frac{1}{x} - 0 \right| = |x| \left| \sin \frac{1}{x} \right| \leqslant |x| < \varepsilon = \delta. \ ▶$$

Recall that the δ-neighbourhood of a point $a \in \mathbf{R}$ is defined as an interval of the form $(a - \delta, a + \delta)$, where $\delta > 0$.

Example 4.2.2. Find out whether the function

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if} \quad x \neq 0, \quad x \in \mathbf{R}, \\ 0 & \text{if} \quad x = 0, \end{cases}$$

is continuous on **R**.

Note that on the set $\{x \in \mathbf{R}\colon\ x \neq 0\}$ the given function coincides with the function $f_2(x)$ from Example 4.2.1 and therefore it is continuous on this set. Let us show that $f(x)$ is not continuous (discontinuous) at the point

$x_0 = 0$, that is, that the equality

$$\lim_{x \to 0} \sin \frac{1}{x} = 0$$

is violated. To this end, let us take $\varepsilon = 1/2$ and consider the sequence of points

$$x_n = \frac{2}{\pi(1+4n)}, \quad n = 1, 2, 3, \ldots,$$

which satisfy the condition $f(x_n) = 1$. Then $|f(x_n) - f(0)| = 1 > 1/2 = \varepsilon$, and any δ-neighbourhood of the point $x_0 = 0$ contains infinitely many such points x_n, that is, the condition of the continuity of the function at the point $x_0 = 0$ is violated. Thus, the function $f(x)$ is *discontinuous* at the point $x_0 = 0$. (Similarly, we can show that no real number b is the limit of the function $\sin \frac{1}{x}$ at the point $x_0 = 0$.) ▶

The computation of many limits is based on the following theorem.

Theorem 4.2 (on the first remarkable limit):

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Proof. Let $0 < |x| < \pi/2$. Then the inequality $|\sin x| < x < |\tan x|$ holds (see Equation (4.1)). Dividing both sides of this inequality by $|\sin x| \neq 0$, we get $1 < \left| \frac{x}{\sin x} \right| < \frac{1}{|\cos x|}$. Taking into account that $\frac{x}{\sin x} > 0$, $\cos x > 0$ for $0 < |x| < \frac{\pi}{2}$, we have $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ and, consequently, $1 > \frac{\sin x}{x} > \cos x$. Hence we get

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x = \cos 0 - \cos x.$$

Since cosine is a continuous function, for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < |0 - x| < \delta \Rightarrow |\cos 0 - \cos x| < \varepsilon.$$

Therefore for the numbers $x \neq 0$, satisfying the inequality $0 < |x| < \delta$, we have:

$$\left|1 - \frac{\sin x}{x}\right| = 1 - \frac{\sin x}{x} < \cos 0 - \cos x$$
$$= |\cos 0 - \cos x| < \varepsilon.$$

This means that $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1.$ ▶

Example 4.2.3. Compute (a) $\lim\limits_{x \to 0} \frac{2x}{\sin 3x}$, (b) $\lim\limits_{x \to 0} \frac{\sin 5x}{x}$, (c) $\lim\limits_{x \to 0} \frac{\tan 2x}{x}$, (d) $\lim\limits_{x \to 0} \frac{\tan 2x}{\sin 3x}$.

◀(a) $$\lim_{x \to 0} \frac{2x}{\sin 3x} = \lim_{x \to 0} \frac{2}{3} \cdot \frac{1}{\frac{\sin 3x}{3x}}$$
$$= \frac{2}{3} \cdot \frac{1}{\lim\limits_{3x \to 0} \frac{\sin 3x}{3x}} = \frac{2}{3} \cdot 1 = \frac{2}{3},$$

(b) $$\lim_{x \to 0} \frac{\sin 5x}{x} = \lim_{x \to 0} 5 \cdot \frac{\sin 5x}{5x} = 5 \lim_{5x \to 0} \frac{\sin 5x}{5x} = 5,$$

(c) $$\lim_{x \to 0} \frac{\tan 2x}{x} = \lim_{x \to 0} 2 \cdot \frac{\sin 2x}{2x} \cdot \frac{1}{\cos 2x}$$
$$= 2 \lim_{2x \to 0} \frac{\sin 2x}{2x} \cdot \lim_{2x \to 0} \frac{1}{\cos 2x} = 2 \cdot 1 \cdot 1 = 2,$$

(d) $$\lim_{x \to 0} \frac{\tan 2x}{\sin 3x} = \lim_{x \to 0} \frac{2}{3} \cdot \frac{\tan 2x}{2x} \cdot \frac{3x}{\sin 3x}$$
$$= \lim_{x \to 0} \frac{2}{3} \cdot \frac{\sin 2x}{2x} \cdot \frac{1}{\cos 2x} \cdot \frac{1}{\frac{\sin 3x}{3x}}$$
$$= \frac{2}{3} \cdot \lim_{2x \to 0} \frac{\sin 2x}{2x} \cdot \lim_{2x \to 0} \frac{1}{\cos 2x} \cdot \frac{1}{\lim\limits_{3x \to 0} \frac{\sin 3x}{3x}}$$
$$= \frac{2}{3} \cdot 1 \cdot 1 \cdot \frac{1}{1} = \frac{2}{3}.$$ ▶

Example 4.2.4. Compute $\lim\limits_{x \to 0} \frac{1 - \cos x}{x^2}$.

◀Note that, by virtue of formula (2.41),

$$1 - \cos x = 2 \sin^2 \frac{x}{2}.$$

Consequently, by the property of the limit of a product,

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}2\cdot\frac{1}{2}\,\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{1}{2}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

$$=\frac{1}{2}\lim_{x/2\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x/2\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}=\frac{1}{2}.\ \blacktriangleright$$

Example 4.2.5. Compute $\lim\limits_{x\to\pi/6}\dfrac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$.

◀Note that for $x=\pi/6$ both the numerator and denominator of the given fraction vanish, therefore, in this case, it is impossible to use directly the theorem on the limit of a ratio. In similar problems we may proceed as follows: we single out in the numerator and denominator of the fraction a common factor which vanishes at the given limit point, but does not vanish in the neighbourhood of this point, and such that after cancelling this factor the denominator of the given fraction no longer vanishes at the point under consideration. Then we can use the property of the limit of a ratio.

In our case:

$$2\sin^2 x+\sin x-1=(2\sin x-1)(\sin x+1),$$

$$2\sin^2 x-3\sin x+1=(2\sin x-1)(\sin x-1),$$

and, consequently,

$$\lim_{x\to\pi/6}\frac{(2\sin x-1)(\sin x+1)}{(2\sin x-1)(\sin x-1)}$$

$$=\lim_{x\to\pi/6}\frac{\sin x+1}{\sin x-1}=\frac{\lim\limits_{x\to\pi/6}(\sin x+1)}{\lim\limits_{x\to\pi/6}(\sin x-1)}$$

$$=\frac{\sin\frac{\pi}{6}+1}{\sin\frac{\pi}{6}-1}=-3$$

(in addition to the theorems on the limits of a quotient and a sum, we have used the fact that the function $\sin x$ is continuous). ▶

4.3. Investigating Trigonometric Functions with the Aid of a Derivative

1. The basic properties of many functions can be successfully studied without the aid of the derivative, and the properties of the derivative are a good illustration of the properties of the function itself. However, in many problems points of extremum, intervals of increase or decrease cannot be determined by elementary means, such problems must be solved using derivatives. Besides, when graphing some functions, one must know more about the behaviour of the function, for instance, whether its graph touches the axis of abscissas at a certain point of intersection with this axis or intersects it and what angle is formed. Such questions can be answered only by considering the derivative.

Let us first consider the rules for finding the derivatives of basic trigonometric functions.

Definition. The *derivative of a function $f(x)$ at a point* x_0 is defined as the number

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

A function having a derivative at a certain point is *differentiable* at this point. Let D_1 be a set of points at which the function f is differentiable. Associating each number $x \in D_1$ with the number $f'(x)$, we get a function defined on the set D_1. This function is called the *derivative* of the function f and is symbolized as $f'(x)$.

Example 4.3.1. Show that the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if} \quad x \neq 0, \quad x \in \mathbf{R}, \\ 0 & \text{if} \quad x = 0, \end{cases}$$

is not differentiable at the point $x = 0$.

Recall that in Example 4.2.1 we proved the continuity of the given function throughout the number line.

◀Let us prove that this function is not differentiable at the point $x = 0$, that is, that no real number b can be equal to

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

Consider the numbers $x_n = \frac{2}{\pi(1+4n)}$, $n = 1, 2, \ldots$ (such that $f(x_n) = x_n$) and $z_n = 1/\pi n$, $n = 1, 2, \ldots$ (such that condition $f(z_n) = 0$). If $b = 0$, then for $\varepsilon = 1/2$ we set $h = x_n$. Then

$$\left|\frac{f(x_n)-f(0)}{x_n} - 0\right| = 1 > \frac{1}{2} = \varepsilon,$$

and the inequality

$$\left|\frac{f(h)-f(0)}{h} - b\right| < \varepsilon$$

does not hold for all the numbers $h = x_n$ (any δ-neighbourhood of the point $h = 0$ contains infinitely many such points). Analogously, if $b \neq 0$, then we take $\varepsilon = |b|/2$ and set $h = z_n$. Then

$$\left|\frac{f(z_n)-f(0)}{z_n} - b\right| = |b| > \varepsilon$$

for infinitely many numbers of the form $h = z_n$ lying in any δ-neighbourhood of the point $h = 0$. ▶

Recall that if a function f has a derivative at a point x_0, then a tangent line to the graph of f is defined, its slope being equal to $f'(x_0)$. The equation of the tangent line is:

$$y = f(x_0) + f'(x_0)(x - x_0). \tag{4.2}$$

Example 4.3.2. Prove that the function

$$f(x) = \begin{cases} |x|^{3/2} \sin \frac{1}{x} & \text{if } x \neq 0,\ x \in \mathbf{R}, \\ 0 & \text{if } x = 0, \end{cases}$$

is differentiable at the point $x = 0$, and $f'(0) = 0$.

◀ For a given positive number ε we set $\delta = \varepsilon^2$. Then

$$\left|\frac{f(h)-f(0)}{h}\right| \leqslant |h|^{1/2} < \varepsilon,$$

if $0 < |h| < \delta$. Consequently,

$$0 = \lim_{h \to 0} \frac{f(h)-f(0)}{h} = f'(0).$$

Note that in this case the tangent line to the graph of $f(x)$ at $x = 0$ coincides with the axis of abscissas. ▶

Theorem 4.3. *On the domains of definition of the functions* $\sin x$, $\cos x$, $\tan x$, $\cot x$ *the following equalities hold true*:

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x,$$

$$(\tan x)' = 1/\cos^2 x \qquad (\cot x)' = -1/\sin^2 x.$$

Proof. For the function $\sin x$ this theorem is deduced from the first remarkable limit (Theorem 4.2). Indeed, if $f(x) = \sin x$, then, by virtue of identity (2.18) for the sine of two real numbers, we have:

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin x}{h}$$

$$= \left(2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}\right)/h.$$

Now, using the properties of limits, we get

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= \lim_{h/2\to 0}\cos\left(x+\frac{h}{2}\right)\lim_{h/2\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x.$$

Here, we have also taken advantage of the continuity of the function $\cos x$.

For the function $\cos x$ the theorem can be proved using the reduction formulas (2.7) and the following rule for differentiation of a composite function:

If a function g *has a derivative at a point* $y_0 = f(x_0)$, *and a function* f *at a point* x_0, *then the composite function* $h(x) = g(f(x))$ *also has a derivative at* x_0, *and*

$$h'(x_0) = g'(y_0)\,f'(x_0) = g'(f(x_0))\,f'(x_0). \qquad (4.3)$$

Let us now represent the function $\cos x$ in the form $\sin\left(\frac{\pi}{2}-x\right)=g(f(x))$ and apply the rule (4.3) for $g(y)=\sin y,\ f(x)=\frac{\pi}{2}-x$:

$$\left(\sin\left(\frac{\pi}{2}-x\right)\right)'=g'(f(x))f'(x)=-\cos\left(\frac{\pi}{2}-x\right)$$
$$=-\sin x.$$

In order to prove the theorem for $\tan x$ and $\cot x$, it suffices to use the rule for differentiating the quotient of two functions. Let us also recall some other differentiation rules. If the functions $f(x)$ and $g(x)$ are differentiable, then:

$$(f(x)+g(x))'=f'(x)+g'(x), \tag{4.4}$$

$$(Cf(x))'=Cf'(x),\ C \text{ is a constant}, \tag{4.5}$$

$$(f(x)g(x))'=f'(x)g(x)+f(x)g'(x), \tag{4.6}$$

$$(f(x)/g(x))'=\frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)} \tag{4.7}$$

for points x such that $g(x)\neq 0$.

Using the rule (4.7) and the aforeproved, we get

$$\tan' x=\left(\frac{\sin x}{\cos x}\right)'=\frac{\sin' x\cos x-\cos' x\sin x}{\cos^2 x}$$
$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x},$$

$$\cot' x=\left(\frac{\cos x}{\sin x}\right)'=\frac{\cos' x\sin x-\sin' x\cos x}{\sin^2 x}$$
$$=\frac{-\sin^2 x-\cos^2 x}{\sin^2 x}=-\frac{1}{\sin^2 x}. \blacktriangleright$$

Example 4.3.3. Find the derivative of the function $y=\frac{\sin x-x\cos x}{\cos x+x\sin x}$.

◀ Using the rules (4.7), (4.4), (4.6), and the formulas of Theorem 4.3, we get

$$\left(\frac{\sin x-x\cos x}{\cos x+x\sin x}\right)'=\frac{u'v-uv'}{v^2},$$

where

$$u = \sin x - x\cos x, \quad v = \cos x + x\sin x,$$
$$u' = \sin' x - (x\cos x)'$$
$$= \cos x - (\cos x - x\sin x) = x\sin x,$$
$$v' = \cos' x + (x\sin x)'$$
$$= -\sin x + (\sin x + x\cos x) = x\cos x,$$
$$u'v - uv' = x\sin x\,(\cos x + x\sin x)$$
$$- (\sin x - x\cos x)\,x\cos x$$
$$= x^2\sin^2 x + x^2\cos^2 x = x^2,$$

whence

$$y' = \frac{x^2}{(\cos x + x\sin x)^2},$$

the derivative exists everywhere in the domain of the given function, that is, for x's such that $\cos x + x\sin x \neq 0$. ▶

Example 4.3.4. Compute the derivative of the indicated functions:

(a) $y = \sin(\sin(\sin x))$,

(b) $y = \sin(\cos^2(\tan^3(x^4 + 1)))$.

◀ (a) By the rule (4.3) for differentiating a composite function applied twice, we get

$$y' = \cos(\sin(\sin x))\cdot\cos(\sin x)\cdot\cos x.$$

(b) Let us write the function in the form of a composite function:

$$y = f_6(f_5(f_4(f_3(f_2(f_1(x)))))),$$

where $f_1(x) = x^4 + 1$, $f_2(x) = \tan x$, $f_3(x) = x^3$, $f_4(x) = \cos x$, $f_5(x) = x^2$, $f_6(x) = \sin x$, and we get $f_1'(x) = 4x^3$, $f_2'(x) = 1/\cos^2 x$, $f_3'(x) = 3x^2$, $f_4'(x) = -\sin x$, $f_5'(x) = 2x$, $f_6'(x) = \cos x$. Consequently, by applying the rule (4.3) several times, we get

$$y'(x) = \cos(\cos^2(\tan^3(x^4+1)))\cdot 2\cos(\tan^3(x^4+1))$$
$$\times(-\sin(\tan^3(x^4+1)))\cdot 3\tan^2(x^4+1)\cdot\frac{1}{\cos^2(x^4+1)}\cdot 4x^3$$
$$= -12x^3\cos(\cos^2(\tan^3(x^4+1)))$$
$$\times\sin(2\tan^3(x^4+1))\,\frac{\sin^2(x^4+1)}{\cos^4(x^4+1)}. \;\blacktriangleright$$

Example 4.3.5. Prove the following formulas for derivatives of inverse trigonometric functions:

(a) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$,

(b) $\left(\arccos \frac{x}{a}\right)' = -\frac{1}{\sqrt{a^2-x^2}}$,

(c) $\left(\arctan \frac{x}{a}\right)' = \frac{a}{a^2+x^2}$.

◀ Let us prove, for instance, formula (a). By the definition of arc sine, for $x \in [-1, 1]$ there holds the identity

$$\sin(\arcsin x) = x.$$

Computing the derivatives of both sides, we get for $|x| < 1$:

$$\cos(\arcsin x) \cdot (\arcsin x)' = 1.$$

But $\cos(\arcsin x) = \sqrt{1-x^2}$, since if $\alpha = \arcsin x$, then $\sin \alpha = x$, $-\pi/2 \leqslant \alpha \leqslant \pi/2$, therefore $\cos \alpha$ is nonnegative and $\cos \alpha = \sqrt{1-x^2}$. Consequently,

$$(\arcsin x)' = 1/\sqrt{1-x^2}.$$

Equalities (b) and (c) are proved in a similar way by applying Theorem 4.3, the rule (4.3), and identities (1.9), (1.10). ▶

2. Applying the Derivative to Investigation of Trigonometric Functions. Let us consider some essential things needed for solving problems with the aid of the derivative.

Sufficient condition of monotonicity. *If a function $f(x)$ is differentiable on the interval (a, b) and $f'(x) > 0$ ($f'(x) < 0$) on (a, b), then $f(x)$ increases (decreases) on this interval.*

Remember that the converse is not always true; for instance, the function $f(x) = x^3$ increases monotonically on the interval $(-1, 1)$, and its derivative $f'(x) = 2x^2$ is not positive everywhere, $f'(0) = 0$.

A point $x_0 \in D(f)$ is called the *point of* (local) *maximum* (*minimum*) of the function f if for all $x \in D(f)$ from some neighbourhood of x_0 the inequality $f(x) \leqslant f(x_0)$ ($f(x) \geqslant f(x_0)$) is fulfilled. The points of maximum and minimum are also called the *points of extremum* of a function. One should remember that if a function $f(x)$ is defined on a certain set X, and x_0 is a point of maximum of f, $x_0 \in X$, then the value $f(x_0)$ is not necessarily the greatest value of the function $f(x)$ on the set X. Consider an example. Figure 33 represents the graph of a function having on the interval $X = (-1, 3)$ three points of maximum x_1, x_2, x_3, however, none of them is the greatest

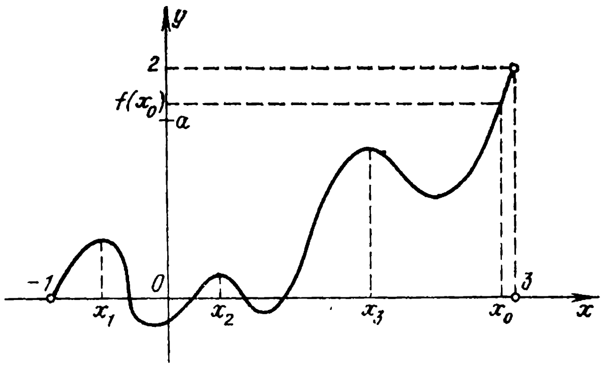

Fig. 33

value of the function f on the set X. Moreover, the given function does not attain the greatest value on X since all the values of the function are less than the number 2, but no number $a < 2$ is the greatest value. Indeed, we can choose a point x_0 such that $a < f(x_0) < 2$ (see Fig. 33). The same note refers to the points of minimum and least values.

Let us note (without proof) that if a function $f(x)$ is defined and continuous on an interval $[a, b]$, then this function takes on the least and greatest values on this interval. This is one of the fundamental theorems in the course of mathematical analysis. Also note that on an open interval, half-open interval or throughout the number line, the function may not take on the greatest or the least value. This is exemplified by the function whose graph is shown in Fig. 33.

A *critical point* of a function is a point from the domain of the function at which the derivative is zero or does not exist at all.

A necessary condition of an extremum. *If a function f differentiable at a point x_0 has an extremum at this point, then $f'(x_0) = 0$.*

A sufficient condition of an extremum. *If, when passing through a critical point x_0, the derivative of the function changes sign from plus to minus, then x_0 is a point of maximum; and if the derivative changes sign from minus to plus, then x_0 is a point of minimum of this function; if the derivative does not change sign, then x_0 is not a point of extremum.*

We often come across problems on finding the greatest or the least value of a continuous function on a closed interval $[a, b]$. When solving such problems, it is not sufficient to find the greatest local maximum of the function (the least minimum), we have also to compare these numbers with the values of the function attained at the end points $x = a$ and $x = b$ of the interval. The greatest (least) value is frequently attained at the end points.

In practice, to find the greatest or the least value of a (continuous) function on a given interval, one finds all the critical points inside the interval and compares the values of the function at the critical points with each other and with the values at the end points of the interval, without further investigating the critical points.

Consider several examples.

Example 4.3.6. Graph the function

$$f(x) = \frac{3}{2} \sin 2x + 2|\cos 2x|$$

and find its greatest and least values on the closed interval $[0, \pi]$.

◀ In solving problems with functions involving a modulus sign, we have always to consider two cases since

$$|a| = \begin{cases} a & \text{if } a \geqslant 0, \\ -a & \text{if } a < 0. \end{cases}$$

The function $\cos 2x$ vanishes if $2x = \frac{\pi}{2} + \pi k$, $k \in \mathbf{Z}$, that is, $x = \frac{\pi}{4} + \frac{\pi k}{2}$, $k \in \mathbf{Z}$, and if the number k is even, then $\cos 2x$ changes sign from plus to minus, and if k is odd, the sign changes from minus to plus. Consequently, the inequality $\cos 2x \geqslant 0$ is equivalent to that

$$-\frac{\pi}{4}+\pi m \leqslant x \leqslant \frac{\pi}{4}+\pi m, \quad m \in \mathbf{Z}, \tag{4.8}$$

and $\cos 2x < 0$ is equivalent to that

$$\frac{\pi}{4}+\pi m < x < \frac{3\pi}{4}+\pi m, \quad m \in \mathbf{Z}. \tag{4.9}$$

Case 1: x satisfies the condition (4.8). Then $\cos 2x \geqslant 0$, $|\cos 2x| = \cos 2x$, and, by Theorem 2.2,

$$f(x)=\frac{3}{2}\sin 2x+2|\cos 2x|=\frac{5}{2}\sin(2x+\varphi),$$

where $\varphi = \arcsin\frac{4}{5}$, $\frac{\pi}{4} < \varphi < \frac{\pi}{3}$ (see Example 4.1.7).

Case 2: x satisfies the condition (4.9). Then $\cos 2x < 0$, $|\cos 2x| = -\cos 2x$ and

$$f(x)=\frac{3}{2}\sin 2x-2\cos 2x=\frac{5}{2}\sin(2x-\varphi),$$

where $\varphi = \arcsin\frac{4}{5}$.

Consequently, to construct the graph of a given function, it is first necessary to graph the functions (see Sec. 4.1)

$$\frac{5}{2}\sin(2x+\varphi) \tag{4.10}$$

and 1248

$$\frac{5}{2}\sin(2x-\varphi) \tag{4.11}$$

using dashed lines, and then for the values of x lying on the intervals (4.8) to outline the graph of (4.10) using continuous lines, and on the intervals (4.9) the graph of (4.11); as a result, we get the graph of the function under consideration (the continuous line in Fig. 34).

To find the critical points, note first that on the interval $(0, \pi)$ $f'(x) = 5\cos(2x+\varphi)$ if $0 < x < \pi/4$ or $3\pi/4 < x < \pi$, and $f'(x) = 5\cos(2x-\varphi)$ if $\pi/4 < x < 3\pi/4$. At the points $x = \pi/4$ and $x = 3\pi/4$ the function is not differentiable. This is proved exactly in the same manner as in Example 4.3.1 by determining the derivative, and geometrically the absence of the derivative means that it is impossible to draw a tangent line to the graph of the given function at these points.

Thus, $f'(x)=0$ (on $(0, \pi)$) at the points $x_1=\frac{\pi}{4}-\frac{\varphi}{2}$, $x_2=\frac{\pi}{4}+\frac{\varphi}{2}$ (both of them are points of maximum since at these points the derivative changes sign from

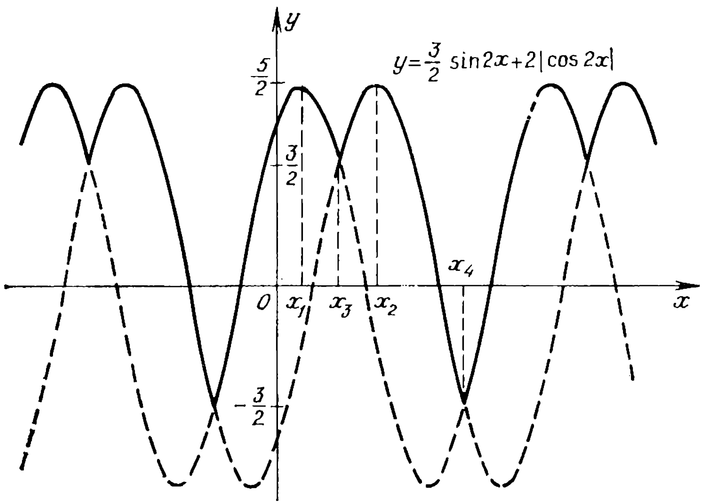

Fig. 34

plus to minus). The derivative is not existent at the points $x_3=\pi/4$ and $x_4=3\pi/4$ (both of them are points of minimum since the derivative changes sign from minus to plus when passing through these points). Evaluating the function at the critical points

$$f(x_1)=f\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)=\frac{5}{2}, \quad f(x_2)=f\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=\frac{5}{2},$$
$$f(x_3)=f\left(\frac{\pi}{4}\right)=\frac{3}{2}, \quad f(x_4)=f\left(\frac{3\pi}{4}\right)=-\frac{3}{2},$$

and also the values $f(0)=f(\pi)=2$ at the end points of the interval $[0, \pi]$, we see that the greatest value of the func-

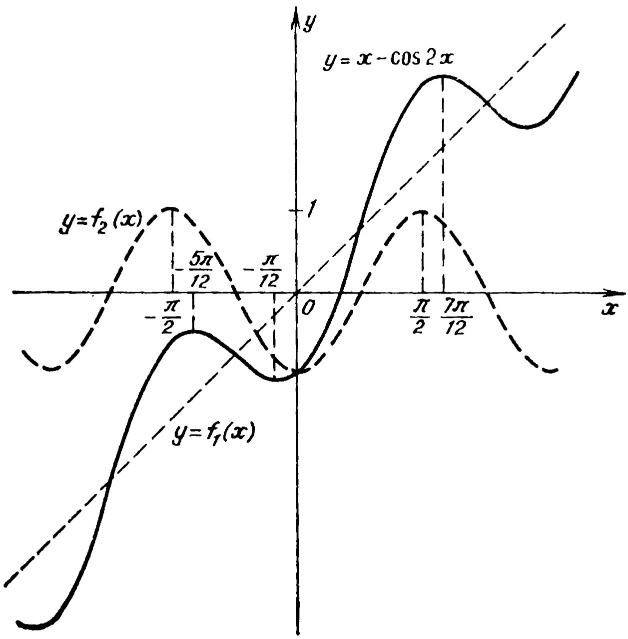

Fig. 35

tion is equal to 5/2, and the least value to —3/2. Note that, by virtue of the periodicity of the function (its period being equal to π), these values are the greatest and least values on the entire number line.▶

Example 4.3.7. Graph the function

$$y = x - \cos 2x$$

and find its points of extremum.

◀To construct the graph of the given function, we apply the method of "adding the graphs" of two functions. Let us construct the graphs of the functions $f_1(x) = x$ and $f_2(x) = -\cos 2x$, on one drawing using dashed lines (Fig. 35). Now, the ordinate of any point of the graph of the function $x - \cos 2x$ is equal to the sum of the ordinates of the points on the graphs of auxiliary functions (for an arbitrary value of x). By adding the ordinates of points, it is possible to construct a sufficient number of points belonging to the graph of the function $x - \cos 2x$, and then to join them with a continuous line. Prior to finding the critical points, let us note that the given function is differentiable everywhere and that $y' = 1 + 2\sin 2x$. The derivative vanishes at the points where $1 + 2\sin 2x = 0$, that is, at the points $x = (-1)^{n+1}\frac{\pi}{12} + \frac{\pi n}{2}$ $(n \in \mathbf{Z})$. Note that if $n = 2m + 1$, $m \in \mathbf{Z}$, then $x = \frac{7\pi}{12} + \pi m$. At these points, the derivative changes sign from plus to minus, and therefore $x = \frac{7\pi}{12} + \pi m \ (m \in \mathbf{Z})$ are points of maximum. If $n = 2m$, $m \in \mathbf{Z}$, then $x = -\frac{\pi}{12} + \pi m$ $(m \in \mathbf{Z})$, and, when passing through these points, the derivative changes sign from minus to plus, therefore these points are points of minimum. ▶

Example 4.3.8. Find the greatest and least values of the function $f(x) = x - \cos 2x$ on the closed interval $[-\pi/2, \pi/12]$.

◀We use the investigation of the critical points of the given function (from Example 4.3.7), and see that the interval $(-\pi/2, \pi/12)$ contains both the point of maximum $x = -5\pi/12$ and the point of minimum $x = -\pi/12$. Consequently, to find the greatest value of the function on the closed interval $[-\pi/2, \pi/12]$, we have to compare the value $f(-5\pi/12)$ attained by the function at the point of maximum with the values reached at the end points $f(-\pi/2)$ and $f(\pi/12)$. We have $f\left(-\frac{5\pi}{12}\right) = -\frac{5\pi}{12} - \cos\left(-\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{5\pi}{12}$, $f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 1$, $f\left(\frac{\pi}{12}\right) = \frac{\pi}{12} - \frac{\sqrt{3}}{2}$, and the following inequalities hold:

$$\frac{\sqrt{3}}{2} - \frac{5\pi}{12} > 1 - \frac{\pi}{2}, \quad \frac{\sqrt{3}}{2} - \frac{5\pi}{12} > \frac{\pi}{12} - \frac{\sqrt{3}}{2},$$

which are derived from the estimate $3 < \pi < 3.2$. Therefore the greatest value of the function on the given interval is equal to $f\left(-\frac{5\pi}{12}\right)=\frac{\sqrt{3}}{2}-\frac{5\pi}{12}$. Similarly, we find that the least value of the function on the given interval is $f\left(-\frac{\pi}{12}\right)=-\frac{\pi}{12}-\frac{\sqrt{3}}{2}$. ▶

Example 4.3.9. Find all the values of the parameter a for each of which the function

$$f(x) = \sin 2x - 8(a+1)\sin x + (4a^2 + 8a - 14)x$$

increases throughout the number line and has no critical points.

◀ For any fixed a the given function is differentiable at every point of the number line. If the function $f(x)$ increases, then the inequality $f'(x) \geqslant 0$ holds at every point. If, in addition, $f(x)$ has no critical points, the relationship $f'(x) \neq 0$ is true for any x, and, consequently, $f'(x) > 0$. On the other hand, if for all x's the inequality $f'(x) > 0$ holds, then, obviously, the function has no critical points and increases.

In view of the fact that

$$f'(x) = 2\cos 2x - 8(a+1)\cos x + (4a^2 + 8a - 14),$$

the problem can now be reformulated as follows: find all the values of the parameter a for each of which the inequality

$$\cos 2x - 4(a+1)\cos x + (2a^2 + 4a - 7) > 0$$

holds for any $x \in \mathbf{R}$. Since $\cos 2x = 2\cos^2 x - 1$, by setting $\cos x = t$, we reformulate the problem as follows: find all the values of the parameter a for each of which the least value of the function

$$2t^2 - 1 - 4(a+1)t + (2a^2 + 4a - 7),$$

or the function

$$g(t) = t^2 - 2(a+1)t + a^2 + 2a - 4,$$

on the closed interval $[-1, 1]$ is positive. The derivative $g'(t) = 2t - 2(a+1)$ vanishes at the point $t_0 = a + 1$. Therefore the least value m of the quadratic function

$g(t)$ on the closed interval $[-1, 1]$ is equal to

$$m = \begin{cases} g(-1) = a^2+4a-1 & \text{if} \quad a+1 \leqslant -1, \\ g(a+1) = -5 & \text{if} \quad -1 < a+1 < 1, \\ g(1) = a^2-5 & \text{if} \quad a+1 \geqslant 1. \end{cases}$$

Since the least value of the function $g(t)$ on $[-1, 1]$ must be positive, the values of the parameter a satisfying the conditions of the problem lie in two intervals: $a \leqslant -2$ and $a \geqslant 0$. If $a \leqslant -2$, then the least value of $g(t)$ on the closed interval $[-1, 1]$ is equal to $a^2 + 4a - 1$, and the desired values of the parameter a satisfy the inequality $a^2 + 4a - 1 > 0$. If $a \geqslant 0$, then the least value of $g(t)$ on $[-1, 1]$ equals $a^2 - 5$, and the sought-for values of the parameter satisfy the inequality $a^2 - 5 > 0$. Thus, the set of the sought-for values of a is the union of solutions of two systems of inequalities

$$\begin{cases} a \leqslant -2 \\ a^2+4a-1 > 0, \end{cases} \qquad \begin{cases} a \geqslant 0, \\ a^2-5 > 0. \end{cases}$$

The set of solutions of the first system is the interval $a < -2-\sqrt{5}$ and the set of solutions of the second system is $a > \sqrt{5}$. Hence the required set of values of a is $(-\infty, -2-\sqrt{5}) \cup (\sqrt{5}, +\infty)$. ▶

Example 4.3.10. Construct the graph of the function

$$f(x) = \arcsin(\sin x)$$

and find all of its critical points.

◀ The given function is defined throughout the number line **R**. By virtue of the periodicity of the function $\sin x$, the function $f(x)$ is also periodic with period 2π, and it suffices to analyze it, say, on the closed interval $[-\pi/2, 3\pi/2]$. By the definition of arc sine, on the closed interval $[-\pi/2, \pi/2]$ we have $\arcsin(\sin x) = x$ (see Sec. 1.4, Item 1), therefore for these values of x there holds the equality $f(x) = x$. If $x \in [\pi/2, 3\pi/2]$, then $\pi - x \in [-\pi/2, \pi/2]$, and the equality $\sin(\pi - x) = \sin x$ implies that $\arcsin(\sin x) = \pi - x$. The final graph is represented in Fig. 36. In order to find the critical points of the given function, it suffices to investigate only those

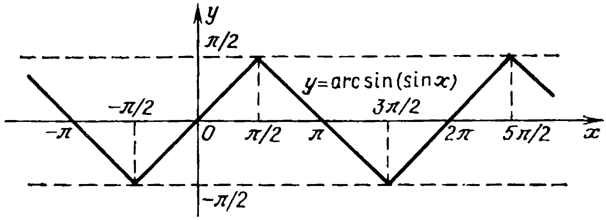

Fig. 36

values of x for which $|\sin x| = 1$, that is, $x = \frac{\pi}{2} + \pi k$, $k \in \mathbf{Z}$. Using the definition of the derivative, we can show that the derivative is nonexistent at these points, the points of the form $x = \frac{\pi}{2} + 2\pi m$, $m \in \mathbf{Z}$, being points of maximum, and the points of the form $x = \frac{\pi}{2} + \pi(2m-1) = -\frac{\pi}{2} + 2\pi m$, $m \in \mathbf{Z}$, being points of minimum.

PROBLEMS

In Problems **4.1** to **4.10**, graph the given functions.

4.1. $y = |\sin 2x| + \sqrt{3}\cos 2x$.

4.2. $y = |\sin 2x| + \sqrt{3}\,|\cos 2x|$.

4.3. $y = \arcsin(\cos x)$. **4.4.** $y = \arccos(\cos x)$.

4.5. $y = \sin x - x$. **4.6.** $y = |x| - \cos 2x$.

4.7. $y = \begin{cases} \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

4.8. $y = \begin{cases} x \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$

4.9. $y = \arcsin \frac{2x}{1+x^2}$.

4.10. $y = 4\sin^4 x + 4\cos^4 x$.

In Problems **4.11** to **4.14**, compute the indicated limits.

4.11. $\lim\limits_{x\to\infty} (x+1)\sin\frac{4}{x}$. **4.12.** $\lim\limits_{x\to 0} \frac{(x^2+3x-1)\tan x}{x^2+2x}$.

4.13. $\lim\limits_{x\to 1} \frac{(x^2-4x+3)\sin(x-1)}{(x-1)^2}$.

4.14. $\lim\limits_{x\to -\frac{\pi}{4}} \frac{\sin x+\sin 5x}{\cos x+\cos 5x}$.

In Problems **4.15** to **4.20**, find the derivatives of the given functions.

4.15. $y=\frac{1-\cos 2x}{1+\cos 2x}$. **4.16.** $y=(\sin^2 x+1)e^x$.

4.17. $y=\tan 2x-\cot 2x$. **4.18.** $y=x^2\cos\frac{1}{x}$.

4.19. $y=x+\sin x\cos x$.

4.20. (a) $y=\tan\sin x$; (b) $y=\tan^3 x$.

In Problems **4.21** to **4.24**, find the critical points and compute the least and greatest values of the given functions.

4.21. $y=|\sin 2x|+\sqrt{3}\cos 2x$.

4.22. $y=|\sin 2x|+\sqrt{3}\,|\cos 2x|$.

4.23. $y=\arcsin(\cos x)$. **4.24.** $y=\arccos(\cos x)$.

In Problems **4.25** and **4.26**, find the intervals of increase and decrease of the given functions.

4.25. $y=|x|-\cos 2x$. **4.26.** $y=\begin{cases}\sin\frac{1}{x}, & x\neq 0,\\ 0, & x=0.\end{cases}$

4.27. Find all the values of x for each of which the tangent lines to the graphs of the functions

$$y(x)=3\cos 5x \text{ and } y(x)=5\cos 3x+2$$

at points with abscissa x are parallel.

4.28. Find all the values of the parameter b for each of which the function

$$f(x)=\sin 2x-8(b+2)\cos x-(4b^2+16b+6)x$$

decreases throughout the number line and has no critical points.

4.29. Find the greatest value of the expression

$$\sin^2\left(\frac{15\pi}{8}-4x\right)-\sin^2\left(\frac{17\pi}{7}-4x\right) \quad \text{for} \quad 0\leqslant x\leqslant \pi/8.$$

4.30. Find the least value of the expression

$$\frac{\cot 2x - \tan 2x}{1+\sin\left(\frac{5\pi}{2} - 8x\right)} \quad \text{for} \quad 0 < x < \pi/8.$$

In Problems **4.31** to **4.35**, find the greatest and least values of the given functions on the indicated intervals.

4.31. $f(x) = x + \cos^2 x, \quad x \in [0, \pi/2]$.

4.32. $f(x) = \tan x + \cot 2x, \quad x \in [\pi/6, \pi/3]$.

4.33. $f(x) = \frac{1}{2}\cos 2x + \sin x, \quad x \in [0, \pi/2]$.

4.34. $f(x) = \frac{x}{2} - \frac{1}{4}\sin 2x + \frac{1}{3}\cos^3 x - \cos x,$

$x \in [-\pi/2, \pi/2]$.

4.35. $f(x) = \cos^2 x + \sin x, \quad x \in [0, \pi/4]$.

Chapter 5

Trigonometric Inequalities

5.1. Proving Inequalities Involving Trigonometric Functions

Problems on proving trigonometric inequalities fall into several types. Some problems require to prove a numerical inequality which is satisfied by some value of a trigonometric function or an expression composed of values of trigonometric functions; while other problems require to prove that an inequality is satisfied for all values of the arguments of a given trigonometric expression or for the permissible values of the arguments that satisfy an additional constraint of the hypothesis. However, in any case the solution is reduced to investigating the values of a trigonometric function on some interval from its domain of definition or on the entire domain. In simple cases, we succeed in transforming a trigonometric expression in similar problems so that it is then possible to directly apply an inequality of the form $|\sin x| \leqslant 1$ or $|\cos x| \leqslant 1$. In other cases, we can transform the trigonometric expression under consideration so that, as a result, it takes the form of a function $F(z)$ in which the argument z is represented by some other, simpler trigonometric function or trigonometric expression. In such a case, the solution of the problem is reduced to investigating the function $F(z)$, bearing in mind that z can meet the additional conditions connected with properties of trigonometric functions. In this case the function $F(z)$ can be analyzed using either the derivative or elementary considerations such as, for instance, the inequality between an arithmetic and a geometric mean of two nonnegative numbers a and b:

$$\sqrt{ab} \leqslant \frac{a+b}{2}, \tag{5.1}$$

in which the equality sign occurs only for $a = b$. An important class of problems comprise inequalities whose proof is reduced to comparison of the size of an acute angle, its sine and tangent (see equation (4.1)). We begin to consider some examples with this inequality.

Example 5.1.1. Prove that for $0 < t < \pi/2$ the following inequalities hold:

$$\sin t < t < \tan t$$

$$\cos t < \frac{\sin t}{t} < 1.$$

◀ Consider the trigonometric circle and mark the point P_t in the first quadrant corresponding to a real number t. Then $\angle AOP_t = t$ (radians). The first inequality is obtained from the comparison of the areas of the triangle OAP_t, the sector $OAEP_t$, and the triangle OAW_t, where W_t is a point on the line of tangents (that is, on the tangent to the trigonometric circle at the point A) corresponding to the point P_t (see Fig. 37 and Sec. 1.2, Item 3). It is known from geometry that the following equalities hold true:

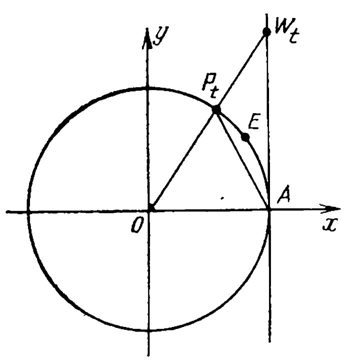

Fig. 37

$$S_{\triangle OAP_t} = \frac{1}{2} |OA| \cdot |OP_t| \sin t = \frac{1}{2} \sin t,$$

$$S_{\triangle OAW_t} = \frac{1}{2} |OA| \cdot |AW_t| = \frac{1}{2} \tan t,$$

$$S_{OAEP_t} = t/2$$

(the last formula follows from the fact that the area of a complete circle of unit radius, equal to $\pi r^2 = \pi$, may be considered as the area of a sector of 2π radians, and, consequently, the area of a sector of t radians is found from an obvious proportion and equals $t/2$).

Since $\triangle OAP_t$ is contained entirely in the sector $OAEP_t$, and the latter is contained in $\triangle OAW_t$, we have $S_{\triangle OAP_t} < S_{OAEP_t} < S_{\triangle OAW_t}$, whence

$$\sin t < t < \tan t. \tag{5.2}$$

Dividing the terms of (5.2) by $\sin t > 0$, we get $1 < \frac{t}{\sin t} < \frac{1}{\cos t}$, or $\cos t < \frac{\sin t}{t} < 1$. ▶

Example 5.1.2. Prove that $\sin 1 > \pi/4$.

◀ Using the reduction formula (2.7), we pass to cosine and make use of formula (2.39) for cosine of double argument:

$$\sin 1 = \cos\left(\frac{\pi}{2} - 1\right) = 1 - 2\sin^2\left(\frac{\pi}{4} - \frac{1}{2}\right).$$

We now use inequality (5.2), which implies that $\sin\left(\frac{\pi}{4} - \frac{1}{2}\right) < \frac{\pi}{4} - \frac{1}{2}$, therefore

$$1 - 2\sin^2\left(\frac{\pi}{4} - \frac{1}{2}\right) > 1 - 2\left(\frac{\pi}{4} - \frac{1}{2}\right)^2,$$

and, to prove the inequality, it suffices to ascertain that $1 - 2\left(\frac{\pi}{4} - \frac{1}{2}\right)^2 > \frac{\pi}{4}$ or $\frac{\pi^2}{8} - \frac{\pi}{4} - \frac{1}{2} < 0$ or $\pi^2 - 2\pi - 4 < 0$. Note that all the solutions of the inequality $x^2 - 2x - 4 < 0$ are specified by the condition $1 - \sqrt{5} < x < 1 + \sqrt{5}$. Using the estimates $\sqrt{5} > 2.2$ and $\pi < 3.2$, we get $\pi < 3.2 < 1 + \sqrt{5}$, $\pi > 0 > 1 - \sqrt{5}$. Consequently, together with the inequality $\pi^2 - 2\pi - 4 < 0$, the original inequality is also true. ▶

Note that inequality (5.2) enables us to calculate approximately the number π. Indeed, it follows from (5.2) that for $n > 2$ the following inequalities hold true:

$$n \sin\frac{\pi}{n} < \pi < n \tan\frac{\pi}{n}.$$

Note that the number $2n \sin\frac{\pi}{n}$ is the perimeter of a regular n-gon inscribed in the trigonometric circle, while the number $2n \tan\frac{\pi}{n}$ is the perimeter of a regular n-gon circumscribed about this circle (see Fig. 38, where $n=5$). It follows from Theorem 4.2 that

$$\lim_{n\to\infty} n\left(\sin\frac{\pi}{n}-\tan\frac{\pi}{n}\right)=\lim_{n\to\infty}\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\left(\frac{1}{1-\cos\frac{\pi}{n}}\right)$$
$$=\pi\cdot 1\,(1-1)=0,$$

which makes it possible to approximate the number π by numbers of the form $n\sin\frac{\pi}{n}$, $n\tan\frac{\pi}{n}$

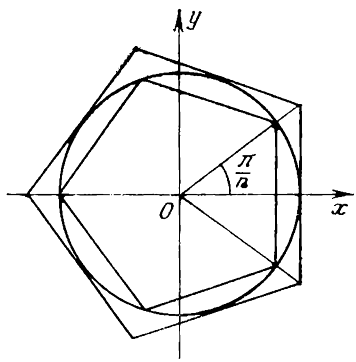

Fig. 38

with an arbitrarily high accuracy. For instance, if $n=2^k$, then the numbers $\sin\frac{\pi}{2^k}$, $\tan\frac{\pi}{2^k}$ can be found from formulas (2.47) and (2.49) for half-argument functions. Thus, if $k=2$, then $\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$, $\tan\frac{\pi}{4}=1$, therefore $2\sqrt{2}<\pi<4$. If $k=3$, then $\sin\frac{\pi}{8}=\frac{\sqrt{2-\sqrt{2}}}{2}$, $\cos\frac{\pi}{8}=\frac{\sqrt{2+\sqrt{2}}}{2}$, $\tan\frac{\pi}{8}=\sqrt{2}-1$, consequently,

$$4\sqrt{2-\sqrt{2}}<\pi<8(\sqrt{2}-1).$$

For $k=4$ we get $\sin\frac{\pi}{16}=\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$, $\cos\frac{\pi}{16}=\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$, $\tan\frac{\pi}{16}=2\sqrt{\sqrt{2}+1}-\sqrt{2}-1$, therefore $8\sqrt{2-\sqrt{2+\sqrt{2}}}<\pi<16(2\sqrt{\sqrt{2}+1}-\sqrt{2}-1)$. In addition, the following estimates hold true:

$$8\sqrt{2-\sqrt{2+\sqrt{2}}}>3.1,$$

$$16(2\sqrt{\sqrt{2}+1}-\sqrt{2}-1)<3.2,$$

which are proved by squaring both sides several times. Hence, in particular, we get the inequality $3.1<\pi<3.2$, which we used repeatedly when solving problems.

Example 5.1.3. Prove that for $0<t<\pi/2$ the inequality $t-\frac{t^3}{4}<\sin t$ holds.

◀Let us use inequality (5.2) which implies that

$$\frac{t}{2}<\tan\frac{t}{2}. \tag{5.3}$$

Multiplying both sides of (5.3) by the positive number $2\cos^2\frac{t}{2}$, we get $t\cos^2\frac{t}{2}<2\tan\frac{t}{2}\cos^2\frac{t}{2}=2\sin\frac{t}{2}\cos\frac{t}{2}=\sin t$, or

$$t\left(1-\sin^2\frac{t}{2}\right)<\sin t. \tag{5.4}$$

Since $\sin\frac{t}{2}<\frac{t}{2}$, by replacing in the left-hand side of (5.4) $\sin\frac{t}{2}$ by $\frac{t}{2}$, we get $t\left(1-\frac{t^2}{4}\right)<\sin t$, that is, $t-\frac{t^3}{4}<\sin t$. ▶

Example 5.1.4. Prove the inequality

$$\sin\alpha\sin 2\alpha\sin 3\alpha<3/4.$$

◀We have

$$\sin\alpha\sin 2\alpha\sin 3\alpha = \sin 2\alpha\,\frac{\cos 2\alpha-\cos 4\alpha}{2}$$
$$=\frac{2\sin 2\alpha\cos 2\alpha-2\sin 2\alpha\cos 4\alpha}{4}$$
$$=\frac{\sin 4\alpha+\sin 2\alpha-\sin 6\alpha}{4}\leqslant\frac{3}{4},$$

since $|\sin t|\leqslant 1$. But an equality would occur only if $\sin 4\alpha=\sin 2\alpha=-\sin 6\alpha=1$. Let us show that this is impossible. To this end, it suffices to show that $\sin 4\alpha$ and $\sin 2\alpha$ are not equal to 1 simultaneously. Indeed, let $\sin 2\alpha = 1$, then $\cos 2\alpha = 0$, therefore $\sin 4\alpha = 2\sin 2\alpha\cos 2\alpha = 2\cdot 0 = 0\neq 1$. ▶

Example 5.1.5. Prove that for $0<x<\pi/4$ the inequality $\frac{\cos x}{\sin^2 x(\cos x-\sin x)}>8$ takes place.

◀Since $\cos x\neq 0$, by dividing the numerator and denominator of the left-hand side by $\cos^3 x$, we get

$$\frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}\left(\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}\right)}=\frac{1+\tan^2 x}{\tan^2 x(1-\tan x)}$$
$$=\frac{(1-\tan x)^2+2\tan x}{\tan^2 x(1-\tan x)}$$
$$=\frac{1-\tan x}{\tan^2 x}+\frac{2}{\tan x(1-\tan x)}.$$

Since $\tan x+(1-\tan x)=1$ (a constant quantity), and the numbers $a=\tan x$, $b=1-\tan x$ are nonnegative by hypothesis, inequality (5.1) implies that for $\tan x=1/2$ the expression $\tan x(1-\tan x)$ takes on the greatest value. Consequently, the least value of the expression $\frac{2}{\tan x(1-\tan x)}$ is equal to 8. But since for $0<x<\frac{\pi}{4}$ the expression $\frac{1-\tan x}{\tan^2 x}>0$, we have

$$\frac{1-\tan x}{\tan^2 x}+\frac{2}{\tan x(1-\tan x)}>8.$$ ▶

Example 5.1.6. Prove the inequality

$$\sin^8 \alpha + \cos^8 \alpha \geqslant 1/8.$$

For what values of α does the equality occur?
◀We have $\sin^2 \alpha + \cos^2 \alpha = 1$,

$(\sin^2\alpha + \cos^2\alpha)^2 = \sin^4 \alpha + \cos^4 \alpha + 2 \sin^2\alpha \cos^2 \alpha = 1$,

but since, according to (5.1), $\sin^4 \alpha + \cos^4 \alpha \geqslant 2 \sin^2 \alpha \cos^2 \alpha$, we have $\sin^4 \alpha + \cos^4 \alpha \geqslant 1/2$, the equality occurring when $\sin \alpha = \pm\cos \alpha$. Further,

$$(\sin^4 \alpha + \cos^4 \alpha)^2 = \sin^8 \alpha + \cos^8 \alpha + 2 \sin^4 \alpha \cos^4 \alpha \geqslant 1/4,$$

and since, according to (5.1), $\sin^8 \alpha + \cos^8 \alpha \geqslant 2 \sin^4 \alpha \cos^4 \alpha$, we have $\sin^8 \alpha + \cos^8 \alpha \geqslant 1/8$. As before, the equality occurs when $\sin \alpha = \pm\cos \alpha$, that is, for $\alpha = \frac{\pi}{4} + \frac{\pi k}{2}$, $k \in \mathbf{Z}$. ▶

Example 5.1.7. Prove the inequality

$$(x + y)(x + y + 2 \cos x) + 2 \geqslant 2 \sin^2 x.$$

For what values of x and y does the equality occur?
◀Let us rewrite the given inequality as follows:

$$(x + y)^2 + 2 (x + y) \cos x + 2 (1 - \sin^2 x) \geqslant 0,$$

or $$(x + y)^2 + 2 (x + y) \cos x + \cos^2 x + \cos^2 x \geqslant 0,$$

that is,

$$((x + y)^2 + \cos x)^2 + \cos^2 x \geqslant 0.$$

In other words, the inequality has been reduced to an obvious one, since both terms are nonnegative. For the equality to occur, it is necessary and sufficient that

$$\begin{cases} x + y + \cos x = 0, \\ \cos^2 x = 0, \end{cases} \quad \text{or} \quad \begin{cases} x + y = 0, \\ \cos x = 0. \end{cases}$$

Consequently, the equality holds when $x = \frac{\pi}{2}(2k + 1)$ $y = -\frac{\pi}{2}(2k + 1)$, $k \in \mathbf{Z}$. For any other values of x, y we have a strict inequality. ▶

Example 5.1.8. Prove that there occurs the inequality $-4 \leqslant y \leqslant 2\frac{1}{8}$, where $y = \cos 2x + 3 \sin x$.

◀We have $y = \cos 2x + 3 \sin x = -2\sin^2 x + 3 \sin x + 1$. Let $\sin x = z$, where $-1 \leqslant z \leqslant 1$, then $y = -2z^2 + 3z + 1$. For $z = \frac{-3}{2(-2)} = \frac{3}{4}$ the function $y(z)$ takes on the greatest value equal to $-2 \cdot \frac{9}{16} + 3 \cdot \frac{3}{4} + 1 = 2\frac{1}{8}$. To find the least value of $y(z)$ on the closed interval $[-1, 1]$, it suffices (by virtue of the properties of the quadratic function) to compare its values at the end points of the interval. We have: $y(-1) = -2 - 3 + 1 = -4$, $y(1) = -2 + 3 + 1 = 2$. Thus, for the numbers z belonging to the closed interval $[-1, 1]$ the least value of y is -4, the greatest value is $2\frac{1}{8}$. ▶

Example 5.1.9. Prove the inequality

$$0 < \sin^8 x + \cos^{14} x \leqslant 1.$$

◀Since $\sin^2 x \leqslant 1$ and $\cos^2 x \leqslant 1$ for any $x \in \mathbf{R}$, we have: $\sin^8 x \leqslant \sin^2 x$ and $\cos^{14} x \leqslant \cos^2 x$. Adding these inequalities termwise, we get $\sin^8 x + \cos^{14} x \leqslant 1$. Since $\sin^8 x \geqslant 0$ and $\cos^{14} x \geqslant 0$, we have $\sin^8 x + \cos^{14} x \geqslant 0$, an equality being satisfied only if $\sin^8 x$ and $\cos^{14} x$ are both equal to zero which is impossible, that is, $\sin^8 x + \cos^{14} x > 0$. ▶

5.2. Solving Trigonometric Inequalities

For some time trigonometric inequalities have not been set at entrance exams, although some problems involve the comparison of values of trigonometric functions. For instance, when solving equations and systems of equations containing, along with trigonometric functions, logarithms or radicals, the domain of permissible values of unknowns is given by conditions having the form of trigonometric inequalities. In such problems, however, the only thing required is, from the set of roots obtained, to choose those which belong to the domain of permissible values without finding this domain itself.

But the ability to solve simple trigonometric inequalities may turn out to be useful, for instance, when it is required to find the intervals of increase and decrease of a function using its derivative, and the function and its derivative are given with the aid of trigonometric expressions. In this section, we consider some examples on solving trigonometric inequalities.

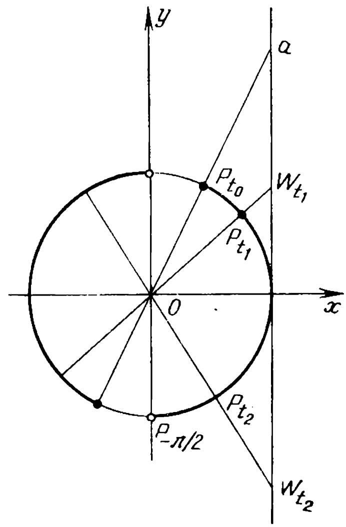

Fig. 39

The technique of solving simple trigonometric inequalities is, in many respects, the same as that of solving corresponding trigonometric equations. For instance, let there be required to solve the inequality $\tan t \leqslant a$. The number $\tan t$ is the ordinate of the point W_t on the line of tangents corresponding to the point P_t (see Sec. 1.4, Item 3 and Fig. 39). Therefore in order to solve this inequality, we have first to find all points P_t on the trigonometric circle such that the ordinates of the corresponding points on the line of tangents are less than, or equal to, a. The set of such points is shown in Fig. 39; in the given case, it consists of two parts, one part being obtained from the other when rotated about the point O through an angle of π (radians). Then we have to pass from the points on the circle to the corresponding real numbers. Since the function $\tan t$ is periodic with period π, it suffices to find all the solutions of the inequality in question belonging to a definite interval of length π, since all the remaining solutions will differ from the found ones by a shift to the right or left by numbers multiple of π. To get the shortest possible answer, it is desired that the initial interval of length π be chosen so that the solutions belonging to that interval, in turn, constitute a continuous interval. In our case, we may take, for instance, $(-\pi/2, \pi/2)$ as such an interval. If $t \in (-\pi/2, \pi/2)$, then the found set of points contains on the circle points P_t such that $-\pi/2 < t \leqslant \arctan a$ (in Fig. 39 the points P_{t_1} and P_{t_2}, for example). Therefore, the set of solutions of the inequality $\tan x \leqslant a$ is a union of an infinite number of intervals

$$-\frac{\pi}{2} + \pi n < x \leqslant \arctan a + \pi n, \quad n \in \mathbf{Z}.$$

Other simplest trigonometric inequalities are solved in a similar way. For the sake of convenience, we give a list of solutions of simple trigonometric inequalities:

(1) $\sin x \leqslant a$. If $|a| \leqslant 1$, then

$$-\arcsin a + \pi(2n - 1) \leqslant x \leqslant \arcsin a + 2\pi n, \quad n \in \mathbf{Z}.$$

If $a \geqslant 1$, then x is an arbitrary real number. If $a < -1$, then there is no solution.

(2) $\sin x \geqslant a$. If $|a| \leqslant 1$, then

$$\arcsin a + 2\pi n \leqslant x \leqslant -\arcsin a + (2n + 1)\pi, \quad n \in \mathbf{Z}.$$

If $a > 1$, then there is no solution. If $a \leqslant -1$, then x is an arbitrary real number.

(3) $\cos x \leqslant a$. If $|a| \leqslant 1$, then

$$\arccos a + 2\pi n \leqslant x \leqslant -\arccos a + 2\pi(n + 1), \quad n \in \mathbf{Z}.$$

If $a \geqslant 1$, then x is an arbitrary real number. If $a < -1$, then there is no solution.

(4) $\cos x \geqslant a$. If $|a| \leqslant 1$, then

$$-\arccos a + 2\pi n \leqslant x \leqslant \arccos a + 2\pi n, \quad n \in \mathbf{Z}.$$

If $a > 1$, then there is no solution. If $a \leqslant -1$, then x is an arbitrary real number.

(5) $\tan x \leqslant a$. $-\frac{\pi}{2} + \pi n < x \leqslant \arctan a + \pi n, \quad n \in \mathbf{Z}.$

(6) $\tan x \geqslant a$. $\arctan a + \pi n \leqslant x < \frac{\pi}{2} + \pi n, \quad n \in \mathbf{Z}.$

(7) $\cot x \leqslant a$. $\operatorname{arccot} a + \pi n \leqslant x < \pi(n + 1)$, $n \in \mathbf{Z}$.

(8) $\cot x \geqslant a$. $\pi n < x \leqslant \operatorname{arccot} a + \pi n$, $n \in \mathbf{Z}$.

In most cases, solution of trigonometric inequalities can be reduced to solution of one or several simple inequalities using identical trigonometric transformations and an auxiliary unknown. Consider some examples.

Example 5.2.1. Solve the inequality

$$\frac{5}{4}\sin^2 x + \frac{1}{4}\sin^2 2x > \cos 2x.$$

◀Using the half-angle formula for sine, we rewrite our inequality in the form $5(1 - \cos 2x) + 2(1 - \cos^2 2x) > 8\cos 2x$, or

$$2\cos^2 2x + 13\cos 2x - 7 < 0.$$

Setting $y = \cos 2x$, we get the quadratic inequality $2y^2 + 13y - 7 < 0$ whose solution is the interval $-7 < y < 1/2$. Thus, the problem has been reduced to solving the inequality $-7 < \cos 2x < 1/2$. The inequality $-7 < \cos 2x$ is satisfied for any x. Solving the inequality $\cos 2x < 1/2$, we get $\frac{\pi}{3} + 2\pi n < 2x < \frac{5\pi}{3} + 2\pi n$, $n \in \mathbf{Z}$, that is, $\frac{\pi}{6} + \pi n < x < \frac{5\pi}{6} + \pi n$, $n \in \mathbf{Z}$. ▶

Example 5.2.2. Solve the inequality

$$5 + 2\cos 2x \leqslant 3\,|\,2\sin x - 1\,|.$$

◀Using the double-argument formula for cosine, we reduce the given inequality to the form

$$7 - 4\sin^2 x \leqslant 3\,|\,2\sin x - 1\,|.$$

Setting $y = \sin x$, we get

$$7 - 4y^2 \leqslant 3\,|\,2y - 1\,|.$$

(a) Let $y \geqslant 1/2$, then $7 - 4y^2 \leqslant 3(2y - 1)$ or $2y^2 + 3y - 5 \geqslant 0$. Solving this inequality, we get $y \geqslant 1$ and $y \leqslant -5/2$, but from the condition $y \geqslant 1/2$ we have $y \geqslant 1$.

(b) Let $y < 1/2$. Then the inequality is rewritten as follows: $7 - 4y^2 \leqslant -3(2y - 1)$ or $2y^2 - 3y - 2 \geqslant 0$. Solving the last inequality, we get $y \geqslant 2$ and $y \leqslant -1/2$ or, by hypothesis, $y \leqslant -1/2$.

Thus, all x's satisfying the inequalities $\sin x \geqslant 1$ and $\sin x \leqslant -1/2$ are solutions of the original inequality. The first inequality holds true only for x's satisfying the equation $\sin x = 1$, that is,

$$x = \frac{\pi}{2} + 2\pi n, \quad n \in \mathbf{Z}.$$

Solving the second inequality, we get

$$-\frac{5\pi}{6} + 2\pi n \leqslant x \leqslant -\frac{\pi}{6} + 2\pi n, \quad n \in \mathbf{Z}.$$

Thus, $x = \frac{\pi}{2} + 2\pi n$, $n \in \mathbf{Z}$, and $-\frac{5\pi}{6} + 2\pi n \leqslant x \leqslant -\frac{\pi}{6} + 2\pi n$, $n \in \mathbf{Z}$. ▶

Example 5.2.3. Solve the inequality

$$\sin^6 x + \cos^6 x > 13/16.$$

◀ Transforming the left-hand side, we have

$$\begin{aligned} &\sin^6 x + \cos^6 x \\ &= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x) \\ &= (\sin^2 x + \cos^2 x)^2 - 3\sin^2 x \cos^2 x \\ &= 1 - \frac{3}{4}\sin^2 2x = 1 - \frac{3}{4} \cdot \frac{1 - \cos 4x}{2} = \frac{5}{8} + \frac{3\cos 4x}{8}. \end{aligned}$$

Hence, the problem has been reduced to solving the inequality $\frac{5}{8} + \frac{3\cos 4x}{8} > \frac{13}{16}$ or $\cos 4x > 1/2$. Hence, $-\frac{\pi}{3} + 2\pi n < 4x < \frac{\pi}{3} + 2\pi n$, $n \in \mathbf{Z}$, whence

$$-\frac{\pi}{12} + \frac{\pi n}{2} < x < \frac{\pi}{12} + \frac{\pi n}{2}, \quad n \in \mathbf{Z}. \ ▶$$

Example 5.2.4. Solve the inequality

$$\sin 2x + \tan x \geqslant 2.$$

◀ The left-hand side is defined for $x \neq \frac{\pi}{2} + \pi n$, $n \in \mathbf{Z}$, and for the same x's the substitution $\sin 2x = \frac{2\tan x}{1 + \tan^2 x}$ leads to the equivalent inequality

$$\frac{2\tan x}{1 + \tan^2 x} + \tan x - 2 \geqslant 0.$$

Setting $y = \tan x$, we have $\frac{2y}{1+y^2} + y - 2 \geqslant 0$, or (since $y^2 + 1 > 0$) $2y + (y - 2)(y^2 + 1) \geqslant 0$. Removing the parentheses, we get $y^3 - 2y^2 + 3y - 2 \geqslant 0$ which can be rewritten as follows:

$$y^2(y - 1) - y(y - 1) + 2(y - 1) \geqslant 0,$$

or

$$(y^2 - y + 2)(y - 1) \geqslant 0.$$

The quadratic function $y^2 - y + 2$ is positive for any y (since the discriminant is negative), therefore the last inequality is equivalent to the inequality $y - 1 \geqslant 0$, or $\tan x - 1 \geqslant 0$, $\tan x \geqslant 1$. Hence

$$\frac{\pi}{4} + \pi n \leqslant x < \frac{\pi}{2} + \pi n, \quad n \in \mathbf{Z}. \blacktriangleright$$

Example 5.2.5. Solve the inequality $\cos^2 x < 3/4$.

◀The given inequality is equivalent to the inequalities $-\sqrt{3}/2 < \cos x < \sqrt{3}/2$ or to the system of inequalities

$$\begin{cases} \cos x > -\sqrt{3}/2, \\ \cos x < \sqrt{3}/2. \end{cases}$$

The set representing the solution of this system is the intersection of the sets which are the solutions of two simple trigonometric inequalities. In order to find this intersection, it is convenient to consider the closed interval $[-\pi, \pi]$ and mark on it separately the solutions of the first and second inequalities. Then we get two subintervals

$$-\frac{5\pi}{6} < x < -\frac{\pi}{6} \text{ and } \frac{\pi}{6} < x < \frac{5\pi}{6}.$$

Noting that one interval can be obtained from the other by shifting the latter by π and taking into account that $\cos x$ is a periodic function with period 2π, we may write the answer in the following form:

$$\frac{\pi}{6} + \pi n < x < \frac{5\pi}{6} + \pi n, \quad n \in \mathbf{Z}. \blacktriangleright$$

PROBLEMS

In Problems 5.1 to 5.18, solve the given inequalities.

5.1. $\cos x \geqslant 1/2$. **5.2.** $\sin 2x < \sqrt{2}/2$.

5.3. $\sin \frac{x}{5} \geqslant -1/2$. **5.4.** $\tan(3x - 1) < 1/\sqrt{3}$.

5.5. $|\tan x| \geqslant \sqrt{3}$. **5.6.** $|\cos x| < 1/2$.

5.7. $\sin^2 x < 1/4$. **5.8.** $\tan^2 x \geqslant 1/3$.

5.9. $2\sin^2 x \leqslant \sin x$. **5.10.** $2\cos^2 x + \cos x < 1$.

5.11. $|\sin x| < |\cos x|$. **5.12.** $|\sin x| > |\cos x|$.

5.13. $|\sin x| \cos x > 1.4$.

5.14. $|\sin x| + |\cos x| > 1$.

5.15. $4(\sin^2 x - |\cos x|) < 1$.

5.16. $\frac{\sin x + \cos x}{\sin x - \cos x} > \sqrt{3}$.

5.17. $\frac{5 - 4(\sin^2 x + \cos x)}{\cos x} \leqslant 0$.

5.18. $\left|\frac{\sin x - \cos x}{\sin x + \cos x}\right| \leqslant 1$.

5.19. Which is greater: tan 1 or arctan 1?

5.20. Which is smaller: (a) $\frac{\pi}{4}$ or $\arctan \frac{1}{4} + \arctan \frac{5}{8}$? (b) $\frac{\pi}{4}$ or $\arcsin \frac{2}{3} + \arccos \frac{2}{3}$?

5.21. Which is greater: sin (tan 1) or tan (sin 1)?

In Problems 5.22 to 5.25, prove the indicated inequalities.

5.22. $-\frac{1}{4} \leqslant \sin x \sin\left(\frac{\pi}{3} - x\right) \sin\left(\frac{\pi}{3} + x\right) \leqslant \frac{1}{4}$.

5.23. $0 \leqslant \cos^2 \alpha + \cos^2(\alpha + \beta) - 2\cos\alpha\cos\beta\cos(\alpha + \beta) \leqslant 1$.

5.24. $(\cot^2 \alpha - 1)(3\cot^2 \alpha - 1)(\cot 3\alpha \tan 2\alpha - 1) \leqslant -1$.

5.25. $\frac{\sin x - 1}{\sin x - 2} + \frac{1}{2} \geqslant \frac{2 - \sin x}{3 - \sin x}$.

Answers*

CHAPTER 1

1.1. (a) In the fourth quadrant, (b) in the second quadrant, (c) in the third quadrant. **1.2.** (a) $\overset{\frown}{A_2A_3}$, (b) $\overset{\frown}{A_1A_2}$, (c) $\overset{\frown}{A_4A_0}$. **1.3.** $\overset{\frown}{B_3B_4}$. **1.4.** $N=2\cdot 11\cdot 13=286$. **1.6.** The number $(\alpha-\beta)/\pi$ must be rational. **1.7.** (a) $\sin 1>\sin\left(1+\frac{2\pi}{5}\right)$, (b) $\cos\left(1+\frac{2\pi}{5}\right)>\cos\left(1+\frac{4\pi}{5}\right)$. **1.8.** (a) The plus sign, (b) the plus. **1.9.** (a) No, (b) yes. **1.10.** The minus. **1.11.** (a) 1/2, (b) $-\sqrt{2}/2$, (c) $\sqrt{2}/2$, (d) $\sqrt{3}/2$. **1.12.** The minus. **1.13.** (a) -1, (b) $-\sqrt{3}$. **1.14.** *Hint.* Consider the coordinates of the sum of vectors $\overrightarrow{OP}_{a+\frac{2\pi k}{N}}$ and prove that this sum is zero since it remains unchanged under the rotation through an angle of $2\pi/N$. **1.15.** 918π. **1.16.** No. **1.19.** 2π. **1.21.** (a) 6, (b) 30π. **1.22.** $\pi/3$. **1.27.** $y=f_0+f_1$, where $f_0=(\sin(x+1)\sin^3(2x-3)+\sin(x-1)\sin^3(2x+3))/2$, $f_1=(\sin(x+1)\sin^3(2x-3)-\sin(x-1)\sin^3(2x+3))/2$. **1.28.** $y=f_0+f_1$, where $f_0=\cos\frac{\pi}{8}\cos x-\sin\frac{\pi}{12}\cos 2x$, $f_1=-\sin\frac{\pi}{8}\sin x+\cos\frac{\pi}{12}\sin 2x$. **1.29.** (a) $a=0$, b is arbitrary, (b) $b=0$, a is arbitrary. **1.30.** (a) The plus sign, (b) the minus. **1.31.** (a) The minus, (b) the plus, (c) the plus, (d) the plus. **1.32.** (a) The plus, (b) the minus, (c) the minus, (d) the minus. **1.33.** Increasing. **1.34.** Increasing. **1.36.** Decreasing. **1.37.** (a) $\cos t=-3/5$, $\tan t=-4/3$, $\cot t=-3/4$, (b) $\cos t=-12/13$, $\tan t=5/12$, $\cot t=12/5$, (c) $\cos t=4/5$, $\tan t=-3/4$, $\cot t=-4/3$. **1.38.** (a) $\sin t=24/25$, $\tan t=24/7$, $\cot t=7/24$, (b) $\sin t=-7/25$, $\tan t=7/24$, $\cot t=24/7$, (c) $\sin t=-8/17$, $\tan t=-8/15$, $\cot t=-15/8$. **1.39.** (a) $\sin t=3/5$, $\cos t=4/5$, $\cot t=4/3$, (b) $\sin t=3/5$, $\cos t=-4/5$, $\cot t=-4/3$. **1.40.** (a) $\sin t=-5/13$, $\cos t=-12/13$, $\tan t=5/12$, (b) $\sin t=-12/13$, $\cos t=5/13$, $\tan t=$

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* The letters k, l, m, n symbolize any integers if otherwise is not stated.

$-12/5$. **1.41.** (a) $\pm\frac{\pi}{3}+2\pi k$, (b) $\pm\frac{2\pi}{3}+2\pi k$, $\pm\arccos\frac{1}{3}+2\pi k$. **1.42.** (b) $-4\pi/3$, $-2\pi/3$, $2\pi/3$, $4\pi/3$, $8\pi/3$, $10\pi/3$, $14\pi/3$, $16\pi/3$. **1.43.** (c) $\frac{\pi}{6}+\frac{\pi k}{2}$, (e) $\frac{\pi}{2}+\pi k$, (f) πk. **1.44.** (a) $\pi/2$, (b) $\pi/2$, (c) $4\pi/3$. **1.48.** (a) $\arccos\frac{4}{5}$, $\arctan\frac{3}{4}$, $\operatorname{arccot}\frac{4}{3}$, (b) $\arcsin\frac{5}{13}$, $\arctan\frac{5}{12}$, $\operatorname{arccot}\frac{12}{5}$, (c) $\arcsin\frac{5}{13}$, $\arccos\frac{12}{13}$, $\operatorname{arccot}\frac{12}{5}$, (d) $\arcsin\frac{4}{5}$, $\arccos\frac{3}{5}$, $\arctan\frac{4}{3}$. **1.49.** (a) $\pi-\arcsin\frac{2\sqrt{2}}{3}$, $\pi-\arctan 2\sqrt{2}$, $\operatorname{arccot}\left(-\frac{\sqrt{2}}{4}\right)$, $\pi-\arccos\frac{1}{3}$, (b) $\arcsin\left(-\frac{7}{25}\right)$, $-\arccos\frac{24}{25}$, $-\operatorname{arccot}\frac{24}{7}$, $\arctan\frac{7}{24}$, (c) $\pi-\arcsin\frac{24}{25}$, $\arccos\left(-\frac{7}{25}\right)$, $\pi-\arctan\frac{24}{7}$, $\pi-\arccos\frac{7}{25}$, $\pi-\operatorname{arccot}\frac{7}{24}$. **1.50.** $-\frac{2\sqrt{5}}{5}$.

CHAPTER 2

2.18. $\sin t\sin 4s$. **2.19.** $-\sin 2t\sin 4s$. **2.20.** $1/4$. **2.21.** $\sin 4t$. **2.22.** $\frac{3}{4}\sin 8t$. **2.23.** $-\cos^2 2t$. **2.24.** $2\sin\left(6t-\frac{\pi}{3}\right)$. **2.25.** $8\sin\left(t-\frac{\pi}{4}\right)\sin\left(t+\frac{\pi}{4}\right)\sin\left(t-\frac{\pi}{3}\right)\sin\left(t+\frac{\pi}{3}\right)\Big/\cos^4 t$. **2.26.** $-\tan t\tan s$. **2.27.** $-2\sin 2t\sin s\cos(2t-s)$. **2.28.** $8\cos 16t\times\cos^3 2t$. **2.33.** $\pi/4$. **2.34.** $1/2$. **2.35.** $-2\sqrt{5}/5$. **2.36.** 1. **2.37.** 1 or $-1/6$.

CHAPTER 3

3.1. $\pi(2k+1)/10$, $(-1)^k\frac{\pi}{12}+\frac{\pi k}{3}$. **3.2.** $\pi(4k-1)/12$. **3.3.** $\pi(2k+1)/2$, $(-1)^k\frac{\pi}{6}+\pi k$. **3.4.** $\pi(2k+1)/4$, $(-1)^k\frac{\pi}{12}+\frac{\pi k}{2}$. **3.5.** $\pi(2k+1)/4$, $\pi(2k+1)/14$. **3.6.** $\pi(2k+1)/16$, $(-1)^{k+1}\frac{\pi}{12}+\frac{\pi k}{3}$. **3.7.** $\pi k/2$, $\pi(6k\pm 1)/12$. **3.8.** $(-1)^{k+1}\frac{\pi}{6}+\pi k$. **3.9.** $\pi(4k+1)/4$, $\arctan 5+\pi k$. **3.10.** $\pi(4k+3)/4$. **3.11.** $-\frac{\pi}{4}+\pi k$, $\arctan\frac{3}{4}+\pi k$. **3.12.** $\pi k/5$, $\pi(4k-1)/2$, $\pi(4k+1)/10$. **3.13.** $-\frac{\pi}{4}+\pi k$, $\arctan 3+$

πk. **3.14.** $\pi(2k+1)/12$, $\pm\frac{2\pi}{3}+2\pi k$. **3.15.** $\frac{\pi}{4}+\pi k$, $\arctan\frac{1}{3}+\pi k$. **3.16.** $\pi k/3$, $\pi(2k+1)/7$. **3.17.** $2\arctan 3+2\pi k$, $-2\arctan 7+2\pi k$. **3.18.** $\pi(2k+1)/6$, $\pi(4k-1)/4$. **3.19.** $\pi(2k+1)/4$. **3.20.** πk, $(-1)^k\frac{\pi}{6}+\frac{\pi k}{2}$. **3.21.** $\pi(4k-1)/4$. **3.22.** πk, $\pi(4k+1)/4$. **3.23.** πk, $\pi k-\arctan 3$. **3.24.** $\pi(4k-1)/4$, $\pi(2k+1)/2$. **3.25.** $\pi(2k+1)/6$, $k\neq 3l+1$, $\pi k/5$, $k\neq 5l$. **3.26.** $\pm\frac{1}{2}\arccos\frac{\sqrt{73}-7}{12}+\pi k$, $\pm\frac{\pi}{3}+\pi k$, $\pm\frac{1}{2}\arccos(-1/3)+\pi k$. **3.27.** $\pi(4k+3)/32$. **3.28.** πk, $\pi(2k+1)/6$. **3.29.** $\pi(3k\pm 1)/3$. **3.30.** $\pi(2k+1)/8$, $\pi(6k\pm 1)/12$. **3.31.** $\pi k/2$. **3.32.** $\frac{\pi}{4}+2\pi k$. **3.33.** $(-1)^k\arcsin\frac{\pi}{4}+\pi k$. **3.34.** πk. **3.35.** $\pi(4k+1)/4$. **3.36.** $\frac{\pi}{2}+\pi k$. **3.37.** $\pi/2$. **3.38.** $\frac{5\pi}{48}+\pi k$, $\frac{17\pi}{48}+\pi k$, $\frac{7\pi}{24}+\pi k$. **3.39.** πk. **3.40.** $\frac{2\pi}{3}+4\pi k$, $\frac{4\pi}{3}+4\pi k$, $\frac{11\pi}{3}+4\pi k$. **3.42.** $\left(2, \pm\frac{2\pi}{3}-2+2\pi k\right)$. **3.43.** $\{(2, -1), (-2, -1)\}$. **3.44.** $(\pi(2k+1)/4, \pi(6l+1)/6)$. **3.45.** $(\pi(4k+1)/4, -\pi(12k+1)/12)$, $(\pi(12k-1)/12, \pi(1-4k)/4)$. **3.46.** $(\pi(2k+3)/2, \pi(6k-1)/6)$. **3.47.** $\left(\frac{\pi}{6}+\pi(k-l), \frac{\pi}{3}+\pi(k+l)\right)$, $\left(-\frac{\pi}{6}+\pi(k-l), \frac{2\pi}{3}+\pi(k+l)\right)$. **3.48.** $((6k-1)/6)$, $(6k+1)/6)$. **3.49.** $\left(\arctan\frac{1}{2}+\pi k, \arctan\frac{1}{3}-\pi k\right)$. **3.50.** $\left(\pm\frac{\pi}{6}+\pi k, \pm\frac{\pi}{4}+\pi l\right)$, $l-k=2m$. **3.51.** $-2+12k$, $2+12k$.

CHAPTER 4

4.1. See Fig. 40. **4.2.** See Fig. 41. **4.3.** See Fig. 42. **4.4.** See Fig. 43. **4.5.** See Fig. 44. **4.6.** See Fig. 45. **4.7.** See Fig. 46. **4.8.** See Fig. 47. **4.9.** See Fig. 48. *Hint.*

$$y=\begin{cases}-\pi-2\arctan x, & x\in(-\infty, -1],\\ 2\arctan x, & x\in[-1, 1],\\ \pi-2\arctan x, & x\in[1, +\infty).\end{cases}$$

4.10. See Fig. 49. *Hint.* $y=3+\cos 4x$. **4.11.** 4. **4.12.** -0.5. **4.13.** -2. **4.14.** 1. **4.15.** $2(\tan x+\tan^3 x)$. **4.16.** $(\sin^2 x+\sin 2x+1)e^x$. **4.17.** $8/\sin^2 4x$. **4.18.** $2x\cos\frac{1}{x}+\sin\frac{1}{x}$. **4.19.** $2\cos^2 x$. **4.20.** (a) $\cos x/\cos^2\sin x$, (b) $3(\tan^2 x+\tan^4 x)$. **4.21.** Critical points: $x=\pi(12k\pm 1)/12$ ($y'=0$ points of maximum), $x=\pi k/2$ (y' is

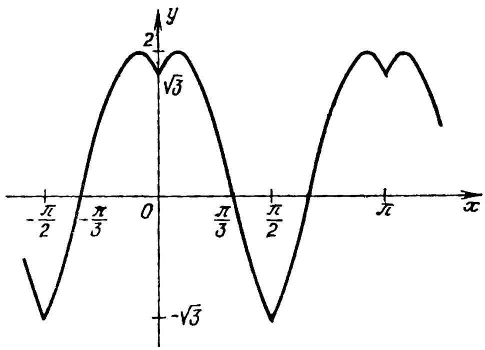

Fig. 40

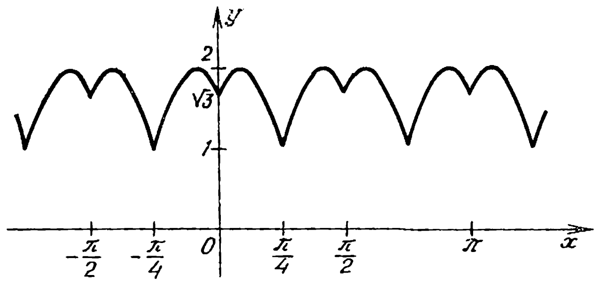

Fig. 41

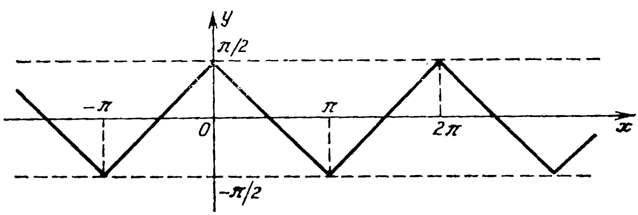

Fig. 42

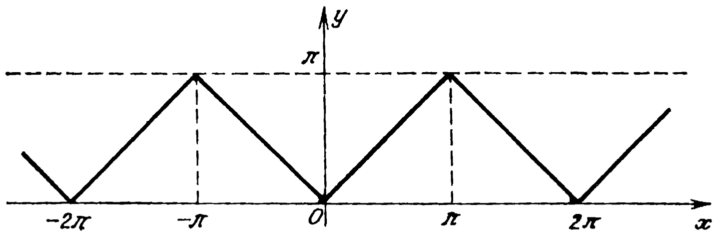

Fig. 43

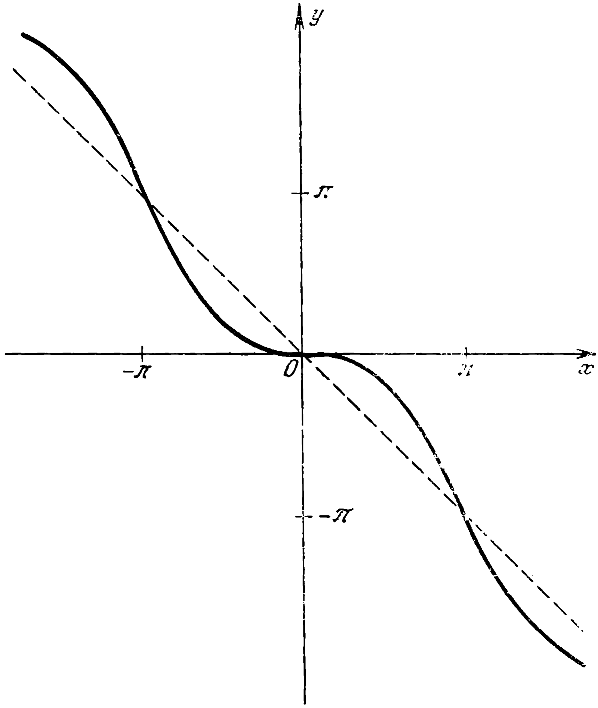

Fig. 44

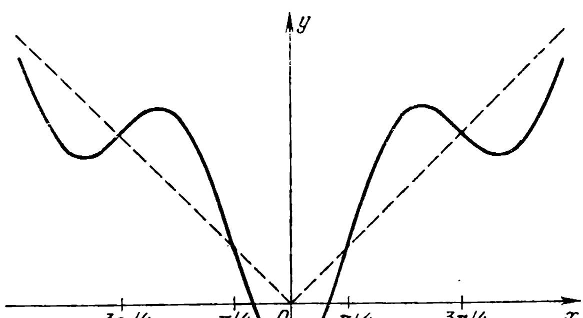

Fig. 45

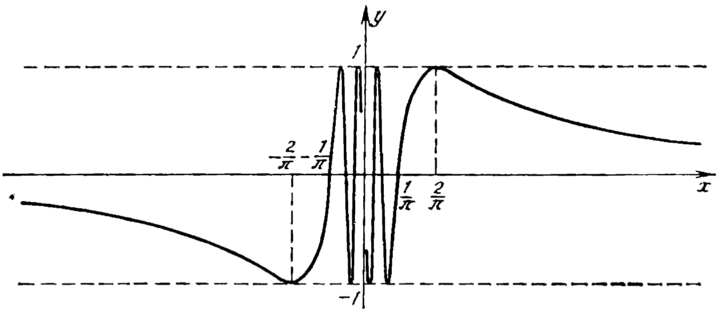

Fig. 46

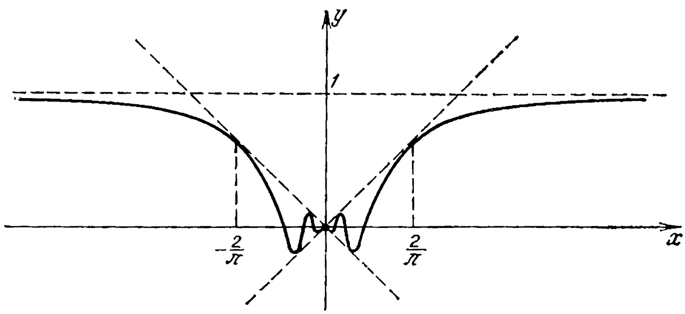

Fig. 47

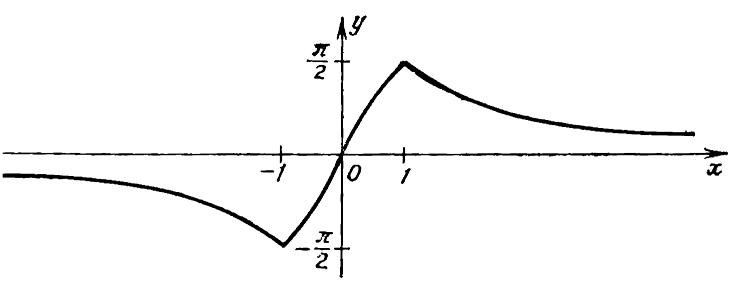

Fig. 48

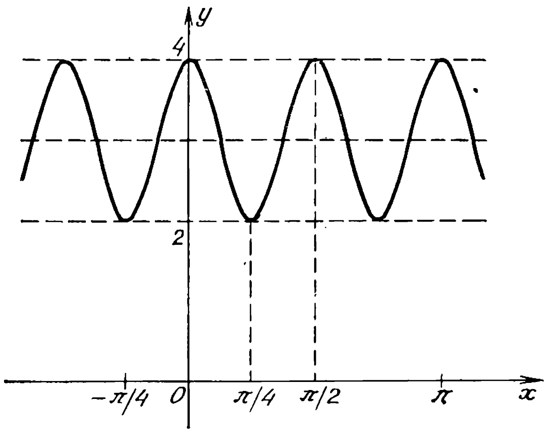

Fig. 49

nonexistent, points of minimum), $\max_{x\in\mathbf{R}} y(x)=2$, $\min_{x\in\mathbf{R}} y(x)=-\sqrt{3}$. **4.22.** Critical points: $x=\pi(6k\pm1)/12$ ($y'=0$, points of maximum), $x=\pi k/4$ (y' is nonexistent, points of minimum), $\max_{x\in\mathbf{R}} y(x)=2$, $\min_{x\in\mathbf{R}} y(x)=1$. **4.23.** Critical points: $x=\pi k$ (the derivative is nonexistent), $\max_{x\in\mathbf{R}} y(x)=\frac{\pi}{2}$, $\min_{x\in\mathbf{R}} y(x)=-\frac{\pi}{2}$. **4.24.** Critical points: $x=\pi k$ (the derivative is nonexistent) $\max_{x\in\mathbf{R}} y(x)=\pi$, $\min_{x\in\mathbf{R}} y(x)=0$.

4.25. Intervals of increase $\left[0,\ \frac{7\pi}{12}\right]$, $\left[\frac{\pi(2k+1)}{4}-\frac{\pi}{6},\ \frac{\pi(2k+1)}{4}+\frac{\pi}{6}\right]$, $k<0$, $\left[\frac{\pi(2l+1)}{4}+\frac{\pi}{6},\ \frac{\pi(2l+3)}{4}-\frac{\pi}{6}\right]$, $l>0$, intervals of decrease $\left[-\frac{7\pi}{12},\ 0\right]$, $\left[\frac{\pi(2l+1)}{4}+\frac{\pi}{6},\ \frac{\pi(2l+3)}{4}-\frac{\pi}{6}\right]$, $l<0$, $\left[\frac{\pi(2k+1)}{4}-\frac{\pi}{6},\ \frac{\pi(2k+1)}{4}+\frac{\pi}{6}\right]$, $k>0$. **4.26.** Intervals of increase $\left[\frac{3}{\pi(4k+3)},\ \frac{2}{\pi(4k+1)}\right]$, intervals of decrease $\left(-\infty,\ -\frac{2}{\pi}\right]$, $\left[\frac{2}{\pi},\ +\infty\right)$, $\left[\frac{2}{\pi(4k+1)},\ \frac{2}{\pi(4k-1)}\right]$, $k\neq0$. **4.27.** πk, $\pi(2k+1)/8$. **4.28.** $b<-3-\sqrt{3}$, $b>-1+\sqrt{3}$. **4.29.** $1/\sqrt{2}$. **4.30.** 2. **4.31.** $y_{\max}=\pi/2$, $y_{\min}=1$. **4.32.** $y_{\max}=2\sqrt{3}/3$, $y_{\min}=1$. **4.33.** $y_{\max}=3/4$, $y_{\min}=0.5$. **4.34.** $y_{\max}=\pi/4$, $y_{\min}=-\pi/4$. **4.35.** $y_{\max}=1.25$, $y_{\min}=1$.

CHAPTER 5

5.1. $\left[-\frac{\pi}{3}+2\pi n,\ \frac{\pi}{3}+2\pi n\right]$. 5.2. $\left(-\frac{5\pi}{8}+\pi n,\ \frac{\pi}{8}+\pi n\right)$. 5.3. $\left[-\frac{5\pi}{6}+10\pi n,\ \frac{35\pi}{6}+10\pi n\right]$. 5.4. $\left(\frac{1}{3}-\frac{\pi}{6}+\frac{\pi n}{3},\ \frac{1}{3}+\frac{\pi}{18}+\frac{\pi n}{3}\right)$. 5.5. $\left(-\frac{\pi}{2}+\pi n,\ -\frac{\pi}{3}+\pi n\right]$, $\left[\frac{\pi}{3}+\pi n,\ \frac{\pi}{2}+\pi n\right)$. 5.6. $\left(\frac{\pi}{3}+\pi n,\ \frac{2\pi}{3}+\pi n\right)$. 5.7. $\left(-\frac{\pi}{6}+\pi n,\ \frac{\pi}{6}+\pi n\right)$. 5.8. $\left(-\frac{\pi}{2}+\pi n,\ -\frac{\pi}{6}+\pi n\right]$, $\left[\frac{\pi}{6}+\pi n,\ \frac{\pi}{2}+\pi n\right)$. 5.9. $\left[2\pi n,\ \frac{\pi}{6}+2\pi n\right]$, $\left[\frac{5\pi}{6}+2\pi n,\ \pi+2\pi n\right]$. 5.10. $\left(-\pi+2\pi n,\ -\frac{\pi}{3}+2\pi n\right)$, $\left(\frac{\pi}{3}+2\pi n,\ \pi+2\pi n\right)$. 5.11. $\left(-\frac{\pi}{4}+\pi n,\ \frac{\pi}{4}+\pi n\right)$. 5.12. $\left(\frac{\pi}{4}+\pi n,\ \frac{3\pi}{4}+\pi n\right)$. 5.13. $\left(\frac{\pi}{12}+2\pi n,\ \frac{5\pi}{12}+2\pi n\right)$, $\left(-\frac{5\pi}{12}+2\pi n,\ -\frac{\pi}{12}+2\pi n\right)$. 5.14. $x \neq \frac{\pi n}{2}$. 5.15. $\left(-\frac{\pi}{3}+\pi n,\ \frac{\pi}{3}+\pi n\right)$. 5.16. $\left(\frac{\pi}{4}+\pi n,\ \frac{5\pi}{12}+\pi n\right)$. 5.17. $\pm\frac{\pi}{3}+2\pi n$, $\left(\frac{\pi}{2}+2\pi n,\ \frac{3\pi}{2}+2\pi n\right)$. 5.18. $\left[\pi n,\ \frac{\pi}{2}+\pi n\right]$. 5.19. $\tan 1$. 5.20. (a) $\pi/4$, (b) $\pi/4$. 5.21. $\tan(\sin 1)$.

Subject Index

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